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# A method for calibrating input (and output) price elasticities*

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We propose a theoretically consistent method for calibrating input (and output) price elasticities (of agricultural crops) from a minimal set of given estimates. Our review of production theory suggests three starting points for the exercise: (a) inputs and outputs have to be classified by input nonjointness, (b) production functions may be assumed to be linearly homogeneous, and (c) given an $n \times n$ (symmetric) matrix of elasticities, which has $n(n+1)/2$ distinct cells, the values of $n(n-1)/2$ of the distinct cells must be known to solve the $n$ unknown elasticities. Exploiting Shephard's Lemma and Euler's Theorem, we work out the method for a cost function with four inputs. We also provide a numerical example involving a $9 \times 9$ matrix of a multiple-output profit function.

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## 1. Introduction

How does a shock of one type or another (e.g., an El Niño or La Niña event, a financial crisis, or a revitalized agrarian reform program) affect agricultural production, in general, and the production of specific crops or livestock, in particular? How are farm incomes affected? What are the impacts on input

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demands and rural employment? Questions such as these are perennially posed to a social planner or policy analyst in the agriculture sector. Since shocks are inevitably felt in terms of price changes, a planner who is tasked to provide more than just vague qualitative answers needs filled-out matrices of price elasticities of input demands and output supplies as a policy tool for working out numerically the behavioral responses of farm households.

Unfortunately, such matrices are hard to come by because of data constraints. Nationally representative cross-section farm-level data on agricultural outputs, inputs, and prices are virtually unheard of. Instead, what are usually available from surveys of the agricultural stations are aggregated data on major crops or livestock for a limited set of inputs, which afford the estimation of production, cost, or profit functions that are much simplified or overly restrictive. (An example for the Philippines is Dumagan and Alba [2010], which estimates multiple-output generalized Leontief revenue and cost functions with regionally aggregated data, in effect assuming a representative farm in each region.) Consequently, what the analysts usually have in hand are a few elasticity estimates that could be econometrically squeezed out of such data sets.

An intriguing set of technical questions for a planner in the agriculture sector then is: From the science (as body of knowledge) of production theory, is it possible to develop a method for filling out matrices of price elasticities from a few given estimates? Are there ways by which the dimensions of the problem can be collapsed as was done in Bouis [1996] for food demand?[1]

An initial attempt to address these questions, this paper proposes a theoretically consistent method for calibrating input (and output) price elasticities of agricultural crops from a minimal set of "givens." The next section discusses certain theoretical considerations that need to be addressed, given the limitations of data, and explores their implications for the task at hand. The third section then presents the proposed method, which is based on Euler's theorem and Shephard's lemma, using the implications of the review of issues as the starting points. The fourth section provides a numerical example.

[1] Unlike food demand, however, whose underlying factors may be grouped into a few attributes such as the demand for energy in the form of caloric intake, taste preferences, and preferences for food variety as in Bouis [1996], the supply of agricultural outputs seems not to lend itself to an analogous treatment. Edmeades [2006], for instance, who uses a hedonic approach for estimating the value of attributes of bananas in the supply of the crop, borrows the attributes from demand. What this means is that fruits such as apples, bananas, and oranges, for example, can be collapsed into desired characteristics such as color, taste, size, ripeness, etc. This does not seem to buy much, however, for the task at hand, unless the production functions of the fruits can be mapped into the desired attributes. It is also doubtful whether consumers substitute between fruits on the basis of these characteristics.

## 2. Preliminary issues

### *2.1. Aggregation over firms*[2]

It is often the case that firm-level data are unavailable, and the researcher has to make do with aggregate—that is, provincial- or regional-level—data for an industry. Since profit, cost, and production functions are firm-level constructs, the researcher needs to address at least two questions: First, what properties does an industry's or a representative firm's profit, cost, or production function have to have to ensure that they are consistent with its firm-level version? Second, what are the implications of these properties? This section considers these aggregation-over-firm problems in the context of cost functions.

Perhaps almost by definition, an industry cost function (for a province or region) must be the sum of firm-level cost functions (in the area). Let $c(\mathbf{w}, y)$ be the industry-level cost function and $c_i(\mathbf{w}, y_i)$ be the cost function of the *i*th firm in the industry, where $\mathbf{w}$ is a vector of input prices and $y_i$ is the output of firm $i$, for $i = 1,\ldots,I$, so that $y = \sum_{i=1}^{I} y_i$. Then this property may be described as

$$c(\mathbf{w}, y) = \sum_{i=1}^{I} c_i(\mathbf{w}, y_i). \tag{1}$$

Let $\mathbf{x}_i(\mathbf{w}, y_i)$ be the vector of conditional input demand functions of firm $i$, so that $c_i(\mathbf{w}, y_i) \equiv \mathbf{w}\mathbf{x}_i(\mathbf{w}, y_i)$. Recall that if the firm-level production function exhibits constant returns to scale (CRS), the cost of producing $y^*$ units of output may be written as

$$c_i(\mathbf{w}, y^*) \equiv \mathbf{w}\mathbf{x}_i(\mathbf{w}, y^*) = \mathbf{w}y^*\mathbf{x}_i(\mathbf{w}, 1) = y^*\mathbf{w}\mathbf{x}_i(\mathbf{w}, 1) = y^*c_i(\mathbf{w}, 1).$$ [3]

Thus, in the trivial case where each firm produces the same output level $y^*$ and the production function exhibits constant returns to scale, equation (1) may be rewritten as

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[2] The discussion here is based on sections 5.6 and 5.7 of Chambers [1988].

[3] The proof for this claim may be given as follows: Let $c(\mathbf{w},1)$ be the cost of producing one unit of output at input prices $\mathbf{w}$. Then $c(\mathbf{w},1) \equiv \mathbf{wx}(\mathbf{w},1)$. But by CRS $f[y^*\mathbf{x}(\mathbf{w},1)] \equiv y^* \cdot 1$, which implies that $y^* = f[\mathbf{x}(\mathbf{w},y^*)] = f[y^*\mathbf{x}(\mathbf{w},1)]$ so that $\mathbf{x}(\mathbf{w},y^*) = y^*\mathbf{x}(\mathbf{w},1)$. Therefore, $c(\mathbf{w},y^*) \equiv \mathbf{wx}(\mathbf{w},y^*) = \mathbf{w}y^*\mathbf{x}(\mathbf{w},1) = y^*\mathbf{wx}(\mathbf{w},1) = y^*c(\mathbf{w},1)$.

$$c(\mathbf{w},y)=\sum_{i=1}^{I}y^{*}c_i(\mathbf{w},1)=y^{*}\sum_{i=1}^{I}c_i(\mathbf{w},1)=y\sum_{i=1}^{I}\frac{c_i(\mathbf{w},1)}{I}$$
$$=y\frac{c(\mathbf{w},I)}{I}=y\frac{Ic(\mathbf{w},1)}{I}=yc(\mathbf{w},1),$$

where $y=Iy^{*}$. In effect, aggregation over firms is not a problem because the industry cost function is simply a multiple of both the industry unit cost function and the arithmetic mean of the firm's unit cost functions, and so reflects the properties of the (average) firm-level cost function in the industry.

Suppose, however, that the production function is not CRS (so that the cost function is no longer linear in output) and firms produce different levels of output, but the distribution of these output levels is not important.[4] Then equation (1) may be written as

$$c(\mathbf{w},y)=c\left(\mathbf{w},\sum_{i=1}^{I}y_i\right)=\sum_{i=1}^{I}c_i(\mathbf{w},y_i) \tag{2}$$

to underscore that the industry-level output is the sum of the output levels of firms in the industry.

What restrictions does equation (2) impose on the industry cost function? Differentiating the equation with respect to $y_j$ gives

$$\frac{\partial c(\mathbf{w},y)}{\partial y_j}=\frac{\partial c(\mathbf{w},y)}{\partial y}\cdot\frac{\partial y}{\partial y_j}=\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}$$
$$\frac{\partial c(\mathbf{w},y)}{\partial y}=\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}\qquad \text{for } j=1,\ldots,I, \tag{3}$$

since $\partial y/\partial y_j=1$. This means that if the industry cost function is to be consistent with the firm-level cost functions that it aggregates, the industry marginal cost must be equal to the marginal cost of the firm that is the source of the change in industry output. Moreover, since equation (3) must be satisfied, whatever is the level of $y_j$ or regardless of which firm changes its output level, it must be that the firm-level marginal cost (of all other firms $i\neq j$) is not affected, i.e., it is independent of, $y_j$. To see this, consider what happens if industry output is redistributed between firms $j$ and $k$ such that $dy_j=-dy_k$. Then the marginal effect is measured by

[4] Alternatively, the researcher may have data only on $y$ but not $y_i$ for $i=1,\ldots,I$.

$$\frac{\partial c(\mathbf{w},y)}{\partial y}\cdot\frac{\partial y}{\partial y_j}dy_j+\frac{\partial c(\mathbf{w},y)}{\partial y}\cdot\frac{\partial y}{\partial y_k}dy_k=\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}dy_j+\frac{\partial c_k(\mathbf{w},y_k)}{\partial y_k}dy_k$$

$$\frac{\partial c(\mathbf{w},y)}{\partial y}dy_j+\frac{\partial c(\mathbf{w},y)}{\partial y}dy_k=\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}dy_j+\frac{\partial c_k(\mathbf{w},y_k)}{\partial y_k}dy_k$$

$$\frac{\partial c(\mathbf{w},y)}{\partial y}\left(dy_j+dy_k\right)=\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}dy_j+\frac{\partial c_k(\mathbf{w},y_k)}{\partial y_k}dy_k$$

$$0=\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}dy_j+\frac{\partial c_k(\mathbf{w},y_k)}{\partial y_k}dy_k$$

$$-\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}dy_j=\frac{\partial c_k(\mathbf{w},y_k)}{\partial y_k}dy_k$$

$$\frac{\partial c_j(\mathbf{w},y_j)}{\partial y_j}=\frac{\partial c_k(\mathbf{w},y_k)}{\partial y_k}.$$

That is, industry marginal cost does not change (nor does industry total cost), so that (regardless of the levels of $y_j$ and $y_k$) the marginal costs of firms $j$ and $k$ must be identical for the effects to be completely offsetting. Note, too, that the marginal costs of firms $i$ ($i \neq j$ and $i \neq k$) are left undisturbed by the redistribution of outputs.

Given the intent of this paper, it turns out that a further and more relevant implication of the foregoing result is that industry marginal cost is independent of industry output. This may be shown by differentiating equation (3) with respect to $y_k$:

$$\frac{\partial^2 c(\mathbf{w},y)}{\partial y_j \partial y_k}=\frac{\partial^2 c(\mathbf{w},y)}{\partial y^2}\cdot\frac{\partial y}{\partial y_j}\frac{\partial y}{\partial y_k}=\frac{\partial^2 c_j(\mathbf{w},y_j)}{\partial y_j \partial y_k}$$

$$\frac{\partial^2 c(\mathbf{w},y)}{\partial y_j \partial y_k}=\frac{\partial^2 c(\mathbf{w},y)}{\partial y^2}=0.$$

In other words, industry marginal cost is unaffected by the level of industry output:

$$\frac{\partial}{\partial y}\left(\frac{\partial c(\mathbf{w},y)}{\partial y}\right)=0,$$

so that industry marginal cost may be written as

$$\lambda(\mathbf{w}) = \frac{\partial c(\mathbf{w}, y)}{\partial y}. \tag{4}$$

Integrating equation (4) with respect to $y$ to recover the form of the industry cost function thus gives

$$\int \frac{\partial c(\mathbf{w}, y)}{\partial y} dy = \int \lambda(\mathbf{w}) dy$$

$$c(\mathbf{w}, y) = \lambda(\mathbf{w}) y + c^0(\mathbf{w}), \tag{5}$$

where $c^0(\mathbf{w})$ is the constant of integration. Equation (5) indicates that an industry cost function that is consistent with (2) must have quasi-homothetic technology.[5]

[5] Quasi-homothetic technology is characterized by straight-line expansion paths (like homothetic technology) that do not emanate from the origin (unlike homothetic technology). This feature can be shown as follows: Applying Shephard's lemma on equation (5) gives

$$\frac{\partial c(\mathbf{w}, y)}{\partial w_n} = x_n(\mathbf{w}, y) = y \frac{\partial \lambda(\mathbf{w})}{\partial w_n} + \frac{\partial c^0(\mathbf{w})}{\partial w_n}.$$

Differentiating with respect to $y$, we get

$$\frac{\partial x_n(\mathbf{w}, y)}{\partial y} = \frac{\partial \lambda(\mathbf{w})}{\partial w_n}.$$

Thus, for two inputs, $n_1$ and $n_2$, we have

$$\frac{\partial x_{n_1}(\mathbf{w}, y)/\partial y}{\partial x_{n_2}(\mathbf{w}, y)/\partial y} = \frac{\partial \lambda(\mathbf{w})/\partial w_{n_1}}{\partial \lambda(\mathbf{w})/\partial w_{n_2}}.$$

The expression on the right-hand side of the equation is not a function of output and therefore constant, just as in homothetic technology. The expression on the left-hand side is the marginal rate of technical substitution as output expands. It is not independent of output, unlike in homothetic technology. [Imagine a graph of isoquants. In homothetic technology, the expansion path of optimal inputs emanates from the origin, so that the ratios of the optimal inputs remain constant as output expands. This is indicated by the fact that if vertical lines are drawn from intersection points of isoquants and the expansion path to the horizontal axis, the heights of the vertical lines relative to the bases of the resulting triangles remain in fixed proportion. In contrast, in quasi-homothetic technology, the expansion path does not emanate from origin. Consequently, as output expands, the slopes of the isoquants at their intersection points with the expansion path remain constant, as indicated by the equation above, but the proportions of the optimal inputs change. This is seen by drawing lines from the origin to the intersection points. Since the slopes of these rays are different, the height-to-base ratios of the resulting triangles do not remain constant, unlike in homothetic technology.]

Quasi-homothetic technology, however, poses difficulties. Note that $c(\mathbf{w},0) \neq 0$, unless $c^0(\mathbf{w}) = 0$. If $c^0(\mathbf{w}) = 0$, however, then the industry cost function is again linear in output, meaning that the production function is CRS, as in the first case considered.

If $c^0(\mathbf{w}) \neq 0$, then $c^0(\mathbf{w})$ may be taken to be a fixed cost, which implies that equation (5) is a short-run cost function. This is just as well, since it has been implicitly assumed that the number of firms in the industry, $I$, is exogenous, i.e., the free-entry-and-exit property of the long-run cost function has not been considered. Nonetheless, it must be that $c^0(\mathbf{w}) > 0$; otherwise, for $0 < y \leq y^0$, where

$$y^0 \equiv -\frac{c^0(\mathbf{w})}{\lambda(\mathbf{w})},$$

costs are nonpositive.

What other restrictions need to be imposed on equation (5) so that it would exhibit the properties of a cost function? Since costs are nondecreasing in output, $\lambda(\mathbf{w})$ must be nonnegative. For equation (5) to reflect linear homogeneity in input prices, it must be that

$$c(t\mathbf{w},y) = \lambda(t\mathbf{w})y + c^0(t\mathbf{w})$$
$$tc(\mathbf{w},y) = t\lambda(\mathbf{w})y + tc^0(\mathbf{w}),$$

which implies that both $\lambda(\mathbf{w})$ and $c^0(\mathbf{w})$ must be homogeneous of degree one in input prices. Applying Shephard's lemma on equation (5) yields

$$x_n(\mathbf{w},y) = y\frac{\partial\lambda(\mathbf{w})}{\partial w_n} + \frac{\partial c^0(\mathbf{w})}{\partial w_n},$$

which suggests that industry input demands must be affine functions of industry output. The elasticity of size[6] of the cost function given in equation (5) is

$$\varepsilon^*(\mathbf{w},y) = \left(\frac{d\ln c(\mathbf{w},y)}{d\ln y}\right)^{-1} = \left[\frac{\lambda(\mathbf{w})y}{\lambda(\mathbf{w})y + c^0(\mathbf{w})}\right]^{-1} = \frac{\lambda(\mathbf{w})y + c^0(\mathbf{w})}{\lambda(\mathbf{w})y}.$$

[6] Elasticity of size is the reciprocal of the output elasticity of cost. In other words, it is the ratio of average cost to marginal cost. Thus, if $\varepsilon^*(\mathbf{w},y) > 1$, average cost must be above the marginal cost at output level $y$, so that there is increasing returns to scale; if $\varepsilon^*(\mathbf{w},y) = 1$, average cost must be equal to marginal cost, so that there is constant returns to scale; if $\varepsilon^*(\mathbf{w},y) < 1$, average cost must be less than marginal cost, so that there is decreasing returns to scale. Elasticity of size is a better measure than elasticity of scale in the following sense: elasticity of scale measures the responsiveness of costs to changes in output levels, holding input ratios fixed (i.e., along a ray emanating from origin); elasticity of size measures the same thing but allows input ratios to change according to their optimal mixes.

Thus, equation (5) exhibits constant returns to size only if $c^0(\mathbf{w})=0$. If $c^0(\mathbf{w})>0$ [$c^0(\mathbf{w})<0$], then $\varepsilon^*(\mathbf{w},y)>1$ [$\varepsilon^*(\mathbf{w},y)<1$], which means that the industry cost function shows increasing [decreasing] returns to size, except when output is so large that returns to size is almost constant, since

$$\lim_{y\to\infty}\varepsilon^*(\mathbf{w},y)=\lim_{y\to\infty}\frac{\lambda(\mathbf{w})y+c^0(\mathbf{w})}{\lambda(\mathbf{w})y}=1.$$

Given these possible ways that $\varepsilon^*(\mathbf{w},y)$ behaves with respect to industry output, it can be concluded that the industry cost function of equation (5) has limited ability to generate the u-shaped average cost curves in economics textbooks.

The upshot of this discussion then is that, when working with industry-level data, the researcher may just as well assume constant returns to scale technology, i.e., $c^0(\mathbf{w})=0$, since it has the least disagreeable implications for an industry-level cost function. In contrast, assuming that $c^0(\mathbf{w})<0$ implies that costs are not positive over a range of industry output, while assuming that $c^0(\mathbf{w})>0$ implies that the production function exhibits increasing returns to scale at finite levels of industry output or that marginal cost is never above average cost at all levels of output.

## *2.2. Separability in production technology*[7]

Variables in data sets are often too limited in both number and variety to adequately represent the full range of inputs and outputs. The researcher then needs to address the following questions: Under what conditions can input-output combinations be considered joint or nonjoint production processes? Under what conditions is technology separable in outputs? Under what conditions is technology separable in inputs? As descriptions of production technology are needed to address these questions, this section starts with a review of these concepts.

Suppose that a firm handles $(M+N)$ possible goods or services. Let $z_\ell^i$ represent units of the $\ell$th good or service that the firm uses as an input and $z_\ell^o$ be units that the firm produces as an output. Then the firm's net output of the $\ell$th good or service is given by $z_\ell = z_\ell^o - z_\ell^i$, with $z_\ell > 0$ $(z_\ell < 0)$ implying that the firm produces more (less) than it uses of the good or service. Collecting the $z_\ell$s into an $(M+N)$-dimensional vector $\mathbf{z}$ gives what is called a production plan.

Some production plans are technologically feasible; that is, the technological know-how, methods, and processes exist to use net inputs to produce net

[7] The discussion here is based on sections 7.1 and 7.4 of Chambers [1988] and chapter 1 of Varian [1992].

outputs. The set of all these technologically feasible production plans is called the production possibilities set: $Z = \left\{ \boldsymbol{z} \in \mathbb{R}_{+}^{M+N} \middle| \boldsymbol{z} \text{ is feasible} \right\}$.

Although as specified the set $Z$ is perfectly adequate, in the discussion that follows it is convenient to maintain a clear distinction between inputs and outputs. A production possibilities set that observes this delineation may be written as $Z = \left\{ (\mathbf{x},\mathbf{y}) \in \mathbb{R}_{+}^{N} \times \mathbb{R}_{+}^{M} \middle| (\mathbf{x},\mathbf{y}) \text{ is feasible} \right\}$, where $\mathbf{x}$ is an $N$-dimensional vector of inputs and $\mathbf{y}$ is an $M$-dimensional vector of outputs.

The set $Z$ is assumed to have the following properties: (a) $Z \neq \varnothing$ ($Z$ is not a null set; some production plans are feasible); (b) $Z$ is a closed set (this is adopted for technical reasons; it guarantees that $Z$ includes its boundaries); (c) $Z$ is a convex set (this implies that any linear combination of two technologically feasible production plans is also technologically feasible); (d) if $(\mathbf{x},\mathbf{y}) \in Z$ and $\mathbf{x}' \geq \mathbf{x}$, then $(\mathbf{x}',\mathbf{y}) \in Z$ (inputs can be freely disposed of; excess inputs do not impede production); (e) if $(\mathbf{x},\mathbf{y}) \in Z$ and $\mathbf{y}' \leq \mathbf{y}$, then $(\mathbf{x},\mathbf{y}') \in Z$ (outputs can be freely disposed of; inputs that can produce a given output bundle can also produce smaller output bundles); (f) for every finite $x$, $Z$ is bounded from above (this guarantees the existence of a production function); and (g) $(\mathbf{x},\mathbf{0}_M) \in Z$, but $(\mathbf{0}_N,\mathbf{y}) \notin Z$ if $\mathbf{y} \geq \mathbf{0}$ (zero output is always technologically feasible, but it is not possible to obtain nonnegative output using no inputs).

Two characterizations of $Z$ are the input requirement set and the producible output set. The input requirement set is defined by

$$V(\mathbf{y}) = \left\{ \mathbf{x} \in \mathbb{R}_{+}^{N} \middle| (\mathbf{x},\mathbf{y}) \in Z \right\},$$

which states that it is the set of all input vectors $\mathbf{x}$ that can produce output bundle $\mathbf{y}$. The producible output set, which is given by

$$Y(\mathbf{x}) = \left\{ \mathbf{y} \in \mathbb{R}_{+}^{M} \middle| (\mathbf{x},\mathbf{y}) \in Z \right\},$$

is the set of all output vectors that can be produced by a given input vector.

Of primary importance to economists and engineers are the technologically efficient production plans. A production plan $(\mathbf{x},\mathbf{y}) \in Z$ is technologically efficient if there is no $(\mathbf{x}',\mathbf{y}') \in Z$ such that $\mathbf{x}' \leq \mathbf{x}$ and $\mathbf{y}' \geq \mathbf{y}$, i.e., it is not possible to produce the same output with less inputs or to produce more output with the same inputs. The set of technically efficient production plans is usually described by a transformation function,

$$T : \mathbb{R}^{N} \times \mathbb{R}^{M} \to \mathbb{R} \quad \text{where } T(\mathbf{x},\mathbf{y}) = 0 \text{ if and only if } (\mathbf{x},\mathbf{y}) \text{ is technologically efficient}$$

The input requirement set and the producible output set may be expressed in terms of the transformation function as follows:

$$V(y)=\left\{x\in \mathbb{R}_+^N \middle| T(x,y)\le 0\right\} \quad \text{and} \quad Y(x)=\left\{y\in \mathbb{R}_+^M \middle| T(x,y)\le 0\right\}.$$

In other words, $(\mathbf{x},\mathbf{y})\in Z$ if $T(\mathbf{x},\mathbf{y})\le 0$ (since $Z$ is bounded from above and $T(\mathbf{x},\mathbf{y})=0$ consists of production plans that yield the maximum $\mathbf{y}$ for a given $\mathbf{x}$ or use the smallest $\mathbf{x}$ to produce a given $\mathbf{y}$).

This completes the review of the basic concepts. In what follows, various descriptions of jointness and separability are tackled.

Suppose that an output index $g(\mathbf{y})$ can be defined and a set $\hat{Z}$ can be specified such that $(\mathbf{x},g(\mathbf{y}))\in \hat{Z}$ if and only if $(\mathbf{x},\mathbf{y})\in Z$. Then production technology is said to be separable in outputs and the input requirement set can be written as if there was only one output:

$$\begin{aligned} V(\mathbf{y}) &= \left\{\mathbf{x}\in \mathbb{R}_+^N \middle| (\mathbf{x},\mathbf{y})\in Z\right\} \\ &= \left\{\mathbf{x}\in \mathbb{R}_+^N \middle| (\mathbf{x},g(\mathbf{y}))\in Z\right\} \\ &= \left\{\mathbf{x}\in \mathbb{R}_+^N \middle| (\mathbf{x},g)\in Z\right\} \\ &= V(g). \end{aligned}$$

Analogously, suppose that an input index $h(\mathbf{x})$ can be defined and a set $\tilde{Z}$ can be specified such that $(h(\mathbf{x}),\mathbf{y})\in \tilde{Z}$ if and only if $(\mathbf{x},\mathbf{y})\in Z$. Then production technology is said to be separable in inputs and the producible output set can be written as though there were only one input:

$$\begin{aligned} Y(\mathbf{x}) &= \left\{\mathbf{y}\in \mathbb{R}_+^M \middle| (\mathbf{x},\mathbf{y})\in Z\right\} \\ &= \left\{\mathbf{y}\in \mathbb{R}_+^M \middle| (h(\mathbf{x}),\mathbf{y})\in Z\right\} \\ &= \left\{\mathbf{y}\in \mathbb{R}_+^M \middle| (h,\mathbf{y})\in Z\right\} \\ &= Y(h). \end{aligned}$$

A problem with the foregoing definitions, however, is that, being rather abstract and quite general, they impose minimal structure on technology to be useful guideposts for empirical research. More instructive for this paper is the following definition: Technology $Z$ is nonjoint in inputs if, for every $(\mathbf{x},\mathbf{y})\in Z$, input vectors $\mathbf{x}_m \ge 0$ can be specified such that

$$y_m \leq f_m(\mathbf{x}_m) \text{ and } \sum_{m=1}^{M} \mathbf{x}_m \leq \mathbf{x} \quad \text{for } m = 1,\ldots,M,$$

where $f_m(\mathbf{x}_m)$ is a production function that satisfies the usual (weak) regularity conditions, viz., properties (a) to (g) of production technology that were stated earlier. A nice feature of the input requirement set of a technology that is nonjoint in inputs is that it decomposes as the sum of the input requirement sets of individual outputs:

$$\begin{aligned} V(\mathbf{y}) &= \left\{ \mathbf{x} \in \mathbb{R}_{+}^{N} \middle| \sum_{m=1}^{M} \mathbf{x}_m \leq \mathbf{x} \text{ and } y_m \leq f_m(\mathbf{x}_m) \text{ for } m = 1,\ldots,M \right\} \\ &= \left\{ \sum_{m=1}^{M} \mathbf{x}_m \in \mathbb{R}_{+}^{N} \middle| y_m \leq f_m(\mathbf{x}_m) \right\} \\ &= \sum_{m=1}^{M} \left\{ \mathbf{x}_m \in \mathbb{R}_{+}^{N} \middle| y_m \leq f_m(\mathbf{x}_m) \right\} \\ &= \sum_{m=1}^{M} V_m(y_m), \end{aligned}$$

where $V_m(y_m)$ is the input requirement set of the $m$th output.

In analogous fashion, nonjointness in outputs may be defined as follows: Technology $Z$ is nonjoint in outputs if, for every $(\mathbf{x},\mathbf{y}) \in Z$, output indices $g_n(\mathbf{y}_n) \geq 0$ can be specified such that

$$\mathbf{x}_n \geq g_n(\mathbf{y}_n) \quad \text{and} \quad \sum_{n=1}^{N} \mathbf{y}_n \geq \mathbf{y} \quad \text{for } n = 1,\ldots,N,$$

where $g_n(\mathbf{y}_n)$ is a nondecreasing function of $\mathbf{y}_n$, has a producible output set that is consistent with the regularity conditions of $Z$, and meets the condition that $g_n(\mathbf{0}_n) = 0$. Similarly, the producible output set of a technology that is nonjoint in outputs has the property that it decomposes as the sum of the producible output sets of individual inputs:

$$
\begin{aligned}
Y(\mathbf{x}) &= \left\{ \mathbf{y} \in \mathbb{R}_+^M \,\middle|\, \sum_{n=1}^{N} \mathbf{y}_n \geq \mathbf{y} \text{ and } x_n \geq g_n(\mathbf{y}_n) \text{ for } n = 1,\ldots,N \right\} \\
&= \left\{ \sum_{n=1}^{N} \mathbf{y}_n \in \mathbb{R}_+^M \,\middle|\, x_n \geq g_n(\mathbf{y}_n) \right\} \\
&= \sum_{n=1}^{N} \left\{ \mathbf{y}_n \in \mathbb{R}_+^M \,\middle|\, x_n \geq g_n(\mathbf{y}_n) \right\} \\
&= \sum_{n=1}^{N} Y_n(x_n)
\end{aligned}
$$

where $Y_n(x_n)$ is the producible output set that is associated with $g_n(\mathbf{y}_n)$.

Nonjointness in outputs somewhat stretches credulity: Is it really possible to specify a producible output set for each input? While sheep (as an input) can be divided into meat, skin, and wool, for example, surely other inputs are needed to effect the transformations. Nonetheless, it need only be pointed out that the definition is consistent with a concept long in use in economics, viz., the production possibilities curve or the transformation function, which gives the varying combinations of output levels that can be obtained from a single input.

To conclude this section, it may be noted that the foregoing definitions may be combined in various ways to provide more specific descriptions of production technology as needed by the researcher. For instance, suppose that $Z = \bigcup Z_{m'}$ where $Z_{m'} = \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbf{Z} \middle| g_{m'}\left(\mathbf{e}_{ym'}\mathbf{y}\right) \leq f_{m'}\left(\mathbf{e}_{xm'}\mathbf{x}\right) \right\}$ for $m' = 1,\ldots,M'$, and $Z_j \cap Z_k = \varnothing$ for all $j \neq k$. That is, production technology can be broken down into $M'$ mutually exclusive technologies, where $f_{m'}\left(\mathbf{e}_{xm'}\mathbf{x}\right) = f_{m'}\left(\mathbf{x}_{m'}\right)$ [$\mathbf{e}_{xm'}$ is a vector that selects the subset of inputs that go into the production of the $m'$th output index] and $g_{m'}\left(\mathbf{e}_{ym'}\mathbf{y}\right) = g_{m'}\left(\mathbf{y}_{m'}\right)$ [$\mathbf{e}_{ym'}$ is a vector that selects the subset of outputs that are produced by the $m'$th production function]. Then the input requirement set can be written as

$$V(\mathbf{y})=\left\{\mathbf{x}\in\mathbb{R}_{+}^{N}\middle|\sum_{m'=1}^{M'}\mathbf{e}_{xm'}\mathbf{x}\le\mathbf{x},\sum_{m'=1}^{M'}\mathbf{e}_{ym'}\mathbf{y}\ge\mathbf{y},\right.$$
$$\left.\text{and } g_{m'}\left(\mathbf{e}_{ym'}\mathbf{y}\right)\le f_{m'}\left(\mathbf{e}_{xm'}\mathbf{x}\right)\text{ for } m'=1,\dots,M'\right\}$$
$$=\left\{\sum_{m'=1}^{M'}\mathbf{x}_{m'}\in\mathbb{R}_{+}^{N}\middle|g_{m'}\left(\mathbf{y}_{m'}\right)\le f_{m'}\left(\mathbf{x}_{m'}\right)\right\}$$
$$=\sum_{m'=1}^{M'}\left\{\mathbf{x}_{m'}\in\mathbb{R}_{+}^{N}\middle|g_{m'}\left(\mathbf{y}_{m'}\right)\le f_{m'}\left(\mathbf{x}_{m'}\right)\right\}$$
$$=\sum_{m'=1}^{M'}V_{m'}\left(g_{m'}\right).$$

That is, the input requirement set is just the sum of the input requirement sets of the $M'$ technologies. Consequently, each of the $M'$ technologies can be separately studied, the only constraint being that the sum of inputs used across technologies to produce $y$ cannot exceed the total amounts available in the whole economy.

### *2.3. Separability in technology redux: multi-stage production*[8]

This section explores an issue that, although categorized under separability and certainly related to what was discussed in the previous section, has a different emphasis. The practical problem is that there tends to be numerous production factors (which can tax understanding or overwhelm degrees of freedom in econometric estimation). If production has a multi-stage structure, however, such that at earlier stages subsets of inputs are used to produce intermediate products that in turn are used to produce the final product, the problem becomes more tractable. The question is, What features of the production function allow it to be specified as having multiple stages?

Assume that the production function is twice continuously differentiable in its inputs. The property that allows it to have a multi-stage structure is that the marginal rate of technical substitution between two inputs is not affected by changes in a third input, that is, if

$$\frac{\partial}{\partial x_{n_3}}\left[\frac{\partial f(\mathbf{x})/\partial x_{n_1}}{\partial f(\mathbf{x})/\partial x_{n_2}}\right]=0, \tag{6}$$

where $n_1$, $n_2$, and $n_3$ refer to different inputs. When a production function $f(\mathbf{x})$ has the property described in equation (6), inputs $x_{n_1}$ and $x_{n_2}$ are said to be

[8] The discussion here is based on section 1.8c of Chambers [1988].

separable from input $x_{n_3}$. Note carefully what equation (6) indicates: The *slope* of the isoquants in $n_1$-$n_2$ space is not affected by changes in $x_{n_3}$. But the isoquant in $n_1$-$n_2$ space that describes the level of output produced may be affected by how much $x_{n_3}$ is available. Hence, the relevant isoquant may be farther away from the origin, the larger is the magnitude of $x_{n_3}$.

Another way to represent equation (6) may be derived from it as follows:

$$\frac{\partial}{\partial x_{n_3}}\left[\frac{\partial f(\mathbf{x})/\partial x_{n_1}}{\partial f(\mathbf{x})/\partial x_{n_2}}\right]=0$$

$$\frac{\dfrac{\partial f}{\partial x_{n_2}}\dfrac{\partial^2 f}{\partial x_{n_1}\partial x_{n_3}}-\dfrac{\partial f}{\partial x_{n_1}}\dfrac{\partial^2 f}{\partial x_{n_2}\partial x_{n_3}}}{\left(\dfrac{\partial f}{\partial x_{n_2}}\right)^2}=0$$

$$\frac{\partial f}{\partial x_{n_2}}\frac{\partial^2 f}{\partial x_{n_1}\partial x_{n_3}}=\frac{\partial f}{\partial x_{n_1}}\frac{\partial^2 f}{\partial x_{n_2}\partial x_{n_3}}$$

$$\frac{\partial^2 f/\left(\partial x_{n_1}\partial x_{n_3}\right)}{\partial f/\partial x_{n_1}}=\frac{\partial^2 f/\left(\partial x_{n_2}\partial x_{n_3}\right)}{\partial f/\partial x_{n_2}}$$

$$\frac{x_{n_3}}{\partial f/\partial x_{n_1}}\square\frac{\partial}{\partial x_{n_3}}\left(\frac{\partial f}{\partial x_{n_1}}\right)=\frac{x_{n_3}}{\partial f/\partial x_{n_2}}\square\frac{\partial}{\partial x_{n_3}}\left(\frac{\partial f}{\partial x_{n_2}}\right)$$

$$\frac{\partial \ln\left(\partial f/\partial x_{n_1}\right)}{\partial \ln x_{n_3}}=\frac{\partial \ln\left(\partial f/\partial x_{n_2}\right)}{\partial \ln x_{n_2}}.$$

This means that the separability of $x_{n_1}$ and $x_{n_2}$ from $x_{n_3}$ requires that the elasticity of the marginal product of $x_{n_1}$ with respect to $x_{n_3}$ be equal to the elasticity of the marginal product of $x_{n_2}$ with respect to $x_{n_3}$.

There are different types of separability in inputs. To differentiate between them, we need a notation for partitioning the input vector. Let $\mathcal{N}=\{1,2,\ldots,N\}$ be the set of input indices. Any scheme for classifying inputs can be described by a partitioning of $\mathcal{N}$ into a set of subsets $\{\mathcal{N}_1, \ldots, \mathcal{N}_s\}$, where $\mathcal{N}_1\cup\cdots\cup\mathcal{N}_s=\mathcal{N}$ and $\mathcal{N}_s\cap\mathcal{N}_{s'}=\varnothing$ for $s\neq s'$. Let $\hat{\mathcal{N}}$ be one particular partitioning of the input matrix. Then the $N$-dimensional input vector $\mathbf{x}$ will have a corresponding partitioning $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(s)}\}$, where a subvector $\mathbf{x}^{(s)}$ will have as elements all inputs $x_n$ such that $n\in\hat{\mathcal{N}}_s$.[9]

[9] Note that partitioning is not at all restrictive. In an agricultural production function, for instance, if the partitioning is done over planting, weeding, and harvesting, labor supply can be defined at the outset as being differentiated over these activities.

Using this partitioning notation, we can define the production function to be weakly separable in partition scheme $\hat{\mathcal{N}}$ if

$$\frac{\partial}{\partial x_{n_3}}\left[\frac{\partial f(\mathbf{x})/\partial x_{n_1}}{\partial f(\mathbf{x})/\partial x_{n_2}}\right]=0 \qquad \text{for } n_1,n_2\in\hat{\mathcal{N}}_s \text{ and } n_3\notin\hat{\mathcal{N}}_s. \tag{7}$$

In other words, weak separability in the partition scheme obtains if the marginal rate of technical substitution between any two inputs in a given input subvector (or partition) is independent of any other input that is not an element of that subvector.

But Goldman and Uzawa [1964] show (in the context of utility functions) that if $f(\mathbf{x})$ is strictly quasiconcave and its marginal products are positive for all $\mathbf{x}\in \square^{N}_{+}$, the condition specified in equation (7) is equivalent to the production function having the form:

$$f(\mathbf{x})=F\left[f^1\left(\mathbf{x}^{(1)}\right),\ldots,f^s\left(\mathbf{x}^{(s)}\right)\right], \tag{8}$$

where $F$ and each $f^s(\mathbf{x}^{(s)})$ are themselves strictly monotonic and quasiconcave. Since $F$ is both strictly monotonic and quasiconcave in its arguments, each $f^s(\mathbf{x}^{(s)})$ may be considered an aggregate input formed from the inputs in its partition; since each $f^s(\mathbf{x}^{(s)})$ is strictly monotonic and quasiconcave in $\mathbf{x}^{(s)}$, it may itself be considered a production function. In effect, a production function that is weakly separable in inputs (in the partition $\hat{\mathcal{N}}$ may be thought of as having two stages: in the first stage, inputs belonging to partition $\hat{\mathcal{N}}_s$ are combined to produce aggregate input $f^s(\mathbf{x}^{(s)})$; in the second stage, the aggregate inputs are then combined to produce the final output.

A point that is not often appreciated, however, is that weak separability in inputs does not merely imply a two-stage production process. This is seen if equation (7) is applied on equation (8), i.e., the derivative of the marginal rate of technical substitution between two inputs in partition $\hat{\mathcal{N}}_s$ is taken with respect to another input that does not belong to the same partition, where we obtain

$$\begin{aligned}\frac{\partial}{\partial x_{n_3}}\left[\frac{\partial f(\mathbf{x})/\partial x_{n_1}}{\partial f(\mathbf{x})/\partial x_{n_2}}\right] &= \frac{\partial}{\partial x_{n_3}}\left(\frac{\dfrac{\partial F}{\partial f^s}\square\dfrac{\partial f^s}{\partial x_{n_1}}}{\dfrac{\partial F}{\partial f^s}\square\dfrac{\partial f^s}{\partial x_{n_2}}}\right)\\ &= \frac{\partial}{\partial x_{n_3}}\left[\frac{\partial f^s\left(\mathbf{x}^{(s)}\right)/\partial x_{n_1}}{\partial f^s\left(\mathbf{x}^{(s)}\right)/\partial x_{n_2}}\right]\\ &= 0 \qquad \text{for } n_1,n_2\in\mathcal{N}_s \text{ and } n_3\notin\mathcal{N}_s.\end{aligned}$$

What this means is that, not only is there a two-stage production process, the first-stage production functions that produce the aggregate inputs—i.e., $f^{s}(\mathbf{x}^{(s)})$ vs. $f^{s'}(\mathbf{x}^{(s')})$—must also be independent of each other. A counterexample for this production structure is an agricultural production function that is partitioned into planting, weeding, and harvesting: To the extent that weeding cannot proceed independently of planting, the production function is not weakly separable in the partitioning.

Another type of separability is strong separability: The production function is strongly separable in partition scheme $\hat{\mathcal{N}}$ if

$$\frac{\partial}{\partial x_{n_3}}\left[\frac{\partial f(\mathbf{x})/\partial x_{n_1}}{\partial f(\mathbf{x})/\partial x_{n_2}}\right]=0 \qquad \text{for } n_1 \in \mathcal{N}_s, n_2 \in \mathcal{N}_{s'} \text{ and } n_3 \notin \mathcal{N}_s \cup \mathcal{N}_{s'}. \tag{9}$$

Equation (9) states that strong separability in the partition scheme obtains if the marginal rate of technical substitution between any two inputs, regardless of the partitions to which they belong—note that $\hat{\mathcal{N}}_s$ may be equal to $\hat{\mathcal{N}}_{s'}$—is independent of any other input that does not belong to either $\hat{\mathcal{N}}_s$ or $\hat{\mathcal{N}}_{s'}$ In effect, a production function that is strongly separable in a partition scheme is also weakly separable in that partition scheme, but the converse proposition is not true.

Again, Goldman and Uzawa [1964] show that if $f(\mathbf{x})$ is strictly quasiconcave and its marginal products are everywhere positive, the condition specified in equation (9) is equivalent to the production function having the form:

$$f(\mathbf{x})=G\left[\sum_{s=1}^{S} g^{s}\left(\mathbf{x}^{(s)}\right)\right], \tag{10}$$

where $G$ and each $g^s$ are also strongly monotonic and strictly quasiconcave. In effect, as in weak separability, each $g^{s}(\mathbf{x}^{(s)})$ may be considered a production function by itself as well as an aggregate input. But equation (10) is more restrictive, because it implies that the aggregate inputs are perfectly substitutable in $G$. This is readily seen if equation (10) is rewritten as

$$f(\mathbf{x})=G\left[\sum_{s=1}^{S} g^{s}\left(\mathbf{x}^{(s)}\right)\right]=G\left(\sum_{s=1}^{S} X_s\right), \tag{11}$$

where $X_s = g^{s}(\mathbf{x}^{(s)})$. Totally differentiate (11) with respect to the aggregate inputs and set the expression to zero. Then, if all derivatives, except those for $X_s$ and $X_{s'}$, are zero, we obtain

$$\frac{dX_s}{dX_{s'}} = -\frac{\partial G/\partial X_s}{\partial G/\partial X_{s'}} = -1.$$

In effect, the first-stage production functions $g^s(\mathbf{x}^{(s)})$ are alternative ways of producing the final output.

A last attribute of a production function that is strongly separable in inputs is that it is homothetic in the aggregate inputs since $\Sigma_{s=1}^{S} X_s$ is linearly homogeneous in the $X_s$s.

A third type of separability is factor-wise separability. Let $N = \mathcal{N}_S$, so that each partition contains only one element. If the production function $f(\mathbf{x})$ is strongly separable in the partition, then it is factor-wise separable. Alternatively, $f(\mathbf{x})$ is factor-wise separable if

$$\frac{\partial}{\partial x_{n_3}}\left[\frac{\partial f(\mathbf{x})/\partial x_{n_1}}{\partial f(\mathbf{x})/\partial x_{n_2}}\right] = 0 \qquad \text{for } n_1 \neq n_2, n_1 \neq n_3, \text{ and } n_2 \neq n_3.$$

Following equation (10), we can write a factor-wise separable production function to have the following form:

$$f(\mathbf{x}) = G\left[\sum_{n=1}^{N} g^n(x_n)\right].$$

But since $g^n(x_n)$ depends only on $x_n$, the production function can be rewritten in terms of the $x_n$s:

$$f(\mathbf{x}) = G^*\left(\sum_{n=1}^{N} x_n\right).$$

Input separability thus provides a way for breaking up the production processes into stages and for aggregating inputs. A crucial restriction of input separability (at least in the sense that it is used in this section rather than in Section 2.2), however, is the independence of the (sub)production functions, given a partitioning of the input vector. This may not be a tenable assumption in agriculture.

## 3. An approach to filling up the substitution matrix

Suppose that a researcher has aggregate-level (i.e., provincial or regional) data on costs, $N$ inputs and input prices, and $M'$ outputs, but he is able to estimate only the own price elasticity of each input for each output. How can he find the cross-price elasticities of all the inputs?

Based on the review of the issues undertaken above, the researcher has three starting points. First, he has to group the inputs and outputs according to input nonjointness. For instance, he may bundle agricultural crops that are intercropped into an output index, treating it as one (composite) output good. Second, he may as well assume linearly homogeneous production functions.

Assuming that there are $M$ nonjoint outputs, the researcher's problem may be cast as follows:

$$\begin{aligned} &\min_{x_m} \mathbf{w} \sum_{m=1}^{M} \mathbf{x}_m \\ &\text{s.t.}\, \mathbf{y}_m \le f_m(\mathbf{x}_m) \quad \text{for } m = 1,\dots,M \\ &\sum_{m=1}^{M} \mathbf{x}_m \le \mathbf{x}, \end{aligned} \tag{12}$$

where, for simplicity, it is assumed that factor markets are competitive, so that firms face the same input vector **w** regardless of their product. Note that the last constraint ensures that input use does not exceed the total amounts available in the economy.

If the last constraint does not bind, the solution to this problem is a set of conditional input demand functions $x_{mn}(\mathbf{w}, y_m)$ for $m = 1,\dots,M$ and $n = 1,\dots,N$, since the production functions are separable.

The practical problem that needs to be specified, however, is the size of the substitution matrix of a given output $m$, since this determines the number of unknown cross-price elasticities of input demands that need to be solved, given the number of own price elasticities that are assumed to be known. With $N$ inputs, the substitution matrix has $N^2$ cells, $N$ of which—the ones on the principal diagonal—are known. Because the matrix is symmetric, there are thus $(N^2 - N)/2$ distinct unknown values. In effect, the researcher can have only three inputs.[10]

Constant returns to scale technology can be exploited, however, so that four inputs can be handled. Let $x_{m1}$ be labor, $x_{m2}$ an intermediate good (e.g., some composite of fertilizer, pesticide, and irrigation), $x_{m3}$ capital, and $x_{m4}$ land. Multiplying the Lagrangean associated with the constrained cost minimization problem of the $m$th output by $w_4 x_{m4}(\mathbf{w}, y_m) / w_4 x_{m4}(\mathbf{w}, y_m)$, where $x_{m4}(\mathbf{w}, y_m)$ is the optimal demand for land that is now assumed to be fixed, gives

[10] That is, $N = 3$ is the solution to $\left(N^2 - N\right)/2 = N$.

$$
\begin{aligned}
\mathcal{L} &= w_4 x_{m4}(\mathbf{w}, y_m)\left\{\left(\sum_{n\neq 4}\frac{w_n}{w_4}\square\frac{x_{mn}}{x_{m4}(\mathbf{w}, y_m)}\right) - \right. \\
&\quad \frac{\lambda}{w_4}\left[f_m\left(\frac{x_{m1}}{x_{m4}(\mathbf{w}, y_m)}, \frac{x_{m2}}{x_{m4}(\mathbf{w}, y_m)}, \frac{x_{m3}}{x_{m4}(\mathbf{w}, y_m)}\right) - \right. \\
&\quad \left.\left.\frac{y_m}{x_{m4}(\mathbf{w}, y_m)}\right\}\right] \\
&= w_4 x_{m4}(\mathbf{w}, y_m)\left\{\left(\sum_{n\neq 4} w_n^* x_{mn}^* - \lambda^*\left[f_m\left(x_{m1}^*, x_{m2}^*, x_{m3}^*\right) - y_m^*\right]\right)\right\},
\end{aligned}
$$

where $w_n^* = w_n/w_4$, $\lambda^* = \lambda/w_4$, $x_{mn}^* = x_{mn}/x_{m4}(\mathbf{w}, y_m)$, and $y_m^* = y_m/x_{m4}(\mathbf{w}, y_m)$. Notice that since $w_4 x_{m4}(\mathbf{w}, y_m)$ is simply a scale parameter, it can be assumed away.

If an interior solution is assumed, the optimal input demands are now given by $x_{mn}^* = [\, w_n^*, y_m^* \,|\, x_{m4}(\mathbf{w}, y_m)]$ for $n = 1, 2, 3$. The resulting cost function has all the necessary properties. In particular, it is homogeneous of degree one in factor prices.

By Shephard's lemma, the substitution matrix of the cost function may be written as follows:

$$
\begin{bmatrix}
\frac{\partial^2 c}{\partial w_1^{*2}} & \frac{\partial^2 c}{\partial w_1^* \partial w_2^*} & \frac{\partial^2 c}{\partial w_1^* \partial w_3^*} \\
\frac{\partial^2 c}{\partial w_2^* \partial w_1^*} & \frac{\partial^2 c}{\partial w_2^{*2}} & \frac{\partial^2 c}{\partial w_2^* \partial w_3^*} \\
\frac{\partial^2 c}{\partial w_3^* \partial w_1^*} & \frac{\partial^2 c}{\partial w_3^* \partial w_2^*} & \frac{\partial^2 c}{\partial w_3^{*2}}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial x_{m1}^*}{\partial w_1^*} & \frac{\partial x_{m1}^*}{\partial w_2^*} & \frac{\partial x_{m1}^*}{\partial w_3^*} \\
\frac{\partial x_{m2}^*}{\partial w_1^*} & \frac{\partial x_{m2}^*}{\partial w_2^*} & \frac{\partial x_{m2}^*}{\partial w_3^*} \\
\frac{\partial x_{m3}^*}{\partial w_1^*} & \frac{\partial x_{m3}^*}{\partial w_2^*} & \frac{\partial x_{m3}^*}{\partial w_3^*}
\end{bmatrix}, \tag{13}
$$

where, by Young's theorem, $\partial x_{mn}^*/\partial w_{n'}^* = \partial x_{mn'}^*/\partial w_n^*$, for $n \neq n'$; that is, the matrix is symmetric. To simplify the notation, rewrite this matrix as

$$
\begin{bmatrix}
a & b & c \\
b & d & e \\
c & e & f
\end{bmatrix}. \tag{14}
$$

The problem is to solve for the unknown variables *b, c, e* from *a, d, f.*

Since the cost function is homogeneous of degree one in input prices, the conditional input demand functions, being derivatives of the cost function, are homogeneous of degree zero. Thus, by Euler's theorem,

$$\begin{aligned}
&\frac{\partial x_{m1}^*}{\partial w_1^*} w_1^* + \frac{\partial x_{m1}^*}{\partial w_2^*} w_2^* + \frac{\partial x_{m1}^*}{\partial w_3^*} w_3^* = 0 \\
&\frac{\partial x_{m2}^*}{\partial w_1^*} w_1^* + \frac{\partial x_{m2}^*}{\partial w_2^*} w_2^* + \frac{\partial x_{m2}^*}{\partial w_3^*} w_3^* = 0 \\
&\frac{\partial x_{m3}^*}{\partial w_1^*} w_1^* + \frac{\partial x_{m3}^*}{\partial w_2^*} w_2^* + \frac{\partial x_{m3}^*}{\partial w_3^*} w_3^* = 0
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
&a w_1^* + b w_2^* + c w_3^* = 0 \\
&b w_1^* + d w_2^* + e w_3^* = 0 \\
&c w_1^* + e w_2^* + f w_3^* = 0.
\end{aligned}$$

The solution to this simultaneous equations system is given by

$$b = -\frac{a w_1^{*2} + d w_2^{*2} - f w_3^{*2}}{2 w_1^* w_2^*}, \quad c = \frac{d w_2^{*2} - a w_1^{*2} - f w_3^{*2}}{2 w_1^* w_3^*},$$

$$e = \frac{a w_1^{*2} - d w_2^{*2} - f w_3^{*2}}{2 w_2^* w_3^*}.$$

The matrix of input price elasticities can now be formed by multiplying each of the "coefficients" in equation (14) by the appropriate wage-to-input-demand ratio, where the quantities of input demands are the values at which the evaluation in equation (13) is done:

$$\begin{bmatrix}
\frac{\partial x_{m1}^*}{\partial w_1^*}\frac{w_1^*}{x_{m1}^*} & \frac{\partial x_{m1}^*}{\partial w_2^*}\frac{w_2^*}{x_{m1}^*} & \frac{\partial x_{m1}^*}{\partial w_3^*}\frac{w_3^*}{x_{m1}^*} \\
\frac{\partial x_{m2}^*}{\partial w_1^*}\frac{w_1^*}{x_{m2}^*} & \frac{\partial x_{m2}^*}{\partial w_2^*}\frac{w_2^*}{x_{m2}^*} & \frac{\partial x_{m2}^*}{\partial w_3^*}\frac{w_3^*}{x_{m2}^*} \\
\frac{\partial x_{m3}^*}{\partial w_1^*}\frac{w_1^*}{x_{m3}^*} & \frac{\partial x_{m3}^*}{\partial w_2^*}\frac{w_2^*}{x_{m3}^*} & \frac{\partial x_{m3}^*}{\partial w_3^*}\frac{w_3^*}{x_{m3}^*}
\end{bmatrix}
=
\begin{bmatrix}
a\frac{w_1^*}{x_{m1}^*} & b\frac{w_2^*}{x_{m1}^*} & c\frac{w_3^*}{x_{m1}^*} \\
b\frac{w_1^*}{x_{m2}^*} & d\frac{w_2^*}{x_{m2}^*} & e\frac{w_3^*}{x_{m2}^*} \\
c\frac{w_1^*}{x_{m3}^*} & e\frac{w_2^*}{x_{m3}^*} & f\frac{w_3^*}{x_{m3}^*}
\end{bmatrix}$$

$$= \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}$$

Note that because the wage-to-input-demand ratios of the off-diagonal elements of the matrix are not equal, the matrix of input price elasticities is no longer symmetric.

Based on the foregoing results, it is possible in principle to recover the $4\times 4$ matrix of price elasticities in the original inputs. Note, for instance, that from cell 21 of matrices (13) and (14), we have $\partial x_{m2}^{*}/\partial w_1^{*}=b,$ so that

$$\int \frac{\partial x_{m2}^{*}}{\partial w_1^{*}} dw_1^{*} = \int b dw_1^{*}$$

$$x_{m2}^{*} = bw_1^{*} + c\left(w_1^{*}\right) = k\left(w_1^{*}\right),$$

where $c\left(w_1^{*}\right)$ is the constant of integration. This result in turn implies that

$$x_{m2} = k\left(\frac{w_1}{w_4}\right)x_{m4}\left(\mathbf{w}, y_m\right).$$

The response of demand for the intermediate good to an increase in the wage rate is thus given by

$$\frac{\partial x_{m2}}{\partial w_1} = k'\left(\frac{w_1}{w_4}\right)\frac{x_{m4}\left(\mathbf{w}, y_m\right)}{w_4} + k\left(\frac{w_1}{w_2}\right)\frac{\partial x_{m4}\left(\mathbf{w}, y_m\right)}{\partial w_1}.$$

The same procedure can be carried out for all the nine cells of matrix (14), allowing us to form the matrix

$$\begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} & \frac{\partial x_1}{\partial w_3} & \frac{\partial x_1}{\partial w_4} \\ \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2} & \frac{\partial x_2}{\partial w_3} & \frac{\partial x_2}{\partial w_4} \\ \frac{\partial x_3}{\partial w_1} & \frac{\partial x_3}{\partial w_2} & \frac{\partial x_3}{\partial w_3} & \frac{\partial x_3}{\partial w_4} \\ \frac{\partial x_4}{\partial w_1} & \frac{\partial x_4}{\partial w_2} & \frac{\partial x_4}{\partial w_3} & \frac{\partial x_4}{\partial w_4} \end{bmatrix} = \begin{bmatrix} a' & b' & c' & g \\ b' & d' & e' & h \\ c' & e' & f' & i \\ g & h & i & j \end{bmatrix},$$

where only cells *g, h, i,* and *j* are unknown. Since the input demands $x_m\left(\mathbf{w}, y_m\right)$ are homogeneous of degree zero in input prices, however, Euler's theorem can be applied on $x_{m1}$ to obtain the value for *g,* on $x_{m2}$ for *h,* and on $x_{m3}$ for *i.* With *g, h,* and *i* known, applying Euler's theorem on $x_{m4}$ allows us to get *j.*

## 4. An example

Two additional points may be noted about the method of calibration being proposed. First, although the procedure outlined here is based on the cost function, it is straightforward to develop a parallel procedure that is based on the profit or production function. Second, the procedure need not be limited to $3\times3$ or $4\times4$ matrices. Rather, the principle is that there can only be as many unknown variables as there are (linearly independent) equations in the system. Thus, with an $n\times n$ symmetric matrix, which has $n(n+1)/2$ unique cells, $n(n-1)/2$ of which are off-diagonal cells whose values come in pairs and $n$ of which are cells on the principal diagonal, the researcher must seek to know $n(n-1)/2$ to be able to solve for the $n$ missing values.

For applied work, the following consistency conditions can be imposed, regarding the $G$ known elasticities:

(i) The netput vector may be rearranged such that the lower diagonal matrix is empty, except for the elements of the principal diagonal itself, which are all filled up.

(ii) In the upper diagonal matrix, each row should have at least one missing element (except for the singleton row), and each column should have at least one missing element (except for the singleton column).

Condition (i) ensures that there are no redundancies; it assumes that the initial estimates are more likely available for the own-price elasticities, rather than the cross-price elasticities. Condition (ii) ensures that there would be no inconsistencies in applying Euler's theorem.

As a specific example, the givens are estimates of quantities and peso values of outputs and inputs of the aquaculture industry (Table 1), which are taken from Garcia et al. [2009]. In addition we have a matrix of hypothetical profit function elasticities (Table 2):

**Table 1. Quantities and values of outputs and inputs of the aquaculture industry**

| *Outputs* | | *Quantity* | *Value* |
|---|---|---|---|
| Freshwater fish | 1 | 342,697 | 14,853,544 |
| Brackish water fish | 2 | 277,230 | 29,497,975 |
| Seaweed | 3 | 1,338,597 | 6,040,899 |
| *Inputs* | | *Quantity* | *Value* |
| Feed | 5 | 672,966 | 15,774,104 |
| Chemicals | 6 | 396,583 | 35,45,504 |
| Other intermediate inputs | 7 | 7,270,989 | 44,964,504 |
| Labor | 8 | 23,755 | 2,737,585,868 |
| Other primary inputs | 9 | 53,171,865 | 27,375,868 |

**Table 2. Matrix of hypothetical profit function elasticities**

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.6 | | –0.1 | 0 | –0.1 | 0 | –0.1 | –0.1 | |
| 2 | | 0.8 | | –0.1 | –0.1 | –0.1 | –0.1 | –0.1 | –0.1 |
| 3 | | | 0.8 | | 0 | –0.1 | –0.1 | –0.1 | –0.1 |
| 4 | | | | 0.8 | | –0.1 | –0.1 | –0.1 | –0.1 |
| 5 | | | | | –1.0 | | –0.1 | –0.1 | –0.1 |
| 6 | | | | | | –1.0 | | –0.1 | –0.1 |
| 7 | | | | | | | –1.0 | | –0.1 |
| 8 | | | | | | | | –1.0 | |
| 9 | | | | | | | | | –0.8 |

Note that in the table, $36 = \left[(9(9-1))/2\right]$ unique elasticities are given, as required.[11] The calibration can be implemented in the General Algebraic Modelling Systems (GAMS) software package using the following program code:

```
SETS
i /1,2,3,4,5,6,7,8,9/
iq(i) /1, 2, 3, 4/
ix(i) /5,6,7,8,9/

ALIAS(i,j); ALIAS(iq,jy); ALIAS (ix,jx);

*Data should be entered in the "natural" values of outputs.
TABLE data(i,*)
        qty         value
1      342697     14853544
2      277230     29497975
3     1338597      6040899
4     2203346     96000531
5      672966     15774104
6      396583      3545504
7     7270989     44964054
8       23755     27375868
9    53171865     54733419

TABLE elast(i,j)
     1      2      3      4      5      6      7      8      9
1    0.6           -0.1   0      -0.1   0      -0.01  -0.01
2           0.8           -0.01  -0.01  -0.01  -0.01  -0.01  -0.01
3                  0.8           0      -0.01  -0.01  -0.01  -0.01
```

[11] That the elasticities are all given in the upper triangular part of the matrix is not a limitation. For as long as there are no redundancies in the required information, an elasticity on the lower triangular part of the matrix can always be translated into its counterpart on the upper triangular part using the appropriate price and quantity information.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | | | | 0.8 | | -0.01 | -0.01 | -0.01 | -0.01 |
| 5 | | | | | -1 | | -0.01 | -0.01 | -0.01 |
| 6 | | | | | | -1 | | -0.01 | 0.01 |
| 7 | | | | | | | -1 | | 0.01 |
| 8 | | | | | | | | -1 | |
| 9 | | | | | | | | | -0.8 |

```
PARAMETER quant(i), y(i), p(i),m(i,j);
quant(i) = data(i,"qty");
p(i)     = data(i,"value")/quant(i);
*The tricky part: translating to "netput" values, to ensure straightforward symmetry of
substitution matrix
y(iq)   = quant(iq);
y(ix)   = -quant(ix);
m(i,j)  = elast(i,j)*y(i)/p(j);

*The following is customized for this problem
PARAMETERS
a11,    a22,    a33,    a44, a  55, a   66,     a77,    a88,    a99,

        a13     , a14,  a15,    a16,    a17,    a18,
                a24,    a25,    a26,    a27,    a28,    a29,
                        a35,    a36,    a37,    a38,    a39,
                                a46,    a47,    a48,    a49,
                                        a57,    a58,    a59,
                                                a68,    a69,
                                                        a79;

a11 = m("1","1"); a22 = m("2","2"); a33 = m("3","3");a44 = m("4","4"); a55 =
m("5","5");
        a66 = m("6","6"); a77 = m("7","7"); a88 = m("8","8");a99 = m("9","9");

            a13 = m("1","3"); a14 = m("1","4"); a15 = m("1","5"); a16 = m("1","6");
        a17 = m("1","7");a18 = m("1","8");

            a24 = m("2","4"); a25 = m("2","5"); a26 = m("2","6"); a27 = m("2","7");
        a28 = m("2","8"); a29 = m("2","9");

            a35 = m("3","5"); a36 = m("3","6"); a37 = m("3","7"); a38 = m("3","8");
        a39 = m("3","9");

            a46 = m("4","6"); a47 = m("4","7"); a48 = m("4","8"); a49 = m("4","9");

            a57 = m("5","7"); a58 = m("5","8"); a59 = m("5","9");

            a68 = m("6","8"); a69 = m("6","9");
```

```
        a79 = m("7","9");

    PARAMETERS p1, p2, p3, p4, p5, p6, p7, p8, p9;
    p1 = p("1"); p2 = p("2"); p3 = p("3"); p4 = p("4");
    p5 = p("5"); p6 = p("6"); p7 = p("7"); p8 = p("8"); p9 = p("9");

    VARIABLES
    x12, x19, x23, x34, x45, x56, x67, x78, x89, OBJ;

    EQUATIONS
    EqEuler1, EqEuler2, EqEuler3, EqEuler4, EqEuler5, EqEuler6, EqEuler7, EqEuler8,
EqEuler9, EqOBJ;

    EqEuler1.. a11*p1 + x12*p2 + a13*p3 + a14*p4 + a15*p5 + a16*p6 + a17*p7 + a18*p8
+ x19*p9 =E= 0 ;
    EqEuler2.. x12*p1 + a22*p2 + x23*p3 + a24*p4 + a25*p5 + a26*p6 + a27*p7 + a28*p8
+ a29*p9 =E= 0 ;
    EqEuler3.. a13*p1 + x23*p2 + a33*p3 + x34*p4 + a35*p5 + a36*p6 + a37*p7 + a38*p8
+ a39*p9 =E= 0 ;
    EqEuler4.. a14*p1 + a24*p2 + x34*p3 + a44*p4 + x45*p5 + a46*p6 + a47*p7 + a48*p8
+ a49*p9 =E= 0 ;
    EqEuler5.. a15*p1 + a25*p2 + a35*p3 + x45*p4 + a55*p5 + x56*p6 + a57*p7 + a58*p8
+ a59*p9 =E= 0 ;
    EqEuler6.. a16*p1 + a26*p2 + a36*p3 + a46*p4 + x56*p5 + a66*p6 + x67*p7 + a68*p8
+ a69*p9 =E= 0 ;
    EqEuler7.. a17*p1 + a27*p2 + a37*p3 + a47*p4 + a57*p5 + x67*p6 + a77*p7 + x78*p8
+ a79*p9 =E= 0 ;
    EqEuler8.. a18*p1 + a28*p2 + a38*p3 + a48*p4 + a58*p5 + a68*p6 + x78*p7 + a88*p8
+ x89*p9 =E= 0 ;
    EqEuler9.. x19*p1 + a29*p2 + a39*p3 + a49*p4 + a59*p5 + a69*p6 + a79*p7 + x89*p8
+ a99*p9 =E= 0 ;

    EqOBJ..   OBJ =E= x12 + x23 + x23 + x34 + x56 + x67 + x78 + x89 ;

    MODEL CALIB /ALL/
    OPTION NLP = minos5;

    SOLVE CALIB minimizing OBJ using NLP;

    m("1","2") = x12.L; m("1","9") = x19.L; m("2","3") = x23.L; m("3","4") = x34.L;
    m("4","5") = x45.L; m("5","6") = x56.L; m("6","7") = x56.L; m("7","8") = x78.L; m("8","9")
= x89.L;

    PARAMETER lodiag(i,j), kron(i,j), matrix(i,j), result(i,j);
    kron(i,i)   = 1;
    lodiag(i,j) = m(j,i);
    matrix(i,j) = m(i,j)*(1-kron(i,j)) + lodiag(i,j);
```

```
result(i,j) = matrix(i,j)*p(j)/y(i);

FILE calelas /c:\AMPLE_GAMS\results\calelas.txt /;
calelas.pc = 6; calelas.nd = 4;
PUT calelas;

PUT ""; LOOP(i, PUT i.tl); PUT /;
LOOP(i,
      PUT i.tl;
            LOOP(j,
                  PUT result(i,j)
                  );
      PUT /;
      );
```

The output is displayed in Table 3.

**Table 3. Complete matrix of elasticities**

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.60 | −0.86 | −0.10 | 0.00 | −0.10 | 0.00 | −0.01 | −0.01 | 0.48 |
| 2 | −0.43 | 0.80 | −0.31 | −0.01 | −0.01 | −0.01 | −0.01 | −0.01 | −0.01 |
| 3 | −0.25 | −1.50 | 0.80 | 0.98 | 0.00 | −0.01 | −0.01 | −0.01 | −0.01 |
| 4 | 0.00 | 0.00 | 0.06 | 0.80 | −0.82 | −0.01 | −0.01 | −0.01 | −0.01 |
| 5 | 0.09 | 0.02 | 0.00 | 4.98 | −1.00 | −4.07 | −0.01 | −0.01 | −0.01 |
| 6 | 0.00 | 0.08 | 0.02 | 0.27 | −18.09 | −1.00 | −4.77 | −0.01 | 0.01 |
| 7 | 0.00 | 0.01 | 0.00 | 0.02 | 0.00 | −0.38 | −1.00 | −0.52 | 0.01 |
| 8 | 0.01 | 0.01 | 0.00 | 0.04 | −0.01 | 0.00 | −0.85 | −1.00 | 1.80 |
| 9 | −0.13 | 0.01 | 0.00 | 0.02 | 0.00 | 0.00 | 0.01 | 0.90 | −0.80 |

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