Philippine Review of Economics and Business
Volume XX, No. 1, March, 1983

REGRESSION BY MINIMUM SUM OF ABSOLUTE ERRORS: A NOTE ON PERFECT MULTICOLLINEARITY

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1. Introduction

Consider the multiple linear regression model

$$y = x\beta + \epsilon \tag{1}$$

where y is the regressand, $x = [x_1, x_2, \ldots, x_k]$ the vector of regressors, $\beta = [\beta_1, \beta_2, \ldots, \beta_k]'$ the vector of unknown coefficients, and ϵ the stochastic disturbance term. The most widely used method of estimating β is by least squares, i.e., by minimizing the sum of squared errors (MSSE). Another method is that of minimizing the sum of absolute errors (MSAE), i.e., the MSAE estimate of β is obtained by minimizing $\Sigma_i \mid \epsilon_i \mid$. Although MSAE estimation was suggested as far back as 1888 by Edgeworth (Bowley, 1928), its use has been limited because of computational difficulties. It was only in the 1950s that articles appeared (Charnes et al., 1955; Wagner, 1959) showing that the MSAE estimator can be obtained as a solution to a linear programming problem.

The MSAE regression problem is stated as follows:

$$\textit{MSAE:}\ \text{Minimize} \quad \sum_{i=1}^{n} \mid \epsilon_i \mid$$

$$\text{s.t.} \quad \sum_{j=1}^{k} x_{ij}\beta_j + \epsilon_i = y_i, \qquad i = 1, 2, \ldots, n.$$

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where x_{ij} is the ith observation on the jth regressor. To transform this into the standard form of the linear programming problem, we introduce the positive and negative parts of a variable ν of arbitrary sign by letting

$$\nu^{+} = \max\{0, \ \nu\}$$
$$\nu^{-} = \max\{0, \ {}^{-}\nu\}$$

Then

$$\nu = \nu^{+} - \nu^{-}$$
$$|\nu| = \nu^{+} + \nu^{-},$$

and

$$\nu^{+} \geqq 0, \nu^{-} \geqq 0.$$

Thus, the MSAE problem can be restated as follows:

MSAE-LP1:

Minimize $$\sum_{i=1}^{n} (\epsilon_i^{+} + \epsilon_i^{-})$$

s.t. $$\sum_{j=1}^{k} x_{ij} (\beta_j^{+} - \beta_j^{-}) + \epsilon_i^{+} - \epsilon_i^{-} = y_i, \qquad i = 1, 2, \ldots, n$$

$$\beta_j^{+}, \beta_j^{-}, \epsilon_i^{+}, \epsilon_i^{-} \geqq 0.$$

If we set

$$u = [1, \ 1, \ldots, \ 1]' \text{ (an n– vector)}$$

$$\beta^{+} = [\beta_1^{+}, \ \beta_2^{+}, \ldots, \beta_k^{+}]'$$

$$\beta^{-} = [\beta_1^{-}, \ \beta_2^{-}, \ldots, \beta_k^{-}]'$$

$$\epsilon^{+} = [\epsilon_1^{+}, \ \epsilon_2^{+}, \ldots, \epsilon_n^{+}]'$$

$$\epsilon^{-} = [\epsilon_1^{-}, \ \epsilon_2^{-}, \ldots, \ \epsilon_n^{-}]'$$

$$y = [y_1, \ y_2, \ldots, y_n]'$$

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots x_{1k} \\ x_{21} & x_{22} & \dots x_{2k} \\ \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots x_{nk} \end{bmatrix}$$

then the MSAE problem can be written in matrix form:

MSAE-LP2: Minimize $u'\epsilon^+ + u'\epsilon^-$

$$\text{s.t. } X\beta^+ - X\beta^- + I\epsilon^+ - I\epsilon^- = y$$

$$\beta^+, \beta^-, \epsilon^+, \epsilon^- \geqq 0.$$

2. The Problem of Perfect Multicollinearity

Any linear programming subroutine can solve MSAE-LP2. Narula and Wellington (1977) developed a procedure that is based on an efficient dual simplex algorithm. The authors, however, remark that "unlike the MSSE regression line, the calculations for the MSAE regression line are not affected by linear dependencies among the regressor variables." This statement is inaccurate. The presence of linear dependencies among regressor variables results in multiple estimates of β under MSAE estimation; in fact, there are an infinite number of estimates. Furthermore, a similar situation also holds under MSSE estimation when the regressor variables are linearly dependent.

The presence of linear dependencies among the regressor variables is called the problem of extreme or perfect multicollinearity. The normal equations under MSSE estimation of (1) is given by

(2) $$X'X\beta_{MSSE} = X'y$$

Perfect multicollinearity makes $X'X$ a singular matrix which makes it impossible to obtain $\hat{\beta}_{MSSE}$ since the ordinary inverse of $X'X$ does not exist. However, the normal equation (2) is always consistent

(Graybill, 1969) and a solution (in fact, an infinite set of solutio for $\hat{\beta}_{MSSE}$ can be obtained by using the generalized inverse of X' The general solution is given by

$$(3) \qquad \hat{\beta}_{MSSE} = (X'X)^g X'y + [I - (X'X)^g (X'X)]\, z$$

where $(X'X)^g$ is the generalized inverse of $X'X$, I is the identi matrix, and z is an arbitrary vector (Graybill, 1969).

We now show that in the presence of perfect multicollinearit the estimate of β under MSAE estimation is not unique, i.e., MSA LP2 has an infinite number of optimal solutions. It is clear th MSAE-LP2 has a feasible solution given by

$$\beta_j^+ = \beta_j^- = 0, \qquad j = 1, 2, \ldots, k;$$

$$\left.\begin{aligned} \epsilon_i^+ &= y_i \\ \epsilon_i^- &= 0 \end{aligned}\right\} \text{ if } y_i > 0;$$

$$\left.\begin{aligned} \epsilon_i^+ &= 0 \\ \epsilon_i^- &= -y_i \end{aligned}\right\} \text{ if } y_i < 0.$$

Moreover, the objective function is bounded below by zero; hence MSAE-LP2 has an optimal solution.

For expositional convenience, consider the case where a colum of X is scalar multiple of another column. Let β_1 and β_2 be the co efficients of the linearly dependent regressors x_1 and x_2, respectiv ly. Then in MSAE-LP2, the variables $\beta_1^+, \beta_1^-, \beta_2^+, \beta_2^-$ have zero coefficients in the objective function while the columns associate with them are pairwise linearly dependent. Suppose that β_1^+ is in the basis of the optimal solution obtained by the simplex algorithm Then $\beta_1^-, \beta_2^+, \beta_2^-$ cannot be in the basis since this would violat linear independence of the basis vectors. The portion of the optima tableau (in canonical form) corresponding to these variables would look like the following:

Basic Variables		β_1^+	β_1^-	β_2^+	β_2^-		Right Hand Side
(Objective Function Row) →	...	0	0	0	0	...	b_0
·	...	0	0	0	0	...	·
·		⋮	⋮	⋮	⋮		·
β_1^+	...	1	1	α	$-\alpha$	...	$\hat{\beta}_1^+$
·		0	0	0	0		·
·		⋮	⋮	⋮	⋮		·

where $-b_0$ is the optimal value of the objective function. The column vector associated with β_1^+ is the unit vector since β_1^+ is in the basis. The other column vectors follow from the fact that they are scalar multiples of the vector associated with β_1^+. Note that β_2^+ is a non-basic variable whose objective function coefficient is zero in the optimal tableau. This implies that the optimal solution is not unique since a necessary and sufficient condition for the uniqueness of an optimal solution is that the objective function coefficients of the nonbasic variables in the optimal tableau are positive (Simmonard, 1966). Another optimal solution can be obtained by pivoting on α (if $\alpha > 0$) or on $-\alpha$ (if $\alpha < 0$). This would put β_2^+ or β_2^- in the basis, replacing β_1^+ which now becomes zero. The new optimal solution has $\hat{\beta}_1 = 0$, $\hat{\beta}_2 = \frac{\hat{\beta}_1^+}{\alpha}$ (if $\alpha > 0$) or $\hat{\beta}_2 = \frac{\hat{\beta}_1^+}{\alpha}$ (if $\alpha < 0$). Since the set of optimal solutions is convex, it follows that there are an infinite number of optimal solutions.

Remarks: (1) The general case is proved in a similar manner. If rank $X < k$, then in the optimal solution, not all of the β's will appear as basic variables since this would violate the linear independence of the basis vectors. Suppose that $\beta_1^*, \beta_2^*, \ldots \beta_{k_l}^*$ (where $\beta_j^* = \beta_j^+$ or β_j^-) are in the optimal basis. This implies that the associated regressors $x_1, x_2, \ldots x_{k_l}$ are linearly independent. Moreover, there is a nonbasic variable β_ℓ^* whose associated regressor x_ℓ is a linear combination of $x_1, x_2, \ldots k_l$. One can show that the column associated with β_ℓ^* in the optimal tableau is a linear combination of the columns associated with $\beta_1^*, \ldots, \beta_k^*$ with a zero objective function coefficient. This shows the existence of nonbasic variable with a zero objective function coefficient in the optimal tableau which implies nonuniqueness of the optimal solution.

(2) In effect, MSAE estimation in the presence of perfect multicollinearity will choose a maximal set of linearly independent regressors (whose number equals the rank of X) and drops the other regressors from the equation by setting their coefficients equal to zero. This is also one of the remedies resorted to by researchers when confronted with perfect multicollinearity under MSSE estimation.

Example Consider the following data set:

y	x_1	x_2
1	1	2
3	2	4
2	3	6
3	4	8
4	5	10

and the regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon \; .$$

Note that $x_2 = 2x_1$. Using the standard simplex algorithm on the MSAE-LP2 of this model, we obtain the optimal tableau shown in Figure 1.

Figure 1

Basic Variables	β_0^+	β_0^-	β_1^+	β_1^-	β_2^+	β_2^-	ϵ_1^+	ϵ_1^-	ϵ_2^+	ϵ_2^-	ϵ_3^+	ϵ_3^-	ϵ_4^+	ϵ_4^-	ϵ_5^+	ϵ_5^-	Right Hand Side
Objective Function Row →	0	0	0	0	0	0	1	1	0	2	2	0	2	0	0	2	-2
β_0^+	1	-1	0	0	0	0	$\frac{4}{3}$	$-\frac{4}{3}$	0	0	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$
ϵ_2^+	0	0	0	0	0	0	$-\frac{2}{3}$	$\frac{2}{3}$	1	-1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{4}{3}$
β_2^+	0	0	$\boxed{\frac{1}{2}}$	$-\frac{1}{2}$	1	-1	$-\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{3}$
ϵ_3^-	0	0	0	0	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	-1	1	$\frac{2}{3}$	$-\frac{2}{3}$	0	0	$\frac{1}{3}$
ϵ_5^+	0	0	0	0	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	0	0	$-\frac{4}{3}$	$\frac{4}{3}$	1	-1	$\frac{1}{3}$

The optimal solution corresponding to this optimal tableau is given by

$$\hat{\beta}_0 = \frac{1}{3} \qquad \hat{\epsilon}_1 = 0$$

$$\hat{\beta}_1 = 0 \qquad \hat{\epsilon}_2 = \frac{4}{3}$$

$$\hat{\beta}_2 = \frac{1}{3} \qquad \hat{\epsilon}_3 = -\frac{1}{3}$$

$$\hat{\epsilon}_4 = 0$$

$$\hat{\epsilon}_5 = \frac{1}{3}$$

where the β_j's are the MSAE regression coefficients and the $\hat{\epsilon}_i$'s are the residuals. Another optimal solution can be obtained by pivoting on the element $\frac{1}{2}$ (enclosed in a rectangle) thus putting β_1^+ into the basis and removing β_2^+ from the basis. This other optimal solution is given by

$$\hat{\epsilon}_1 = 0$$

$$\hat{\beta}_0 = \frac{1}{3} \qquad \hat{\epsilon}_2 = \frac{4}{3}$$

$$\hat{\beta}_1 = \frac{2}{3} \qquad \hat{\epsilon}_3 = -\frac{1}{3}$$

$$\hat{\beta}_2 = 0 \qquad \hat{\epsilon}_4 = 0$$

$$\hat{\epsilon}_5 = \frac{1}{3}$$

resulting in another set of MSAE estimates. Convex combinations of these two optimal solutions are also optimal solutions.

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