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NONNEGATIVE APPROXIMATE SOLUTIONS TO AN ECONOMETRIC MODEL WITH PRESCRIBED GOALS

By Rolando A. Danao*

1. The Goal Attainment Problem

From the reduced form of a linear econometric model, a policymaker might be interested in a subsystem of the form

(1) $$Y = Ax + b$$

where:

Y : an M x 1 vector of goals
A : an M x N matrix of impact multipliers
x : an n x 1 vector of instruments
b : an m x 1 vector of constants

If a policymaker prescribes his goals, say y^*, his problem is to find an instrument vector x^* that attains his goals, i.e., to find a solution to the equation

(2) $$Ax = z^*$$

where $z^* = y^* - b$. A solution to equation (2) exists if and only if

(3) $$AA^- z^* = z^*$$

where A^- is the generalized inverse of A (Graybill, 1969). If a solution exists, then

(4) $$x^* = A^- z^*$$

is a solution; in general, the solution is not unique. The general solution is of the form

(5) $$x = A^- z^* + (I - A^- A)v$$

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* Associate Professor of Economics, University of the Philippines.

where I is the identity matrix and v is an arbitrary vector (Graybill, 1969; Ijiri, 1965). A necessary and sufficient condition for the uniqueness of the solution is that $A^{-}A = I$ (Graybill, 1969). If a solution does not exist, the goal attainment problem is that of finding an "approximate solution" $\hat{x}$ so that the vector $\hat{y} = A\hat{x} + b$ is "as close as possible" to the prescribed goal y^* (Ijiri, 1965) or equivalently, $\hat{z} = \hat{y} - b$ is as close as possible to $z^* = y^* - b$.

"Closeness" between vectors may be measured by means of distance functions defined on the vector space (Sfeir-Younis, 1977). One such distance is the ordinary Euclidean distance (or L_2-metric) defined by

$$(6) \qquad d(z^1, z^2) = \left[\sum_{i=1}^{m} (z_i^1 - z_i^2)^2\right]^{\frac{1}{2}}.$$

If $Ax = z^*$ has no solution, then an approximate solution with respect to the Euclidean distance is a vector $\hat{x}$ that minimizes $d(z,z^*)$ over all vectors $z = Ax$, i.e.,

$$(7) \qquad d(\hat{z}, z^*) = \min_{z = Ax} d(z, z^*)$$

Since $\hat{x}$ minimizes the sum of the squares of the deviations between z_i and z_i^*, it is called a *least squares solution.* It has been shown that

$$(8) \qquad \hat{x} = A^{-} z^*$$

is a least squares solution to equation (2) (Graybill, 1969). Hence, a least squares solution always exists; in general, it is not unique. The general form of the least squares solution can be obtained by noting that every least squares solution must satisfy $Ax = \hat{z}$ and, therefore, must have the general form of equation (5):

$$\begin{aligned} x &= A^{-}\hat{z} + (I - A^{-}A)v \\ &= A^{-}A\hat{x} + (I - A^{-}A)v \\ &= A^{-}AA^{-}z^* + (I - A^{-}A)v \\ &= A^{-}z^* + (I - A^{-}A)v. \end{aligned}$$

Hence a necessary and sufficient condition for the uniqueness of the least squares solution is that $A^{-}A = I$.[1]

The necessary and sufficient conditions for the existence and uniqueness of solutions or of least squares solutions, however, are silent on the nonnegativity of these solutions. A policymaker's vector of instruments is usually nonnegative (e.g., government expenditures, tax revenue). It would, therefore, be useful to determine if a nonnegative solution or a nonnegative least squares solution exists and if it does, to obtain such a solution. Moreover, it would also be useful to know if the nonnegative solution obtained is unique, since nonuniqueness implies the existence of alternative instrument policies for attaining the same goal. This paper examines these problems via a linear programming problem similar to the artificial problem in Phase I of the two-phase simplex method. Furthermore, when multiple nonnegative instrument vectors exist, linear programs may be used to select the desired vector.

2. Existence and Uniqueness of Nonnegative Solutions

The problem posed in Section 1 is that of determining the feasibility of the system

$$Ax = z^* \quad , \qquad x \geqq 0 \tag{9}$$

and of obtaining a feasible solution if it exists. Without loss of generality, we may assume that $z^* \geqq 0$. (If $z_i^* < 0$, multiply the *ith* equation by -1).

The feasibility of system (9) can be determined by solving the following artificial linear programming problem:

LP1: Minimize $e'u$

subject to $Ax + Iu = z^*$

$x, u \geqq 0$

[1] By defining $\hat{z} = A\hat{x}$, the system $Ax = \hat{z}$ is consistent. Consequently, the solution $x = A^{-}\hat{z} = A^{-}A\hat{x} = A^{-}AA^{-}z^* = A^{-}z^* = \hat{x}$ is unique if and only if $A^{-}A = I$.

where e' is the m—dimensional vector of 1's and $u = [u_1, u_2, \ldots, u_m]'$ is a vector of artificial variables. Note that LP1 has a feasible solution $x = 0$, $u = z^*$. Since the objective function is bounded below by zero, LP1 has an optimal solution. It has been shown that system (9) is feasible if and only if the optimal objective function value of LP1 is zero (Simmonard, 1966). Consequently, the optimal simplex tableau of LP1 will show if system (9) is feasible or not, and if feasible, the same tableau gives a feasible solution.

Using LP1, we can show the uniqueness of a feasible solution to system (9) by means of the following theorem.

THEOREM 1. If the system $Ax = z^*$, $x \geqq 0$ has a feasible solution, then it is unique if and only if LP1 has a unique optimal solution.

Proof: ($\Rightarrow$)

If x^* is the unique feasible solution to (9), then $[x^*, 0]$ is an optimal solution of LP1 since its objective function value is zero, which is the minimum possible value of LP1's objective function. Hence, every optimal solution $[x, u]$ of LP1 must satisfy $u = 0$, which implies that x is feasible in (9). Consequently, $x = x^*$.

($\Leftarrow$)

Let $[x^*, u^*]$ be the unique optimal solution of LP1. Since system (9) is feasible, then the objective function value of $[x^*, u^*]$ is zero. Hence, $e'u^* = 0$ or $u_1^* + u_2^* + \ldots + u_m^* = 0$. Since $u^* \geqq 0$, it follows that $u_1^* = u_2^* = \ldots = u_m^* = 0$. If x is any feasible solution to (9), then $[x, 0]$ is an optimal solution of LP1. Hence, $[x, 0] = [x^*, u^*] = [x^*, 0]$ and so $x = x^*$.

Remark: It is easy to determine whether an optimal solution of LP1 is unique or not. This is seen from the elements of the optimal simplex tableau when the simplex algorithm is applied to LP1. The necessary and sufficient conditions for the uniqueness of an optimal solution can be found in Simmonard (1966).

3. Existence and Uniqueness of Nonnegative Least Squares Solutions

When the goals y^* cannot be attained simultaneously, i.e., $Ax = y^* - b$ has no solution, then we consider the set of least squares solutions to $Ax = y^* - b = z^*$. We may also use a linear programming problem to determine the existence and uniqueness of nonnegative least squares solutions.

Let $\hat{x}$ be a least squares solution to (2). Then $\hat{y} = A\hat{x} + b$ is as close as possible to y^* or equivalently, $\hat{z} = A\hat{x}$ is as close as possible to z^*. To determine the existence of nonnegative least squares solutions, we solve the linear programming problem:

LP2: Minimize $e'u$

subject to
$$Ax + Iu = \hat{z}$$
$$x, u \geqq 0 .$$

A problem may arise here. If $\tilde{x}$ is a least squares solution distinct from $\hat{x}$, then the vector $\tilde{z} = A\tilde{x}$ is also as close to z^* as $\hat{z}$. Another linear programming problem can be formulated thus:

Minimize $e'u$

subject to
$$Ax + Iu = \tilde{z}$$
$$x, u \geqq 0.$$

There is, therefore, the possibility of solving more than one, possibly an infinite number of linear programs. But this possibility is ruled out since we can show that $\hat{z} = \tilde{z}$. This follows from the fact that every least squares solution to any system $Ax = d$ satisfies the equation $Ax = AA^{-}d$ (Graybill, 1969). Applying this to our problem, we must have

$$\hat{z} = A\hat{x} = AA^{-}z^* = A\tilde{x} = \tilde{z}\text{[2]}.$$

4. Selecting a Desired Instrument Vector

The existence of multiple nonnegative solutions provides the policymaker with alternative instrument vectors for achieving the same goal. Selecting an instrument vector requires a criterion for choice. For example, a policymaker might be particularly interested in a solution in which x_i is minimum. In this case, he solves the linear program

[2] A geometric proof of the uniqueness of $\hat{z}$ is given in Simmons (1963).

Minimize $\quad x_i$

subject to $\quad Ax = z^*$

$$x \geqq 0.$$

Another criterion could be that of finding a solution in which the sum of several x_i's is minimum. The above linear program may be used with a new objective, namely, to minimize $x_{i_1} + x_{i_2} + \ldots + x_{i_n}$.

The preceding remarks apply to the selection of a least squares solution. The linear program is the same as above with z^* replaced by $\hat{z}$.

5. Numerical Examples

Example 1.

$$2x_1 - x_2 - x_3 = 5$$

$$x_1 + x_2 - 2x_3 = 1$$

$$3x_1 + 2x_2 - 5x_3 = 4$$

Here, $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 3 & 2 & -5 \end{bmatrix} \qquad y^* = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

and $A = \dfrac{1}{177} \begin{bmatrix} 58 & -7 & 3 \\ -65 & 17 & 18 \\ 7 & -10 & -21 \end{bmatrix}$

Since $AA^- y^* = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = y^*$, the given system of linear equation has a solution. One solution is given by

$$x^* = A^- y^* = \frac{1}{177} \begin{bmatrix} 295 \\ -236 \\ -59 \end{bmatrix} \ngeqq 0.$$

$$\text{Since } \pi\pi^{-} = \frac{1}{177}\begin{bmatrix} 174 & -21 & 9 \\ -21 & 30 & 63 \\ 9 & 63 & 150 \end{bmatrix} \neq I$$

the solution x^* is not unique. To find a nonnegative solution we solve the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & u_1 + u_2 + u_3 \\ \text{subject to} & 2x_1 - x_2 - x_3 + u_1 = 5 \\ & x_1 + x_2 - 2x_3 + u_2 = 1 \\ & 3x_1 + 2x_2 - 5x_3 + u_3 = 4 \\ & x_1, x_2, x_3, u_1, u_2, u_3 \geqq 0 \end{array}$$

The optimal simplex tableau is given by the following table:

Basic Variables	x_1	x_2	x_3	u_1	u_2	u_3	Right Hand Side
	0	0	0	$\frac{6}{7}$	0	$\frac{10}{7}$	0 ⟵ Objective Function Row
u_2	0	0	0	$\frac{1}{7}$	1	$-\frac{3}{7}$	0
x_1	1	-1	0	$\frac{5}{7}$	0	$-\frac{1}{7}$	3
x_3	0	-1	1	$\frac{3}{7}$	0	$-\frac{2}{7}$	1

An optimal solution is given by

$$\begin{array}{ll} x_1 = 3 & u_1 = 0 \\ x_2 = 0 & u_2 = 0 \\ x_3 = 1 & u_3 = 0, \end{array}$$

and its objective function value is zero. Hence, $x_1 = 3$, $x_2 = 0$, $x_3 = 1$ is a solution to the given system of equations.

The objective function row of the optimal tableau shows that a nonbasic variable, namely x_2, has a zero coefficient. This implies that the linear programming problem has multiple optimal solutions. Consequently, the given system of linear equations has multiple nonnegative solutions. Various criteria may be used to select a desired solution. For example, one can show that the nonnegative solution $x_1 = 3, x_2 = 0, x_3 = 1$ is the solution that minimizes x_1, x_2, x_3, and $x_1 + x_2 + x_3$. If the choice criterion is a nonnegative solution that minimizes $x_1 + x_2 + x_3$ with the added condition that x_2 is at least equal to 1, then the desired solution is $x_1 = 4$, $x_2 = 1, x_3 = 2$.

Example 2

$$\begin{aligned} 2x_1 - x_2 - x_3 &= 7 \\ x_1 + x_2 - 2x_3 &= 2 \\ 3x_1 + 2x_2 - 5x_3 &= 6 \end{aligned}$$

Here, $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 3 & 2 & -5 \end{bmatrix}$ $\quad y^* = \begin{bmatrix} 7 \\ 2 \\ 6 \end{bmatrix}$

Since $AA^-y^* = \dfrac{1}{177}\begin{bmatrix} 1230 \\ 291 \\ 1089 \end{bmatrix} \neq y^*$, the system does not have a solution. However, it has a least squares solution given by

$$\hat{x} = A^-y^* = \frac{1}{177}\begin{bmatrix} 410 \\ -313 \\ -97 \end{bmatrix}$$

The corresponding goal vector $\hat{y}$ that is as close as possible to y^* is given by

$$\hat{y} = A^-y^* = \frac{1}{177}\begin{bmatrix} 1230 \\ 291 \\ 1089 \end{bmatrix} \approx \begin{bmatrix} 6.9491 \\ 1.6441 \\ 6.1525 \end{bmatrix}$$

Since $AA^- \neq I$, the least squares solution $\hat{x}$ is not unique. To find a nonnegative least squares solution, we solve the linear programming problem:

$$\begin{aligned} \text{Minimize} \quad & u_1 + u_2 + u_3 \\ \text{subject to} \quad & 2x_1 - x_2 - x_3 + u_1 = 6.9491 \\ & x_1 + x_2 - 2x_3 + u_2 = 1.6441 \\ & 3x_1 + 2x_2 - 5x_3 + u_3 = 6.1525 \\ & x_1, x_2, x_3, u_1, u_2, u_3 \geqq 0. \end{aligned}$$

An optimal solution is given by

$$\begin{aligned} x_1 &= 4.0847 & u_1 &= 0 \\ x_2 &= 0 & u_2 &= 0 \\ x_3 &= 1.2203 & u_3 &= 0. \end{aligned}$$

It follows that

$$\hat{x} = \begin{bmatrix} 4.0847 \\ 0 \\ 1.2203 \end{bmatrix}$$

is a nonnegative least squares solution. The optimal simplex tableau also shows that this nonnegative least squares solution is not unique. For example,

$$\hat{x} = \begin{bmatrix} 5.5847 \\ 1.5000 \\ 2.7203 \end{bmatrix}$$

is another nonnegative least squares solution.

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