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PRELIMINARY NOTES ON MODES OF PRODUCTION

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For the same technology, historically observed modes of organizing labor — slavery, serfdom, and freeholding — are compared in terms of Pareto-superiority. Conditions are sought under which one is superior to the others.

In general we take output, q, to be a function of labour, a, so that $q = f(a)$. f is assumed to be twice differentiable, with $f' > 0$, $f'' \leq 0$ in the relevant range. It will be understood, however, that production contains a random element, so that while the labour input itself might be directly known (say by the worker) the output is subject to a distribution (owing, say, to weather) which is independent of the input.

The basic unit of analysis is the relation between two agents, who are identical and both expected-utility maximisers. Utility, U, is a function of labour and real income, y, i.e. $U = u(a, y)$, where $u_1 < 0$, $u_{11} < 0$, $u_2 > 0$, $u_{22} < 0$. In some cases it will be assumed that u is also separable, so that $u_{12} = u_{21} = 0$.

1. Slavery

Denote by p the probability that the slave will be detected "shirking", in which event he suffers punishment, which is associated with an extremely low level of "utility", U^o. On the other hand, the slave is maintained through a subsistence ration, $\overline{w}$, which is his only income. p is determined by a function, v, which is increasing in monitoring costs, m, (measured in units of output), and which is decreasing in a, the time the slave chooses actually to be at work. Thus, $p = v(m, a)$, $v_1 > 0$, $v_2 < 0$, with $0 \leq p \leq 1$. We leave open the question of the sign of v_{22} given any m, and also assume for simplicity that $v_{12} = v_{21} = 0$. A restriction that occurs naturally, however, is that for any $m > 0$, $v(m, 0) = 1$, while $v(m, T) = 0$, where T is the slave's total available time.

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For given m, the slave's problem is to choose the level of effort which maximises the following:

(1) $EU^w = [1 - v(m, a)]u(a, \bar{w}) + v(m, a)U^o$

Since m, $\bar{w}$, and U^o are given, the only control variable is a, and maximisation yields:

(2a) $(1 - p)u_1 + v_2(U^o - u) = 0$

(2b) $(1 - p)u_{11} - 2u_1v_2 + v_{22}(U^o - u) < 0$

as first and second order conditions, respectively, which may be used to define the optimal $a = a^*$ for given m. Defining effort, a, as an implicit function, h, of m, i.e., $a = h(m)$, we have:

(3) $h'(m) = u_1v_1 /\{(1 - p)u_{11} - 2u_1v_2 + v_{22}(U^o - u)\} > 0$

whose sign derives from the denominator being identical to (2b) and the numerator being negative as well. In other words, an increase in the level of monitoring raises the slave's labour effort. (A similar argument shows that raising U^o would reduce effort, i.e., that a less severe punishment would reduce effort, a point that will not be pursued much further, however.)

With knowledge of the function, h, which we might call the "labour-extraction function" (after Bowles), the owner, who is assumed to engage in no labour, proceeds to choose that level of m which maximises his expected utility:

(4) $EU^b = Eu(0, q - \bar{w} - m) = Eu[0, f(h(m)) - \bar{w} - m]$

Since the owner does not work, however, maximising (4) reduces to maximising the expected value of net income with respect to m, a procedure which yields the following conditions:

(5a) $Ef'h'(m^*) - 1 = 0$, or $Ef' = 1/h'(m^*)$

(5b) $[h'(m^*)]^2f'' + h''(m^*)f' < 0$

Equation (5a) states merely that at a maximum the expected marginal product of labour must equal the marginal cost (in output units) of extracting an additional unit of effort. On the other hand, (5b) is fulfilled if $f'' \leq 0$ and-or $h'' < 0$ at m^*.

2. Private Production

The question whether a given production relationship will be entered into can be answered only through a comparison with others, given some postulate regarding social choice. (If the answer was always that the stronger are able to dictate upon the weaker, then economic analysis would become superfluous; but this is a view which was rejected by Engels himself in *Anti-Dühring*.) Taking a cue from game theory, we impose restrictions on the outcome which guarantee that the benefits from participating in the relationship are no worse than can be obtained by the individual on his own. In other words, association must be blocked by coalitions of one.

For this reason, one needs to specify the lower bounds for the payoffs in associated production, which in this case are determined by the alternative of private production. Assume for the moment that the production function under private production is given by g, with $g' > 0$, $g'' \leq 0$, and where g is not necessarily the same as f in the previous section. Then maximising $EU = Eu(a, q)$, where $q = g(a)$, results in:

(6a) $E(u_1 + u_2 g') = 0$

(6b) $E(u_{11} + g' u_{21} + u_{22} g'') < 0$

which when used to solve for the optimal value of $a = \hat{a}$, also gives the optimal value of $q = \hat{q}$ through g. Furthermore, the results hold for both agents, since they are assumed identical. These results must then be compared to those obtained in the previous section.

First we assume that the private mode is productive at least in the sense that the expected optimal output must cover subsistence, that is,

(7) $Eg(\hat{a}) \geq \bar{w}$

On the other hand, the productivity of the slave mode may be defined as the expected excess of net output (i.e. net of subsistence wages and monitoring costs) over what is achievable in private production:

(8) $Ef(a^*) - \bar{w} - m^* \geq Eg(\hat{a})$.

(This may be too strong a requirement; an alternative would be to require that the slaveowner make enough to survive, i.e. replace $Eg(\hat{a})$

in (8) by $\bar{w}$. In view of (7), this is also implied by, but does not imply (8).) In any event, using (8), we obtain the equivalent expression:

(8') $Ef(a^*) - Eg(\hat{a}) \geq \bar{w} + m^*$.

Or, in order to be productive, the slave mode must produce an excess over what is obtainable from private production which at least covers subsistence and monitoring costs. (Here it will be seen it does not really matter whether the owner gets $Eg(\hat{a})$ or $\bar{w}$. If technology is identical in both modes, i.e., if $f = g$, then it follows that $a^* > \hat{a}$, or the worker must work longer under the slave mode.

For slavery to be "necessary," however, it must not only be productive but also in the core, i.e., be no worse than what the individual agent can secure on his own. Hence we require, from the worker's viewpoint:

(9) $Eu[\hat{a}, g(\hat{a})] \leq (1-p)u(a^*, \bar{w}) + pU^o$,

or that the expected utility of the worker under slavery less than that under private production (and p is understood to be evaluated according to (1)–(5).) Likewise, from the owner's view:

(10) $Eu[\hat{a}, g(\hat{a})] \leq Eu[0,.f(a^*) - w - m^*]$.

In regard to (10), it is evident that a situation of no-effort *cum* uncertain income is preferable to labour *cum* uncertain income, so long as (in this case) the expected values of the incomes do not diverge too widely, i.e., so long as

$$E[f(a^*) - w - m^*] \geq Eg(\hat{a}), \text{ or } Ef(a^*) - Eg(\hat{a}) \geq \bar{w} + m.$$

But this is exactly what is required by productivity in (8) above, so this is fulfilled by hypothesis.

On the other hand, one might question why the worker should enter into a slavery agreement. At first glance, and assuming $\hat{a} < a^*$, it might seem the tradeoff is between less effort with uncertain income, as against more effort with certain income. Upon closer examination, however, the matter is not as straightforward. To see this, note that since the labour-effort argument in the utility function is not a random variable, we may rewrite (9) as follows:

(9') $u[\hat{a}, Eg(\hat{a})] \leq (1-p)u(a^*, w) + pU^o$.

For given $(a^*, \bar{w})$, the right-hand side of this inequality increases as p approaches zero, attaining in the limit a value of

(10) $\bar{U} = u(a^*, \bar{w})$.

It is evident that $\bar{U}$ is always no less than the right-hand side of (9'). We already know, however, that $Eg(\hat{a}) \geq \bar{w}$ from (7), so that the only way $\bar{U}$ might exceed $u[\hat{a}, Eg(\hat{a})]$ is if $\hat{a} > a^*$, which contradicts the hypothesis. In particular, if $f = g$, then one must have $\hat{a} < a^*$ and $\bar{U} < u[\hat{a}, Ef(\hat{a})]$, and a fortiori, $u[\hat{a}, Ef(\hat{a})]$ must exceed the right-hand side of (9'). By virtue of this, we have shown:

Proposition 1. If the same technology is available in private production, slavery is Pareto-inferior.

The content of Proposition 1 may not be as innocent as might first appear. From it may be drawn that, in order for slavery to assert itself, the conditions of private production which may make the latter a viable alternative to the worker must be drastically reduced or denied to him. Taking this view, it perhaps becomes no surprise that slavery has historically first been extended to war-captives or to foreigners, for whom individual production was difficult or impossible for material or juridical reasons. A historical example is provided by the early American colonies, which found it difficult to retain white slaves owing to an open land frontier. By contrast, the later stability of Southern black slavery owed in part to its extension over a wider area, which effectively denied the alternative of private production to the slaves.

More constructively, (or "destructively," depending on one's point of view), we prove the following:

Proposition 2. For some conditions of private production which are inferior enough (in a sense to be defined below), some slavery-equilibrium is in the core.

In effect, we are required to show that (9) and (10') can be simultaneously fulfilled. If we rewrite both, however, they can be expressed purely in terms of monitoring costs, m. To wit:

(12) $[1 - v(h(m), m)]u[h(m), \bar{w}] + v(h(m), m)U^o \geq Eu[\hat{a}^k, g^k(\hat{a}^k)]$

(13) $Ef(h(m)) - \bar{w} - m \geq Eg^k(\hat{a}^k)$

where the k-superscript on the production function g^k denotes it is a member of a certain set of functions G, for each of which some private-production equilibrium $(\hat{a}^k, \hat{q}^k)$, according to (6a)-(6b) may be defined.

For any given $g^k \in G$, we may define the following:

$$M^w(g^k) = \{m \mid m \text{ is a solution to (12)}\}$$

$$M^b(gk) = \{m \mid m \text{ is a solution to (13)}\}$$

The proposition will have been proven if we can show that for some $g^{k'} \in G$, the intersection of M^w and M^b is not empty.

Now consider the sequence of functions $\{g^k\}$, $k = 1,2, \ldots$, where each $g^k \in G$, and for $i > j$, it is true that

$$g^i(a) < g^j(a), \text{ for all } a.$$

In this sense, the technology of i is "inferior" to that of j, and the series $\{g^k\}$ represents increasingly inferior technology in private production. This implies in turn that for $i > j$ in the sequence:

$$Eu[\hat{a}^i, g^i(\hat{a}^i)] < Eu[\hat{a}^j, g^j(\hat{a}^j)].$$

Hence, the sequence $k = 1, 2, \ldots$ is also associated with decreases in the right-hand side of (12). Since the left-hand side is a continuously decreasing function of m, however, a lower right-hand side expression must increase the intervals representing the allowable values of m, i.e., for $i > j$,

$$M^w(g^i) \supset M\,(g^j).$$

Of course we know from Proposition 1 that for some g^k (e.g. those in the neighbourhood of f), $M^w(g^k)$ may have no intersection with the nonnegative real numbers. Therefore we must first show that for some function in the sequence, say g^o, $M^w(g^o)$ intersects the nonnegative real numbers.

There is nothing keeping us from defining g^o so that it fulfills $Eu[a^o, g^o(a^o)] = U^o$. Then, as $g^k \to g^o$, $Eu[(\hat{a}^k, g^k(\hat{a}^k)] \to U^o$. After rewriting (12) somewhat and taking limits:

$$\lim_{g^k \to g^o} \{[1 - v\,(h\,(m), m)]u\,[h\,(m), \varpi]\} \geq \lim_{g^k \to g^o} \{Eu\,[\hat{a}^k, g^k\,(\hat{a}^k)] - v\,(h(m\,, m)U^o\}$$

Or, equivalently,

(14) $(1 - v)u \geq U^o - vU^o$, or simply, $u \geq U^o$.

But (14) is fulfilled for any $m \geq 0$. Thus for at least some subsequence of $\{g^k\}$. say for $k = k'+1, k'+2, \ldots$, where $g^k \rightarrow g^o$, $M^w(g^k) \rightarrow \Omega$, the nonnegative real numbers. Hence we are able to map the indices $k'+1$, $k'+2, \ldots$ onto intervals of the form:

$[0, m_{k'+1}]$, $[0, m_{k'+2}]$, . . . , where $m_i > m_j$ for $i > j$.

On the other hand, looking at (13), one finds that, since the left-hand side is a concave function of m (as required by (5b)), then $M^b(g^k)$ is a closed interval of the form $[m^1_k, m^2_k]$. Furthermore, as the right-hand side decreases, these intervals enlarge, so that for $i > j$, $M^b(g^i) \supset M^b(g^j)$. Disregarding portions of those M^b which intersect with the negative numbers, we obtain a sequence of nested intervals in Ω associated with the sequence $\{g^k\}$.

We have shown, therefore, that for at least a subsequence $g^{k'} \rightarrow g^o$ in the defined sequence $\{g^k\}$, $M^w(g^k) \rightarrow \Omega$, while for all k in the subsequence, $M^b(g^k)$ is a subset of Ω. For some k^* large enough, therefore, $M^w(g^{k*}) \cap M^b(g^k)$ is nonempty, as was to be shown.

3. Serfdom

Hinton defines the essence of serfdom as "the transference to the use of the lord of the labour of the peasant family which was surplus to that needed for the family's subsistence and economic reproduction."

In our established convention, we may define the worker's expected utility under serfdom as:

(15) $EU^w = Eu\,[a, S(a-k)]$ where $(a - k) > 0$

k is the amount of labour effort demanded by the lord, a is the worker's total labour effort, and therefore $(a - k)$ is the labour input into the worker's production for himself, which is governed by the production function S.

Maximising (11) with respect to the only control variable, a, yields the following conditions:

(16a) $E(u_1 + u_2S') = 0$

(16b) $E(u_{11} + u_{21}S' + u_{22}S'') < 0.$

(16a) and (16b) may be used to determine the optimal $a = \tilde{a}$ for given k, and define a as an implicit function of k, say, $\tilde{a} = r(k)$, with

(17) $r'(k) = E(u_{12}S' + u_{22}(S')^2 + U_2S'') / E(u_{11} + u_{21}S' + u_{22}S'').$

A sufficient (though not necessary) condition for $r'(k)$ to be positive is that $u_{12} = u_{21} = 0$, which is fulfilled if the utility function is separable. This makes the numerator negative, while the denominator is negative by virtue of (16b).

An additional result that proves useful later is that (rather trivially) an increase in k should result in a reduction of utility. This is a consequence of the envelope-theorem. Hence, if the initial level of utility is $u\ [r(k), ES(r(k) - k]$ for some k, a parametric change in k must result in:

(18) $u_1r' + Eu_2S'(r' - 1) < 0$

The decision problem of the lord, on the other hand, is to choose that level of k which maximises his own expected utility:

(19) $EU^b = Eu[0, F(k) - c(k)]$

where once more the lord is assumed not to engage in production, F is the production function for the demesne, and c is a monitoring-cost function which is increasing in k, the labour-input into the demesne. Owing to the no-labour assumption, maximising (19) once more reduces to maximising the expected value of net income, $F(k) - c(k)$, which gives:

(20) $EF'(\tilde{k}) - c'(\tilde{k}) = 0$

as a first-order condition. Assuming second-order conditions are fulfilled as well, (20) suffices to determine the optimal k and to close the model.

It will be noted that the function c acts to set an upper bound on the level of k, for in view of (17), it will always be in the lord's interest to set k as high as possible. An alternative way to close the model is as follows: suppose c rises only gradually, so that c' is low for all k. Then ultimately a high k would be set according to (20); but on the other hand, this $\tilde{k}$ may be incompatible with the serf's survival. Even before $\tilde{k}$ is reached, therefore, a least-upper bound on k will have been given by k_{max}, where:

(21) $ES[r(k_{max}) - k_{max}) = \overline{w}.$

Further, since $k_{max} < \tilde{k}$, $EF'(k_{max}) - c'(k_{max}) > 0$. At this point, it will be noticed, we have joined with Hinton's definition quoted above.

4. Slavery and Serfdom Compared

Following our procedure, the shift to serfdom requires that this be chosen by both worker and owner. Hence, for the worker, one stipulates:

(22a) $u[\tilde{a}, ES(\tilde{a} - \tilde{k})] > (1 - p)u(a^*, \overline{w}) + pU^o$, or

(22b) $p > \{u(a^*, \overline{w}) - u(\tilde{a}, ES(\tilde{a} - \tilde{k})\} / \{u(a^*, \overline{w}) - U^o\}$

where p is understood to be evaluated as $v(m^*, a^*)$, $a^* = h(m^*)$, and $\tilde{a} = r(\tilde{k})$. Since under equilibrium in serfdom $ES(\tilde{a} - \tilde{k}) = \overline{w}$, the value of the numerator hinges upon the relationship between a^* and a, while the denominator is positive. We summarize our observations based on (22b) in the following:

Proposition 3. The worker's preference for serdom over slavery is the greater if:

(a) the likelihood and severity of punishment under slavery are high (i.e., p is high, and $u(a^*, \overline{w}) - U^o$ is high, which really means U^o is low);

(b) either $a^* \geq \tilde{a}$, or $\tilde{a} > a^*$ by an amount that is not "too large".

In regard to (b), if $a^* \geq \tilde{a}$, then the numerator is nonpositive, and the inequality (22b) is immediately fulfilled. For some $\tilde{a} > a^*$, however, it is still possible that the right-hand side of (22b) will be a positive number smaller than p. From this we draw the following:

Corollary. The worker may choose serfdom over slavery, although the former may be associated with inferior working conditions, i.e., working longer hours for the same expected real income. (Is this the premium on "freedom"?)

For the owner to prefer serfdom, on the other hand, one requires that the net income exceed that under slavery, or:

(23) $EF(\tilde{k}) - c(\tilde{k}) > Ef(a^*) - \bar{w} - m^*$, or

transferring $\bar{w}$ to the left-hand side, and noting that $ES(\tilde{a} - \tilde{k}) = \bar{w}$, and moving $c(\tilde{k})$ to the right:

(24) $EF(\tilde{k}) + ES(\tilde{a} - \tilde{k}) - Ef(a^*) > c(\tilde{k}) - m^*$.

The left-hand side of (20) is simply the difference between total output under serfdom and that under slavery, while the right-hand side is the difference in monitoring costs. Hence, (20) merely states that

Proposition 4. A necessary condition for serfdom to be chosen over slavery is that any losses in total output must be less than savings in monitoring costs owing to the shift.

The essential information in (20) is that serfdom need not be associated with higher output compared with slavery if in the process monitoring costs are sufficiently lowered. For the latter implies that the right-hand side of (20) is negative, implying that the left-hand side may also be negative, albeit less negative than the former. The innovation introduced by serfdom is that, by allowing the worker some production decisions that relate to his own survival, the owner reduces the amount of monitoring to that which is needed only to cover "surplus" labour, k.

The next task is again to show what the core looks like. If we return to (22a), we see that its boundary may be written so that $\tilde{k}$ may be regarded as an implicit function of m^*:

$$u[r(\tilde{k}), ES(r(\tilde{k}) - \tilde{k})] - u[h(m^*),\ \bar{w}]\,[1 - v(m^*, h(m^*)] - v(m^*, h(m^*))U^o = 0.$$

This gives the locus of points $(m^*, \tilde{k})$ where the worker is just indifferent between the two modes. Calling this implicit function W, we obtain:

$$W'(k) = -\ \{(1 - v)u_1 h' + u(v_1 + v_2 h') - U^o(v_1 + v_2 h')\}/\{u_1 r' + u_2 ES'(r' - 1)\}$$

$$= -\ \{[(1 - v)u_1 + v_2(U^o - u)]h' + (u - U^o)v_1\}/\{u_1 r' + u_2 ES'(r' - 1)\}$$

But the first two terms in the numerator sum up to zero (see (2a)), while the denominator is negative according to (18). Hence, we are able to write:

(25) $W'(k) = -(u - U^o)v_1 / (u_1 r' + u_2 ES'(r' - 1) > 0$

The Figure shows this curve *W* drawn on the (m, k) space. Points above *W* represent combination of m^* and k, i.e., equilibria, where the worker prefers slavery to serfdom; points below represent equilibria where serfdom is preferred.

Following the same procedure for the owner, we express (23) in terms of (m, k) and look at its boundary:

$$EF(k) - c(k) - Ef(h(m^*)) - \bar{w} - m^* = 0$$

Calling *B* the implicit function which relates k to m^*, we derive:

(26) $B'(k) = (Ef'h' - 1) / (ES' - c')$

But the numerator is zero, as required by (5a), while the denominator is positive by (21). Thus $B'(k)$ is zero. This curve is also shown in the Figure with points above representing combinations where the owner prefers serfdom, and those below where he prefers slavery. If the curves intersect at all, then the core exists and contains serfdom-equilibria. In the Figure below the core contains both slavery- and serfdom- equilibria, represented by the areas lined and cross-hatched, respectively.