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STRICT CONVEXITY OF THE UPPER LEVEL SETS OF STRICTLY QUASICONCAVE FUNCTIONS

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This paper shows that a necessary condition for strict quasiconcavity is that each level set is contained in the boundary of the associated upper level set. This condition and the strict convexity of upper level sets are sufficient for strict quasiconcavity and are also necessary when the function is continuous with a strictly convex domain.

1. Introduction

Strict quasiconcavity of a utility function or of a production function is often assumed in order to ensure a unique global utility-maximizer or a unique global cost-minimizer. In these cases, the consumer demand function and the conditional input demand function are single-valued mappings. (For the role of strictly quasiconcave functions in economic theory, see Diewert, 1981).

Quasiconcave functions are characterized by the convexity of their upper level sets (Mangasarian, 1969; Avriel, 1976). It is tempting to conjecture that strictly quasiconcave functions are characterized by strictly convex upper level sets but it is easy to construct examples showing that this is not true. Thompson and Parke (1973) presented some conditions relating strict quasiconcavity to the strict convexity of the upper level sets. In particular, they claimed that a continuous strictly quasiconcave function has strictly convex upper level sets. We give an example to show that this is not true unless the domain of the function is itself strictly convex.

This paper examines the relationship between strict quasiconcavity and the strict convexity of the upper level sets by looking at the relationship between the level set and the boundary of the associated upper level set. We first present a line segment characterization of strictly quasiconcave functions which is analogous to the line segment characterization of quasiconvex functions found in Berge (1963). Then we show that a necessary condition for strict quasiconcavity is that each level set is contained in the boundary of the associated upper level

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set. This condition and the strict convexity of the upper level sets are shown to be sufficient for strict quasiconcavity and are also necessary when the function is continuous with a strictly convex domain.

Notations. R^n denotes the n-dimensional Euclidean space. For any sets S in R^n, the boundary of S is denoted by $b[S]$. If x, y $\in R^n$, the line segment joining x and y is denoted by [x, y]. If we exclude x from [x, y] we use the symbol [x, y]. Functions are real-valued.

2. Line Segment Properties of Strictly Quasiconcave Functions

2.1. Definition. A function f defined on a convex subset C of R^n is quasiconcave (strictly quasiconcave) on C iff x, y $\in$ C, x $\neq$ y, $\theta \in (0,1)$, and $f(x) \leq f(y)$ imply that $f(x) \leq (<) f[(1-\theta)x + \theta y]$.

2.2. Theorem. A function f is strictly quasiconcave on a convex subset C of R^n iff for all x, y $\in$ C, x $\neq$ y, the function g defined by $g(t) = f[(1-t)x + ty]$ is strictly quasiconcave on [0,1].

Proof: ($\Rightarrow$) Let $t_1, t_2 \in [0,1]$, $t_1 \neq t_2$, $\theta \in (0,1)$, and $g(t_1) \leq g(t_2)$. From the definition of g, we have

$$(1)\quad g[(1-\theta)t_1 + \theta t_2] = f[\{1-(1-\theta)t_1 - \theta t_2\}x + \{(1-\theta)t_1, + \theta t_2\}y]$$

$$= f[\{\theta + (1-\theta) - (1-\theta)t_1 - \theta t_2\}x + \{(1-\theta)t_1 + \theta t_2\}y]$$

$$= f[\{(1-\theta)(1-t_1) + \theta(1-t_2)\}x + \{(1-\theta)t_1 + \theta t_2\}y]$$

We note that $(1-t_1)x + t_1y$ and $(1-t_2)x + t_2y$ are elements of C. Since $t_1 \neq t_2$ and x $\neq$ y, then

$$(1-t_1)x + t_1y \neq (1-t_2)x + t_2y.$$

Since

$$(2)\qquad g(t_1) = f[(1-t_1)x + t_1y]$$

and

$$(3)\qquad g(t_2) = f[(1-t_2)x + t_2y],$$

the the condition $g(t_1) \leq g(t_2)$ implies that

$$f[(1-t_1)x + t_1y] \leq f[(1-t_2)x + t_2y].$$

By the strict quasiconcavity of f,

$$f[(1-t_1)x + t_1y] < f[(1-\theta)\{(1-t_1)x + t_1y\} + \theta\{(1-t_2)x + t_2y\}]$$

$$= f[\{(1-\theta)(1-t_1) + \theta(1-t_2)\}x + \{(1-\theta)t_1 + \theta t_2\}y]$$

or, by (1) and (2), $g(t_1), < g[(1-\theta)t_1, + \theta t_2]$.

Hence, g is strictly quasiconcave on [0,1].

($\Leftarrow$) Let $x, y \in C$, $x \neq y$, $\theta \in (0,1)$, and $f(x) \leq f(y)$.

Note that $\qquad g(0) = f(x), \quad g(1) = f(y).$

By assumption, $g(0) = f(x) \leq f(y) = g(1)$. By the strict

quasiconcavity of g, $\qquad g(0) < g[(1-\theta)(0) + \theta(1)]$

or $\qquad f(x) = g(0) < g(\theta) = f[(1-\theta)x + \theta y].$

Hence, f is strictly quasiconcave on C. □

2.3. *Remark.* Theorem 2.2 implies that the restriction of a strictly quasiconcave function to a convex subset of its domain remains strictly quasiconcave.

2.4. *Theorem.* Let f be quasiconcave on a convex subset C of R^n. If f attains its minimum at an interior point of C, then $f(x)$ is constant on some line segment in C.

Proof: Let x^* be a minimizer of f on C, i.e.,

$$f(x^*) = \min \{f(x): x \in C\}.$$

If $f(x)$ is constant on C, then there is nothing to prove. Otherwise, there is an $x^1 \in C$ such that

$$f(x^*) < f(x^1). \tag{4}$$

Since x^* is an interior point of C, then there is an $x^0 \in C$ and a $t^* \in (0,1)$ such that $x^* = (1-t^*)x^0 + t^*x^1$. For each $x \in [x^0, x^*)$, there is a $\theta \in$

$(0,t^*]$ such that $x^* = (1-\theta)x + \theta x^1$. If $f(x^1) \leq f(x)$, then by the quasiconcavity of f,

$$f(x^1) \leq f[(1-\theta)x + \theta x^1] = f(x^*),$$

contradicting (4). Hence, $f(x) < f(x^1)$. Again by the quasiconcavity of f, this implies that

$$f(x) \leq f[(1-\theta)x + \theta x^1] = f(x^*).$$

Since $f(x^*)$ is minimum on C, then $f(x) = f(x^*)$. Hence, $f(x)$ is constant on the line segment $[x^o, x^*]$. □

2.5. *Corollary.* A strictly quasiconcave function defined on a convex subset C of R^n cannot attain its minimum at an interior point of C.

Proof: Follows from Theorem 2.4 and the fact that a strictly quasiconcave function cannot be constant on a line segment.

3. The Upper Level Sets of Strictly Quasiconcave Functions

3.1. *Definitions.* A convex subset C of R^n is strictly convex iff $x, y \in C$, $x \neq y$, and $\theta \in (0,1)$ imply that $(1-\theta)x + \theta y$ is an interior point of C.

Let f be a function defined on a convex subset C of R^n and let α be in the range of f. The upper level set of f at α is defined as

$$UL_f(\alpha) = \{x \in C: f(x) \geq \alpha\}.$$

The level set of f at α is defined as

$$L_f(\alpha) = \{x \in C: f(x) = \alpha\}.$$

3.2. *Theorem.* Let f be strictly quasiconcave on a convex subset C of R^n. Then $L_f(\alpha) \subseteq b[UL_f(\alpha)]$ for every α in the range of f.

Proof: Let $x^* \in L_f(\alpha)$. Then $x^* \in UL_f(\alpha)$. Since f is quasiconcave, then $UL_f(\alpha)$ is a convex subset of C (Mangasarian, 1969). Hence, the restriction of f to $UL_f(\alpha)$ is strictly quasiconcave. Note that

$$f(x^*) = \min \{f(x): x \in UL_f(\alpha)\}$$

since $f(x^*) = \alpha$. By Corollary 2.5, x^* is not an interior point of $UL_f(\alpha)$. Hence, $x^* \in b[UL_f(\alpha)]$. □

3.3. *Example.* Let f be defined on the closed interval [0,3] by

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & 1 \le t \le 2 \\ 3 - t, & 2 \le t \le 3 \end{cases}$$

The upper level sets, being closed intervals, are strictly convex but f is not strictly quasiconcave (Figure 1). We have $UL_f(1) = [1,2]$, $b[UL_f(1) = \{1,2\}$ and $L_f(1) = [1,2]$. Hence, the level set at $\alpha = 1$ is not contained in the boundary of the upper level set. We prove next that if the upper level sets are strictly convex and each level set is contained in the boundary of the upper level set, then f is strictly quasiconcave.

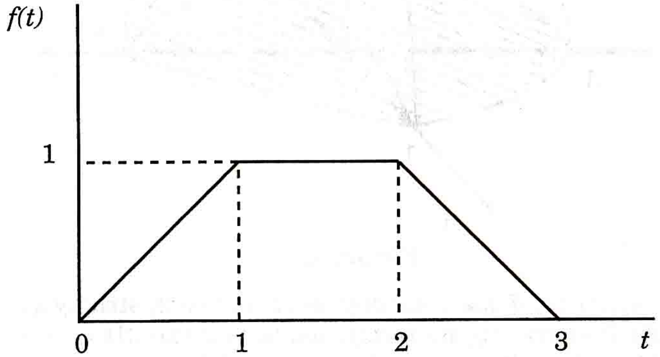

Figure 1

3.4. *Theorem.* Let f be defined on a convex subset C of R^n. If $UL_f(\alpha)$ is strictly convex and $L_f(\alpha) \subseteq b[UL_f(\alpha)]$ for all α in the range of f, then f is strictly quasiconcave on C.

Proof: Let $x, y \in C$, $x \neq y$, $\theta \in (0,1)$, and $f(x) \le f(y)$. Let $f(x) = \alpha$. Then $x, y \in UL_f(\alpha)$. By the strict convexity of $UL_f(\alpha)$, the point of $z \equiv (1-\theta)x + \theta y$ is an interior point of $UL_f(\alpha)$; hence, $z \notin b[UL_f(\alpha)]$ and so $z \notin L_f(\alpha)$. Hence, $f(z) > \alpha = f(x)$ and so f is strictly quasiconcave on C. □

3.5. *Remarks.* Strict quasiconcavity does not necessarily imply strict convexity of the upper level sets even under continuity. Let

$$C = \{(x_1, x_2): x_1^2 + x_2^2 \leq 1, x_1 + x_2 \leq 1\}$$

and define f on C by

$$f(x_1, x_2) = (1 - x_1^2 - x_2^2)^{\frac{1}{2}}$$

Note that f is continuous on C. The graph of f is the portion of the hemispherical dome above C (Figure 2).

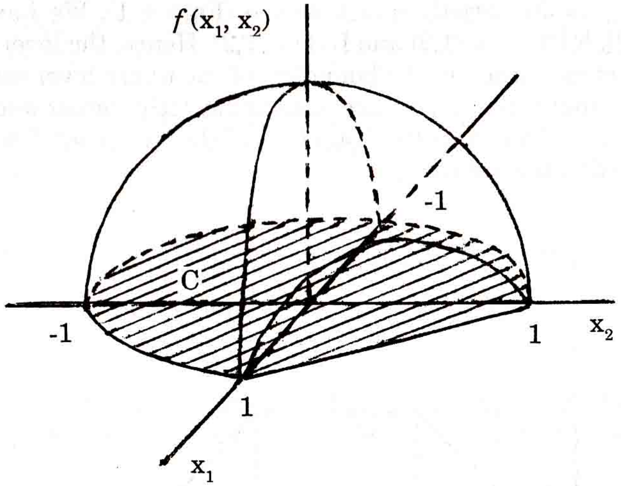

Figure 2

A hemispherical dome is strictly concave, hence, strictly quasiconcave. By Remark 2.3, its restriction to C is strictly quasiconcave on C. Note that $UL_f(0) = C$ which is not strictly convex. However, we prove in the next theorem that if a strictly quasiconcave function is continuous on a strictly convex domain, then its upper level sets are strictly convex.

3.6. *Theorem.* Let f be defined and continuous on a strictly convex subset C of R^n. If f is strictly quasiconcave on C, then $UL_f(\alpha)$ is strictly convex for all α in the range of f.

Proof: Let $UL_f(\alpha)$ be an upper level set of f. Let $x, y \in UL_f(\alpha)$, $x \neq y$, $\theta \in (0,1)$ and $f(x) \leq f(y)$. Let $z = (1-\theta)x + \theta y$. By the strict quasiconcavity of f, $\alpha \leq f(x) < f(z)$. This implies that $z \in UL_f(\alpha)$. By the strict convexity of C, z is an interior point of C. By the continuity of f, there is a neighborhood $N(z)$ such that for all $u \in N(z)$, $f(u) > \alpha$. This implies that $N(z) \subseteq UL_f(\alpha)$. Hence, z is an interior point of $UL_f(\alpha)$. It follows that $UL_f(\alpha)$ is strictly convex. □

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