Philippine Review of Economics and Business
Volume XVIII, Nos. 3 & 4, September & December 1981

GROUP CHOICE WITH LEXICOGRAPHIC PREFERENCES

By José Encarnación, Jr.*

1. Introduction

This paper considers two models of group choice within a framework of lexicographic preferences. Since the members of the group may have different preference orderings over the alternatives, one or another of the requirements on social choice functions posited by Arrow (1963) must be violated. However, not all of those requirements are as compelling as they might seem to be, especially since they were formulated by Arrow with only real-valued "ordinal" utility functions in mind (1963, p. 11) while vector-valued functions permit a richer preference structure. Using that structure, one can define group preference without basing this directly on individual preferences among the alternatives as such.

Section 2 gives a brief review of lexicographic preferences. Section 3 presents the two models: Model I seems to have some explanatory value, at least in some group decision situations, while Model II may have more normative appeal in preserving the Pareto principle. Section 4 makes concluding remarks.

2. Lexicographic Preferences[1]

In this conception of choice, account is taken of the fact that there are various noncomparable criteria of choice which are ranked in order of importance or priority. Depending on the decision context, these criteria may correspond to different wants or needs (as in the case of the consumer, whose need for food cannot be served by clothing or shelter) or more generally to objectives that permit no trade-offs (as in the case of a country that would not give up its sovereign status for the sake of gaining economic benefits from another country). To each element x (y, z, etc.) in the choice space X corresponds a vector $(u_1(x), u_2(x), \ldots)$ where u_i $(i = 1, 2, \ldots)$ is a real-valued function such that $u_i(x) > u_i(y)$ if x

*Professor of Economics, University of the Philippines.

[1] For a review of this literature, see Fishburn (1974). The major references are Georgescu-Roegen (1954) and Chipman (1960). Georgescu-Roegen's idea of satisfactory levels for the various components of the utility vector is formalized in Encarnación (1964) and, in a more general setting, Day and Robinson (1973).

is preferred to y on the basis of the ith criterion.[2] It is assumed that there exist values u_i^* such that if $u_i(x) \geqq u_i^*$, x is considered satisfactory as regards the ith criterion. Define the relation L^* by:[3]

(1) xL^*y iff the first nonvanishing difference
$\min\{u_i(x), u_i^*\} - \min\{u_i(y), u_i^*\}$, $i = 1, 2, \ldots$, is positive.
Defining R^* by xR^*y iff $\sim yL^*x$, R^* is a relation on X that is complete[4] and transitive.

We will say that the preference ordering is an L^*-ordering if for all x, $y \in X$, xPy iff xL^*y, where P means preference. Writing $v_i(x) = \min\{u_i(x), u_i^*\}$, the preference ordering of the x's is then given by the lexicographic ordering of the corresponding vectors $v(x) = (v_1(x), v_2(x), \ldots)$. Accordingly, if X_0 is the set of feasible alternatives and

$$(2) \quad X_i = \{x \in X_{i-1} \mid u_i(x) \geqq \max_y \}v_i(y) \mid y \in X_{i-1}\}\}$$

$i = 1, 2, \ldots$, and if j is the smallest index such that X_j is a one-element set, then the decision problem is to maximize u_j subject to $x \in X_i$, $i = 1, \ldots, j-1$. One thus goes through the choice criteria sequentially, beginning with the most important, and considers only those alternatives that belong to (2) at each stage. The search is thus narrowed in successive stages until only one alternative is left.

In a discussion of public investment appraisal, for example, Dorfman noted that one "must be concerned with many kinds of consequences, not all measurable in monetary units and not all comparable among themselves in any natural unit," and after surveying alternative approaches, suggested the possibility of "maximizing performance with respect to some one objective, subject to meeting targets with respect to the other dimensions of performance" (1965, pp. 191, 199). Presumably, if the targets cannot all be met, one

---

[2] To the extent that trade-offs exist among several objectives, a single real-valued function suffices to represent all of them as choice criterion, and they could all fall under a single u_i.

[3] The following notation is used: iff = if and only if; $\sim \ldots$ = it is not the case that ...; ϕ = the null set; $A - B$ = the set of elements in A that are not in B.

[4] I.e., xR^*y or yR^*x for all x, y.

would relax the least important and turn it into the objective. Such a procedure is rationalized by L*-ordering, as is also Tinbergen's discussion of macroeconomic policy targets (1956, pp. 59-60). Simon's (1955) concept of "satisficing" behavior, which leaves choice indeterminate when not all satisficing levels can be reached, is similarly made precise by L*-ordering.

3. Group Choice

Consider a group of n individuals indexed by $k = 1, \ldots, n$. We will use the notation of Section 2 with an index k, if it is necessary to do so, to refer to individual k; otherwise, without an index, the notation will refer to the group or else it is the same for all the members. We now assume that for each k, k's preference ordering (in regard to the group decision problem at hand) is an L^{*k}-ordering, but for each i, $u_i^k = u_i$ for all k; that is in (1), u_i^{*k} replaces u_i^* throughout. In other words, the members have the same choice criteria and rank them in the same way (which is what we require of them as a group), but the parameters u_i^{*k} generally differ with k for any given i. Accordingly, they would have different preference orderings in general.

For example, in the case of an economic planning commission, we assume that the members have the same priority ranking of (say) less unemployment, lower price inflation, higher per capita income, etc. as objectives to be promoted, but they may hold different views as to what constitutes a tolerable level of unemployment or acceptable rate of price inflation, etc. As Arrow has remarked in the context of society as the group, "it must be demanded that there be some consensus on the ends of society, or no social [choice] function can be formed" (1963, p. 83).

Arrow has also pointed out that "the alternatives, among which social preference is to be defined, may be interpreted in (at least) two ways: (1) each alternative is a vector whose components are values of the various particular decisions actually made by the government, such as tax rates, expenditures, antimonopoly practice, and price policies of socialized enterprise; (2) each alternative is a complete description of the state of every individual throughout the future" (1963, p. 87). As will be apparent, it is the first interpretation that we follow in this paper, for each vector component can be the subject of a majority decision but not normally the vector itself. It will also be apparent that, following the classification of social choice problems in Sen's (1977) very useful review, we will be concerned with aggregation of individual *judgments*, not interests, to arrive at group *decisions*, not welfare propositions.

Model I

In this model we assume that the group preference ordering is an L*-ordering. Then all that needs be done is to determine, for each i, u_i^* from the u_i^{*k} $(k=1,\ldots,n)$ in a reasonable way. For this purpose, let u_1^* be the median of the u_1^{*k}; similarly, $u_2^* = \text{med}(u_2^{*k})$, etc. The rationale is that with respect to any particular criterion u_i, Black's (1948) theorem on single-peaked preferences (over possible "candidates" for u_i^*) applies, for each k will want the group's u_i^* value to be as close as possible to his own u_i^{*k} value, since any higher value implies an unnecessarily high constraint on u_i while any lower value is less than satisfactory. (Thus there is no advantage for anyone to misrepresent his true u_i^{*k} values.) Therefore if the members were to vote for the value of u_i that is to serve as u_i^*, only the median u_i^{*k} could win by simple majority rule over any other candidate, assuming an extra vote by the chairman if necessary.

Noting that since $u_i^* = \underset{k}{\text{med}}(u_i^{*k})$, $u_i(x) \geqq u_i^*$ if a majority considers x satisfactory as regards u_i, it follows that xL^*y iff: (a) for some j, $u_j(x) > u_j(y)$ and a majority finds y less than satisfactory with respect to u_j, and (b) for every $i < j$, $u_i(x) = u_i(y)$ or else a majority considers both x and y satisfactory on the basis of u_i. Therefore the group could reach its choice through a series of decisions, each of which is by simple majority rule. Confronted with a choice between two alternatives, the members consider them first according to u_1; if a majority finds them satisfactory or equal in this regard, the alternatives are then examined according to u_2. Again, if a majority (which may be different from the preceding majority) finds them satisfactory or equal, the third criterion of choice becomes relevant. The procedure continues until one alternative comes out superior by majority vote. Group choice is thus determinate even without any explicit agreement on the group parameters u_i^*.

By definition, a group choice function g selects from any given set of feasible alternatives A, which we assume to be finite, the group choice g(A). In Model I above, we write the particular g as C* and[5]

---

[5] In writing C*(A), for notational simplicity we omit explicit reference to its dependence on the u_i^* and indirectly on the u_i^{*k}. The same applies to C(A) later.

(3) $C^*(A) = \{x \in A \mid \forall y(y \in A \rightarrow \sim yL^*x)\}$

which may be called the L^* choice, and which seems to have some explanatory value in the way it gives a role to simple majority rule in deciding on the vector components constituting an alternative. Majority decision is applicable to each x_i in $x = (x_1, x_2, \ldots)$ but not to x itself. In effect, group choice is built up from separate decisions on the components $x_1, x_2, \ldots$, that make up x.

The formal properties of (3) need to be examined in relation to Arrow's conditions for a social welfare function. In a later formulation of Arrow's impossibility theorem, Murakami (1961) showed that no group choice function g can satisfy all the following:

(i) *Free triple condition:* There is at least one subset of three alternatives in X over which each individual in the group can have any possible preference ordering. Call such a subset T.

(ii) *Nondictatorship:* For some T, there is no k such that for all $x, y \in T, \{x\} = g(\{x, y\})$ if xP^ky.

(iii) *Independence of irrelevant alternatives:* For all $A \subset X$, $g(A)$ is invariant with respect to any changes in individual preference orderings on X that do not affect individual preference orderings on A.

(vi) *Pareto principle:* For all $x, y, \{x\} = g(\{x, y\})$ if xP^ky for all k.

(v) *Collective rationality:* For all A, $g(A) = \{x \in A \mid \forall y (y \in A \rightarrow xRy)\}$, where R is defined by xRy iff $\sim yPx$, and R is a complete and transitive relation on X.

It is easy to see that conditions (i), (ii) and (v) are satisfied by the group choice function C^* of Model I. However, the group's L^*-ordering depends on parameters which are functions of individual parameters: $u_i^* = \underset{k}{\text{med}}(u_i^{*k}), i = 1, 2, \ldots$, and there is no necessary connection between the members' preferences regarding x and y, say, and xL^*y. Accordingly, (iii) and (iv), which make group choice between alternatives depend *only* on individual preferences between them, fail to be satisfied. Condition (iii) is unduly restrictive and has not received general acceptance as a requirement; (iv), on the other hand, has a wide appeal as a normative principle. They need discussion.

Regarding (iii), as Little (1952) has earlier argued, there is no compelling reason for the group to maintain any particular pattern of preferences after individual orderings have changed. Arrow's own argument is twofold: first, that ordinal utility suffices for the representation of preferences, in which case (without measurability of utility and interpersonal comparisons) it would seem that there is no basis for group preference between alternatives except information on individual preferences between them; second, that "social decision processes which are independent of irrelevant alternatives have a strong practical advantage" (1963, p. 110). While we can accept the second point, it is clear that ordinal utility does not always suffice, and in fact Model I shows how group choice can depend on group parameters that depend on the parameters of individual preference orderings but not on individual preferences among the alternatives as such. Moreover, it is obvious that group choice in Model I does not depend on alternatives outside the feasible set (cf. Encarnación, 1969). Our conclusion therefore is that (iii), which was intended as a formalization of this admittedly desirable property, does not accomplish its purpose and is not a compelling requirement.

The Pareto principle, (iv), seems just the opposite of (iii) in finding general acceptance. Yet one could argue, as Arrow himself has done, that the decision process can itself have a value which may override occasional dissatisfactions with its results. "For example, the belief in democracy may be so strong that any decision on the distribution of goods arrived at democratically may be preferred to such a decision arrived at in other ways, even though all individuals might have preferred the second distribution of goods to the first if it had been arrived at democratically ... In such a case ... our social welfare problem may be regarded as solved since the unanimous agreement on the decision process may resolve the conflicts as to the decisions themselves" (1963, p. 90). Thus occasional violations of the Pareto principle need not be a compelling reason for rejecting a group decision process, especially if we consider that a group decision often affects individuals who are not members of the group (as in the case of a national legislature) and whose preferences may be different. Indeed, even if all the living members of a society should be of one mind on a matter, future generations would still be unrepresented in the decision.

How often might such violations be? Write xDy (for x dominates y) iff $xP^k y$ for all k, and consider the simplest case under which yPx but xDy in Model I:

(a) $u_i(x) > u_i(y)$ $(i = 1, 2, 3)$, $u_4(x) < u_4(y) \leqq u_4{}^*$;

(b) $G = m_1 \cup m_2 \cup m_3$

$= M_i \cup m_i$ $(i = 1, 2, 3)$

where M_i is a majority in the group G that finds both x and y satisfactory and m_i a minority that considers y less than satisfactory as regards u_i $(i = 1, 2, 3)$. Then yL^*x because of u_4. However, individuals in m_i prefer x to y on account of u_i (if not on account of a prior criterion), so xDy.

In order to get some idea of the relative frequency of such a case, assume that for any given i: $u_i(x) > u_i(y)$ is just as likely as $u_i(x) < u_i(y)$; $u_i{}^{*k} \neq u_i{}^{*h}$ for $k \neq h$; and any particular ordering of the $u_i{}^{*k}$ by $<$ is just as likely as another (e.g. with 3 members in the group, $u_i{}^{*1} < u_i{}^{*2} < u_i{}^{*3}$ is just as likely as $u_i{}^{*2} < u_i{}^{*3} < u_i{}^{*1}$). Then the probability of (a) is 1/8. With 3 members, the probability of (b) is 2/9, so the relative frequency of such a violation is 1/36 or 2.78%. If there are 5 members, the corresponding figure is 18/800 or 2.25%. With 7 members, it is 1.73% . As one might expect, violations become more infrequent with larger groups.

Nonetheless, condition (iv) is sufficiently appealing that we may want it necessarily satisfied. We therefore consider a modified model that satisfies (iv), but at the cost of failing (v).

Model II

We maintain all the assumptions of Model I except the assumption that the group preference ordering is an L*-ordering, so that (3) is no longer necessarily the group choice. However, we want the group choice to be still $C^*(A)$ in (3) unless it is dominated by an available alternative; more precisely, we wish to define the group choice to be the best undominated L* choice.

For notational convenience, write $A = A^1 = F(A^0)$ and let

(4) $F(A^r) = \{(z \in A | \forall x(x \in C^*(A^r) \rightarrow zDx)\}$

$A^{r+1} = A^r - C^*(A^r)$, $r = 1, 2, \ldots$

Suppose $F(A^1) \neq \phi$. Then we consider $A^2 = A^1 - C^*(A^1)$ and obtain $C^*(A^2)$. If $F(A^2) \neq \phi$ also, we get $C^*(A^3)$, and so on. It is clear that $F(A^{r-1}) \neq \phi$ and $F(A^r) = \phi$ for some $r \geqq 1$; otherwise, every element of A would be dominated by some other element of A. We can then define the group choice in Model II as

$$(5)\quad C(A) = C^*(A^s), \quad s = \max\{r \geqq 1 \mid F(A^{r-1}) \neq \phi\}$$

which assures that no element of C(A) is dominated by any feasible alternative.[6]

Arrow (1963, p. 105) has used the term "constitution" for his concept of a social welfare function. The term seems particularly appropriate for Model II. A constitution expresses social objectives, sets constraints on what is permissible, and indicates priorities when there is conflict among desired ends. Individuals in the group choose the parameters of the constitution and the latter then determines the group choice. The result is that the group choice becomes the decision of the group itself, so to speak, rather than that of any particular majority or minority in regard to the alternatives available. This should make the group decision more generally acceptable to the members, as it is arrived at by evaluating the alternatives in terms of the group's objectives instead of by asking who prefers what. A minority view could thus prevail, if it is in line with the group's objectives, and the majority need not find the result objectionable. The problem of "strategy"—expressing preferences over the alternatives different from one's true preferences—does not arise, since it is the group parameters that determine the choice. Finally, as in a constitution (which ideally presupposes unanimous agreement in its adoption), a suspension of the rules is provided for when there is unanimity for the purpose. Thus while $C^*(A)$ would normally be the group choice, it is eliminated if dominated by some feasible alternative.

"Collective rationality" is however failed by Model II, since there is in it no (reasonable) relation R on X with the properties

---

[6] It might be suggested that one could select the group choice from F(A) if $F(A) \neq \phi$. Following such an approach, one could consider $C^*(F(A))$ as the choice unless this is also dominated by a feasible alternative. In the latter event, one would then try $C^*(F(F(A)))$, where $F(F(A))$ obtains from (4) by putting $F(A)$ in place of A^r. If this is also dominated, the next possibility would be $C^*(F(F(F(A))))$, and so on. Such a procedure seems rather artificial however.

called for by condition (v), but we can define a preference ordering on A. Consider $A_2 = A - C(A)$ and the corresponding $C(A_2)$; writing $A = A_1$, we have the recursive relation $A_{t+1} = A_t - C(A_t)$, $t = 1, 2, ...$ We then define the group preference relation on A, P(A), by:

(6) $xP(A)y$ iff $x \in C(A_t)$ and $y \in C(A_{t+c})$ for some $t \geqq 1$, $c \geqq 1$. Defining $xR(A)y$ iff $\sim yP(A)x$, R(A) is symmetric and transitive. We can then also write

(7) $C(A) = \{(x \in A | \forall y (y \in A \rightarrow xR(A)y)\}$

since the right-hand sides of (7) and (5) are clearly the same set.

The collective rationality requirement has not found general acceptance (see e.g. Kemp, 1953; Plott, 1973; Blair and Pollak, 1979). Arrow's argument for requiring transitivity of P (or of R) is that this would make group choice independent of the particular sequence in which the feasible alternatives are presented for choice: "the basic problem is ... the independence of the final choice from the path to it ["path independence", as this property has since been called]. Transitivity will insure this independence; from any [feasible set] there will be a chosen alternative" (Arrow, 1963, p. 120). But then, transitivity on A and not necessarily on X would do for the purpose. More basically of course, as Plott (1973) has pointed out, if path independence is the objective of transitivity, one may dispense with the latter property if the former can be had without it.

At any rate, we have transitivity of P(A) and R(A) on A in Model II, and path independence is also satisfied. For C*(A) is clearly independent of the sequence in which the feasible alternatives are presented, after which one determines whether or not F(A) is nonnull. The answer to this question is obviously independent of the sequence in which the feasible alternatives are presented, since unanimity is required. $C^*(A^2)$ is then obtained, $F(A^2)$ determined, and so on. In such a multi-stage decision process, the *entire* feasible set is essential for checking whether or not the L* choice is dominated by some feasible alternative. Consequently,

(8) $g(A) = g(g(A_1) \cup g(A_2))$, $A_1 \cup A_2 = A$,

is not necessarily satisfied by $g = C$. Since (8) has been proposed as a formalization of the property of path independence (Plott, 1973; Kelly, 1978, p. 24), it is worth remarking that such a formalization takes no account of the possibility of a multi-stage decision process like that of Model II.

Related to this point, consider the possible requirement that

(9) $A \subset B \rightarrow A \cap g(B) \subset g(A)$

called Property α by Sen (1969, 1977) which has been considered basic to rational or consistent choice by many writers (see the references cited by Sen, 1969, p. 384, n. 1, and Kelly, 1978, p. 26, n. 2). Let $B = A \cup \{y\}$ $C^*(A) = C^*(B) = \{x\}$, $F(A) = \phi$, $F(B) = \{y\}$, $C^*(B^2) = \{z\}$ and $F(B^2) = \phi$. Then $A \cap C(B) = \{z\}$ and $C(A) = \{x\}$, so $g = C$ fails (9).[7] The point is that (9) is a very natural property for g to satisfy if the value placed by the group on an alternative is unidimensional. Doing Venn diagrams for (9)—as also for (8)—shows its reasonableness if to each point in A or B corresponds an "elevation", representing its desirability or value that is independent of A and B, so that contour lines could be drawn to locate g(A) and g(B). But in Model II, choice is conditional on nonviolation of the Pareto principle, so the value of an alternative is not invariant with respect to changes in the feasible set. Since Model II is internally consistent and appears to be otherwise reasonable, our conclusion is that (9) should not be considered as a general requirement on g. Similar remarks apply to Plott's (1973) Axioms 1 and 2, stated below as (10) and (11) for quick reference.

(10) $g(g(A - B) \cup g(A \cap B)) = g(A)$

(11) $g(B) \cap A \neq \phi \rightarrow g((A - B) \cup (g(B) \cap A)) = g(A)$.

4. Concluding Remarks

Model I seems to have some explanatory value as an idealized model of actual group decision-making. It provides a role for the use of majority rule in deciding the components that make up an alternative, considering that issues are decided one at a time and complete alternatives (states-of-the world) do not normally come up for single decisions. The fact that a group does not usually go through all the basic criteria of choice simply means that the available alternatives that are considered already usually satisfy the constraints set by the basic criteria. Violations of the Pareto principle (which would be

[7] Since a choice function that satisfies (8) satisfies (9)—see Sen (1977, p. 68) — this also shows failure of (8).

relatively infrequent) would go unrecognized, since the choice of a complete alternative results only after all the separate majority decisions on the components have been made, and there is no voting on complete alternatives.

Model II, on the other hand, appears to have some normative appeal in making the group choice satisfy the Pareto principle. From an analytical viewpoint, what comes out of Model II is the idea that the group's valuation of an alternative may depend on the feasible set. Accordingly, a number of otherwise very reasonable requirements, which apparently assume that the group's valuation of an alternative is independent of the feasible set, cannot be considered as fundamental and necessary for group choice. That the value of an alternative to a group should depend on what alternatives are available seems, from purely logical considerations, to be more natural and less restrictive than that it should not. And in a sense this idea pursues further the logic of the "independence of irrelevant alternatives" requirement. The latter demands that irrelevant alternatives should not matter. In Model II they do not, and all the relevant alternatives matter.

REFERENCES

Arrow, K.J. (1963), *Social Choice and Individual Values*, 2nd ed., New York: Wiley (1st ed., 1951).

Black, D. (1948), "On the Rationale of Group Decision-Making," *Journal of Political Economy*, 56: 23-34.

Blair, D.H. and Pollak, R.A. (1979), "Collective Rationality and Dictatorship: The Scope of the Arrow Theorem," *Journal of Economic Theory*, 21: 186-194.

Chipman, J.S (1960), "The Foundations of Utility," *Econometrica*, 28: 193-224.

Day, R.H. and Robinson, S.M. (1973), "Economic Decisions with L** Utility," in *Multiple Criteria Decision-Making*, ed. J.L. Cochrane and M. Zeleny, Columbia, S.C.: University of South Carolina Press, pp. 84-92.

Dorfman, R. (1965), "Econometric Analysis for Assessing the Efficacy of Public Investment," in *Study Week on the Econometric Approach to Development Planning*, Amsterdam: North-Holland, pp. 187-205.

Encarnación, J. (1964), "A Note on Lexicographical Preferences," *Econometrica*, 32: 215-217.

Encarnación, J. (1969), "On Independence Postulates Concerning Choice," *International Economic Review*, 10: 134-140.

Fishburn, P.C. (1974) "Lexicographic Orders, Utilities and Decision Rules: A Survey," *Management Science*, 20: 1442-1471.

Georgescu-Roegen, N. (1954), "Choice, Expectations and Measurability," *Quarterly Journal of Economics*, 68: 503-534.

Kelly, J.S. (1978), *Arrow Impossibility Theorems*, New York: Academic Press.

Kemp, M.C. (1953), "Arrow's General Possibility Theorem," *Review of Economic Studies*, 21: 240-243.

Little, I.M.D. (1952), "Social Choice and Individual Values," *Journal of Political Economy*, 60: 422-432.

Mukarami, Y. (1961), "A Note on the General Possibility Theorem of the Social Welfare Function," *Econometrica*, 29: 244-246.

Plott, C.R. (1973), "Path Independence, Rationality, and Social Choice," *Econometrica*, 41: 1075-1091.

Sen, A.K. (1969), "Quasi-transitivity, Rational Choice and Collective Decisions," *Review of Economic Studies*, 36: 381-393.

Sen, A.K. (1977), "Social Choice Theory: A Re-Examination," *Econometrica*, 45: 53-89.

Simon, H.A. (1955), "A Behavioral Model of Rational Choice," *Quarterly Journal of Economics*, 69: 99-118.

Tinbergen, J. (1956), *Economic Policy: Principles and Design*, Amsterdam: North-Holland.