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THE SOURCE OF TIME INCONSISTENCY IN OPTIMAL PLANS

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Kydland-Prescott (1977) and Calvo (1978) have argued that optimal dynamic plans are time inconsistent in general. It is shown that their demonstrations of this proposition are not valid. However, the proposition is correct because a later optimization problem always drops some condition requred of the original plan. When that condition is not satisfied by, since not required of, the solution to the later problem, the result is time inconsistency.

1. Introduction

An optimal dynamic plan is said to be time inconsistent if at some time after its start, its continuation is not optimal although nothing unforeseen had occurred. If an individual's utility function changes over time, as in the Strotz (1956) formulation, time inconsistency is of course to be expected. If preferences are unchanging but there is another decisionmaker, the government, whose plan or policy influences the choices of the (representative) individual, time inconsistency is not particularly remarkable if the government's objective function is different from that of the individual's. What has attracted much attention is the proposition of Kydland and Prescott (1977) and Calvo (1978) that an optimal policy is in general time inconsistent even when the government maximizes the individual's fixed utility function. Fischer (1980, p. 94, n. 2) has in effect already pointed out that the Kydland-Prescott proof of this proposition is not valid but has not made a similar observation as regards that of Calvo.

In what follows it will be seen why the original demonstrations of Kydland-Prescott and Calvo of the time inconsistency proposition are not valid. However, the proposition is correct, and we will indicate the precise source of time inconsistency in a very simple way.

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2. The Kydland-Prescott Argument

Let the plan or policy $z = (z_1, z_2, \ldots, z_T)$ be the government actions planned for time periods 1 to T, and $x = (x_1, \ldots, x_T)$ the corresponding decisions of the individual, $T = \infty$ being possible.

Assume that

(1) $\quad x_t = f^t(x_1, \ldots, x_{t-1}; z)$

for $t = 1, \ldots, T$ and that there is a social objective function $U(x, z)$, the same as that of the representative individual's, which is maximized by a plan if it is optimal. Suppose $T = 2$ so

(2) $\quad x_1 = f^1(z_1, z_2)$

(3) $\quad x_2 = f^2(x_1; z_1, z_2).$

Kydland and Prescott assume differentiability and an interior solution, so an optimal plan must satisfy the condition $A + B = 0$ where

$$A = \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial z_2} + \frac{\partial U}{\partial z_2}$$

and

$$B = \left(\frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2}\frac{\partial x_2}{\partial x_1}\right)\frac{\partial x_2}{\partial z_2}.$$

If the plan is to be time consistent, one must also have $A = 0$ since z_2 should maximize U when x_1 and z_1 are already given by history. Accordingly, only if $B = 0$ "would the consistent policy be optimal," and with this remark Kydland and Prescott (1977, p. 476) conclude that the "inconsistency of the optimal policy is easily demonstrated by a two-period example." Evidently they mean to say that $B = 0$ is an independent condition that does not hold in general, in which case it would be fortuitous for a policy to be both optimal and time consistent.

It should be clear however that since the individual has the same utility function U, the decision functions (2)-(3) must be such that (x_1, x_2) maximizes U given (z_1, z_2). This means that with an interior solution, $\partial U/\partial x_1 = \partial U/\partial x_2 = 0$ hence $B = 0$ from the start— it is not an independent condition but an implication of utility maximization and an interior solution. (It should be carefully noted that the individual's optimization problem is not to maximize $U(x, z)$ subject to some constraint say $\Phi(x, z) = 0$. In this case obviously one would not have $\partial U/\partial x_1 = \partial U/\partial x_2 = 0$. His problem as put by Kydland and Prescott is simply to maximize $U(x, z)$. We conclude therefore that the Kydland-Prescott proof, which relies on the wrong supposition that $B = 0$ would be fortiuitous, is incorrect. (It should perhaps be emphasized that this remark applies to the "demonstration" presented on p. 476 of their paper; the illustrative examples they give do exhibit time inconsistency.)

3. The Calvo Argument

In Calvo's model of a monetary economy, the demand for money in real terms, m^d, is given by

$$\log m^d = -a\pi^* \quad (a > 0) \tag{4}$$

where π^* is the expected rate of inflation at time t. Let p denote the price level. With perfect foresight and market clearing, $\pi^* = \pi \equiv \dot{p}/p$ and $m^d = m \equiv M/p$, M being the nominal money stock which the government uses to influence m. Other assumptions in the model give

$$M_t = M_0 - \int_0^t p_\tau x_\tau \, d\pi \tag{5}$$

$$x = (m \log m)/a - \dot{m} \tag{6}$$

where x is net real taxes (we follow Calvo's notation here for easier reference). The time path of m is chosen so as to maximize

$$\int_0^\infty (u(c_t) + v(m_t))e^{-\delta t} \, dt$$

where δ is the discount rate and output $c = f(x)$ with $f(x)$ taking on a maximum at $x = 0$. There is an amount m^F, the optimum quantity of money (OQM), such that $v'(m^F) = 0$.

An optimal m path in the model requires $x_t = 0$ and $v'(m_t) = 0$ at $t = 0$, and time consistency requires $m_t = \overline{m}$ (const) for all t. A time consistent optimal policy thus means $x_0 = 0$ and $\dot{m} = 0$ in (6), so log

$\overline{m} = 0$ or $\overline{m} = 1$. Therefore $v'(m_0) = v'(1) = 0$, which implies $m^F = 1$. Calvo (1978, p. 1419) claims that $m^F = 1$ is "a condition that cannot be derived from the assumptions of the model. We can then assert that optimal policies will not generally be time consistent—the exception possibly being when ... $m^F = 1$", but he considers such a "special case where the *OQM* is attained at [this] specific value" (p. 1420) to be in the nature of a fluke.

To see why the argument is not valid, it will suffice to show that $m^F = 1$ is actually a consequence of an implicit normalization in the model. Suppose an optimal m path so $m_0 = m^F$ which for consistency implies $\overline{m} = m^F$. Then M is constant in (5) since $x = 0$ throughout in (6). Meanwhile output $c = f(0) = \overline{c}$ is constant at its maximum value, and therefore $\pi^* = \pi = 0$. But (4) in effect makes the unit for measuring m the amount demanded when $\pi^* = 0$, and since that amount is $m^d = \overline{m} = m^F$, it follows that $m^F = 1$. In other words, this is not at all a special case but a result of the fact that (4) implicitly defines what is the unit of m.

We note for later reference that the optimal policy in the Calvo model puts $m = m^F$ and $c = \overline{c}$ throughout, so $u(\overline{c}) + v(m^F)$ in the objective function is always at its maximum. (Much of the discussion in the mathematical appendixes of Calvo's paper is vitiated by his wrong assumption that $m^F > 1$.)

4. The Source of Time Inconsistency

Returning to the notation of Section 2 but permitting each element of x and of z to be a vector, the government's optimization problem at $t = 1$—call it Problem 1—was to maximize $U(x, z)$ subject to (1) for $t = 1,2, \dots$. Let (x^1, z^1) denote a solution. Obviously this also solves Problem 1': maximize $U(x, z)$ subject to

(7) $\quad x_1 = x_1^1 \,, z_1 = z_1^1$

and (1) for $t = 1, 2, \dots$. Now at $t = 2$ when (7) is already a fact of history, the problem is to maximize $U(x, z)$ subject to (7) and (1) for $t = 2, \dots$—call this Problem 2— whose solution we denote by (x^2, z^2). (Of course, $x_1^2 = x_1^1$ and $z_1^2 = z_1^1$.) The important point is that Problem 2 does not require

(8) $\quad x_1 = f^1(z)$

which Problem 1' does, so

(9) $U(x^2, z^2) \geq U(x^1, z^1)$

from the basic fact that the value of the objective function in a constrained maximization problem is no less but may be higher if one of the constraints is dropped.[1]

Time inconsistency arises at $t = 2$ if and only if $U(x^2, z^2) > U(x^1, z^1)$, for in this case z^2, being a better plan (as it might be called) at $t = 2$ than the original plan z^1, displaces the latter. Observe that $U(x^2, z^2) > U(x^1, z^1)$ if and only if $x_1^1 \neq f^1(z^2)$. (*Proof.* If $x_1^1 = f^1(z^2)$, the solution to Problem 2 satisfies (8) and therefore also solves Problem 1', giving $U(x^2, z^2) = U(x^1, z^1)$. If the latter equation holds, (x^2, z^2) solves Problem 1' as well, implying $x_1^1 = f^1(z^2)$. The result then follows from (9)). We thus have

Proposition 1. Time inconsistency obtains at $t = 2$ if and only if $x_1\ [=x_1^1] \neq f^1(z^2)$.

Since there is no reason in general why a solution to Problem 2 should satisfy (8) which is not a condition of the problem, time inconsistency is therefore generally to be expected.[2] Capital accumulation, taxes, strategic considerations and the like, provide the substance in the examples in the literature that show time inconsistency but do not identify the source of the latter, which is simply a formal property. The time aspect of the inconsistency has only to do with the fact that a later Problem τ drops those equations of the form (1) in the original problem which are no longer needed to determine the already historically given $x_1, \ldots, x_{\tau-1}$. When a better plan is then made possible, the result is time inconsistency.

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[1]That (8) does not appear in Problem 2 (which is therefore different from Problem 1) is obvious and acquires significance only because of the implication (9), which is key to the source of time inconsistency.

[2]Recall that only one utility function is involved. Accordingly the claim of Chari, Kehoe and Prescott (1989, p. 269) that "if all agents' preferences coincide with those of society, then there can be no time-consistency problem" cannot be correct. Their argument is based simply on the absence of inconsistency in a single-period model (which is hardly suitable for bringing out a time-dependent phenomenon) and the unsupported assertion that the result extends to multi-period formulations.

5. Time Consistent Models

What might need more explanation, because less to be expected, is time consistency. Let (x^τ, z^τ) solve Problem τ: maximize $U(x, z)$ subject to $x_t = x^1_t$ and $z_t = z^1_t$ for $t = 1, \ldots, \tau - 1$ and (1) for $t = \tau, \tau + 1, \ldots$. We have

Proposition 2. The optimal plan z^1 is time consistent if and only if

$$(10) \qquad (\forall\ \tau > 1\ \&\ t = 1, \ldots, \tau\text{-}1)\text{: } x^1_t = f^t(x^1_1, \ldots, x^1_{\tau-1}; z^\tau).$$

Proof. If (10) holds, (x^τ, z^τ) also solves Problem 1 hence $U(x^\tau, z^\tau) = U(x^1, z^1)$; therefore no z^τ is better than z^1 to supersede the latter, and no time inconsistency can arise. If z^1 is time consistent, it will be followed through, i.e. $z^\tau = z^1$ for all τ, and (10) is satisfied.

Time consistency thus requires one of the following (not mutually exclusive) conditions, any of which is clearly sufficient: (i) there is no better plan at any later time; (ii) there are no equations of the form (1) in the model, so there are none to be dropped in any later problem; (iii) the model is formulated in a way that in effect maintains those equations. All three cases have figured in the literature.

(i) No better plan can be had if the original plan puts the system in a maximal steady state, as in the time consistent models of Turnovsky and Brock (1980). As noted earlier, this is also true of the Calvo (1978) model, which is actually time consistent.

(ii) In a "command economy" (Fischer, 1980) where the government directly controls x, the equations (1) are dispensed with. Choosing (x, z), $U(x, z)$ is maximized in straightforward fashion. If the government has "a sufficiently rich set of policy instruments" (Hillier and Malcomson, 1984, p. 1437) to control x indirectly, again dispensing with (1), time consistency obtains as well.

(iii) This case is exemplified by the time consistent models of Lucas and Stokey (1983) and Persson, Persson and Svensson (1987). Using a model without money, the Lucas-Stokey innovation is a commitment to honor government debt of various maturities in real terms, which ingeniously restores the effect of dropped constraints. "Normally" (8) would not be needed at $t = 2$ because the value of x_1, which is determined by (8) when the original plan is formulated, is history by then. However, because of the commitment to honor debt due at $t = 2, \ldots$, the Lucas-Stokey framework makes (8) in effect a requirement in Problem 2, whose solution must take account of the commitments in z^1 which have resulted from the private decision x_1^1. Although the government is free at $t = 2$ to restructure the debt, Lucas and Stokey show that under their assumptions there is a z^1 which remains optimal at $t = 2$, etc. The Persson-Persson-Svensson extension, which adds money and nominal debt of various maturities, also has a time consistent solution.

6. Conclusion

Time inconsistency occurs when some constraint in a dynamic optimization problem is not satisfied by the solution to, since not required by, a later optimization problem. It is therefore generally to be expected. If the government and the representative individual have the same utility function, time inconsistency is thus a turn for the better simply because utility is then higher.

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