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CONTINUITY IN TRANSFORMATION INVARIANT SOCIAL ORDERINGS: TWO IMPOSSIBILITIES

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We show that (1) a social ordering on $R \geq 0$ that satisfies Strong Paretoness, Invariance with respect to a Positive Proportional Transformation and Lower Semi-Continuity does not exist and (2) that a social ordering on $R > 0$ that satisfies Weak Paretoness, Invariance with respect to an Affine Transformation and Lower Semi-Continuity is trivial.

Transformation Invariance (TI) as a condition for social welfare functions was originally motivated by the result that the Von Neumann-Morgenstern utility functions are unique only up to a linear transformation (Von Neumann and Morgenstern, 1947; Owen, 1968). The condition was made popular in the group choice context by J. Nash (1950, 1953) whose famous solution to the two-person bargaining game was invariant with respect to any affine transformation of the individual utility functions. The idea then is to insulate the social ordering from transformations of the utility functions that remain true to the underlying individual preference orderings. Social orderings and welfare functions that satisfy TI will be the focus of this paper. By transformation here, we will mean linear transformation.

Explorations in this universe are done normally in conjunction with a natural fixture in the area of group choice, namely, the Paretian axiom. Osborne (1976) proved the following interesting result. It is impossible to find a real-valued function $F: R^n_{\geq 0} \rightarrow R^1$ which satisfies the Weak Pareto axiom and the Axiom of Invariance with respect to an Affine Transformation. In order to prove the claim, he proved a lemma, now known as the "Osborne lemma":

Let G be a real-valued function on R^n_+.

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Suppose the following properties:

(A) If $x_i \geq y_i$, then $G(x_1 \ldots x_n) \geq G(x_1 \ldots x_{i-1}, y_i, x_{i+1}, \ldots x_n)$ for all $i \varepsilon N$

(B) $G(x) \geq G(y)$ iff $G(\lambda x) \geq G(\lambda y)$, $\forall\ x, y \varepsilon R^n_+$ and positive real vector λ.

Then there are non-negative real constants $c_1 \ldots, c_n$ and monotone increasing function v over R^1 such that

$$G(x) = v(\underset{i \varepsilon N}{\pi} x_i c_j)$$

Note that (A) is the Weak Pareto condition while (B) is the TI axiom. The proof of the lemma is rather involved. Kaneko and Nakamura (1979) using the Osborne lemma showed that a certain set of desirable axioms mimicking the Nash axioms allow a unique numerical representation which they called the "Nash Social Welfare Function".

In this paper, we set ourselves the task of showing that social orderings satisfying combinations of TI and the Paretian axiom may be inconsistent but we do this without resorting to the Osborne lemma. Instead, we tack on another property of interest and controversy in group choice: continuity. In other words, our result would not be that a certain set of axioms is not representable by a function which is the Osborne result but that the set of axioms are not inconsistent and thus *a priori* not representable. Osborne does not show the latter. But why continuity?

Lexicographic ordering has two non-features: it is not real-valued and it is not continuous. So continuity suggests itself naturally. But more than that, continuity in a function allows a variety of operations that render analysis much easier, the very same consideration that prompted G. Debreu (1959) to remark of a utility function: "In fact, this function would be of little interest if it were not continuous ..." He was consistent when he relegated to the footnotes his remark of a few lines on the most famous non-continuous ordering, the lexicographic one. We start with a representable and *a priori* a consistent system WILC.

Definition 1: An ordering R over $\Omega = R^{n}_{\geq 0}$ is called WILC if it satisfies the following:

(A1) Weak Pareto Condition: $\forall\, A, B\, \varepsilon\, \Omega, A \geq B \Rightarrow ARB$

(A2) Invariance with respect to a Positive Proportional Transformation:

$\forall\, A, B\, \varepsilon\, \Omega, ARB \Leftrightarrow (\lambda A)\, R\, (\lambda B), \lambda > 0$
if $APB \Leftrightarrow ARB$ and $B\overline{R}A$ (– for 'not')
then $APB \Leftrightarrow (\lambda A)\, P\, (\lambda B), \lambda > 0$

(A3) Lower Semi-Continuity: $\forall\, A\, \varepsilon\, \Omega$, the set $\{B\text{: } B\, \varepsilon\, \Omega, APB\}$ is open.

Remark 1: R may be rendered as "no worse than" and P as "better than".

Remark 2: Condition (A2) assumes that one can in fact perform the transformation stated. When Ω is a space of vectors of real-valued functions, this operation is feasible. It is not clear whether the same operation is permissible for representations of preferences over, say, an extension of the real line ["generalized utility functions," (Richter, 1959)]. The lexicographic ordering, for example, allows a generalized utility representation.

We now show the following:

Proposition 1: WILC is representable.

Proof: Consider the function $f(x) = \prod_{i=1}^{n} x_i$, $x_i \geq 0$. It obviously satisfies (A1). Let $A, B\, \varepsilon\, \Omega$. Let $f(A) \geq f(B)$ · $f(cA) = \prod_{i=1}^{n} c_i a_i = \prod_{i=1}^{n} c_i \prod_{i=1}^{n} a_i \geq \prod_{i=1}^{n} c_i\, \Pi b = f(cB)$. Thus,

(A2). Let $f(A) > f(B)$. Let $\varepsilon^* = \{\varepsilon\} > 0$, be such that $f(A) > f(B + \varepsilon^*)$. Let the open sphere be $S(B, \varepsilon^*)$. Then $f(A) > f(C)\ \forall C \,\varepsilon\, S(B, \varepsilon^*)$ which satisfies (A3).
Q.E.D.

We now prove the following lemma.

Lemma: $\exists A, B \,\varepsilon\, \Omega$ such that $A \geq B$, all i and $A > B$ some i and ARB and BRA if WILC.

Proof: Suppose APB whenever $A \geq B$, $\forall i$ and $A > B$ some i. Consider $A = (a_1, \text{o}, \ldots, \text{o})$ and $B = (b_1, \text{o}, \ldots, \text{o})$, $a_1, b_1 \geq \text{o}$. Thus, $A, B \,\varepsilon\, \Omega$. Let $a_1 > b_1$, so APB by assumption. By (A3), there exists an ε^*-neighborhood $N(B, \varepsilon^*)$, $\varepsilon^* > \text{o}$, around B, such that $A\ P\ B^\circ \,\varepsilon\, N(B, \varepsilon^*)$. Consider $B' \,\varepsilon\, N(B, \varepsilon^*)$ for small enough $\lambda > \text{o}$. Consider $\lambda = (\lambda_1, \lambda_i = 1)$, $\forall i \neq 1$. Find λA and $\lambda B'$. Consider the origin $0 = (\text{o}, \text{o}, \ldots, \text{o})$. Let $N(0, \varepsilon^*)$ be the ε^*-neighborhood around 0. For small enough λ_1, $\lambda_1 A \,\varepsilon\, N(0, \varepsilon^*)$. Consider the set $B'' = (\text{o}, \lambda, \text{o}, \ldots, \text{o})$. By our assumption $B''\ P\ \text{O}$. Note as well that for any $\lambda_1 > \text{o}$, $(\lambda B') \geq B''$, $i = 2$. Thus $(\lambda B')\ P\ B''$ and by transitivity of WILC, $(\lambda B')\ P\ \text{O}$. Thus $(\lambda B')\ P\ (\lambda A')$ contradicting (A2). Q.E.D.

Remark 3: One should note that the proof resorts to the null vector $0 = (\text{o}, \text{o}, \ldots, \text{o})$. This is a particular feature of WILC which resists generalization. The lemma on the basis of the null vector says that there are elements in Ω for which Pareto domination does not lead to strict preference.

Depending on one's philosophical persuasion, WILC can present a moral problem. If Ω is the space of utilities, what Proposition 2 says is that, if faced with a choice between sacrificing one member ($U_k = 0$) and sacrificing all members ($U_i = 0$, V_i), society tosses a fair coin to determine its course of action. If society's survival is paramount, then tossing a fair coin is not sensible. Apparently, societies do choose to uphold the survival of the collective to the detriment of the one. The story of Theseus and the Minotaur shows this. Human sacrifices to the gods were gory witnesses to this societal preference. Capital punishment in modern times attests to this as well. We thus need to construct a choice rule that reflects this tendency. In other words, we need to strengthen the Weak Pareto condition (A1). We have the following:

Definition 2: An ordering R over $\Omega = R^n_{\geq 0}$ is SILC if its satisfies (A2), (A3) and (A1′) Strong Pareto: $\forall$ A, B ε Ω, $A \geq B$, $\forall i$ and $A > B$, some $i \rightarrow APB$.

Proposition 2: (Impossibility) SILC is inconsistent.

Proof: The set satisfying SILC is a subset of the set satisfying WILC since $A1'$ is a strengthening of (A1). By Lemma above, $\exists$ A, B ε Ω such that $A \geq B$ some i and $A \sim B$. Thus the set satisfying SILC is empty. Q.E.D.

Remark 4: Domain is an important ingredient in the inconsistency of SILC. Recall that the proof requires the null vector 0 = (o, o, . . ., o). If the lowest possible individual position is represented by utility zero, one cannot construct a choice rule that decides in favor of the majority. We have the classic utilitarian dilemma: do you surrender a member who you know is innocent to keep the horde from descending on the neighborhood and putting everyone to the sword? The problem is however solved by just defining $\Omega = R^n_{\geq 0}$ so that $\prod_{i=1}^{n} u_i$ is a candidate choice function. This is consonant with a result that can be deduced immediately from the Osborne lemma, namely, there does not exist a real-valued function satisfying Strong Paretoness and TI over $R^n_{\geq 0}$.

Because of the null vector requirement, Proposition 3 is a weak result. It only serves to show that the domain of definition may be crucial for certain choice rules. We now turn to a strengthened TI axiom. We substitute affine transformation in lieu of positive proportional transformation.

Definition 3: An ordering R over $\Omega = R^n_{\geq 0}$ is WIAC if it satisfies the following:

(A1) Weak Paretoness

(A2) Invariance with respect to an affine transformation:

$A, B \varepsilon \Omega$, ARB $\Leftrightarrow (\lambda A + \delta)\ R\ (\lambda B + \delta)$, $\lambda > 0$, $\delta \varepsilon R^n$. This is true also for P.

(A3) Lower semi-continuity: $\forall A \varepsilon \Omega$, the set $\{B: B \varepsilon \Omega, APB\}$ is open.

Remark 5: A rule that satisfies WIAC satisfies WILC since (A2′) is a strengthening of (A2).

Proposition 3: (Impossibility of a nontrivial choice rule). A choice rule satisfying WIAC is trivial, i.e., ($\forall$ A, B ε Ω, A ~ B).

Proof:

(a) Let APB whenever $A \geq B$, $\forall i$, $A > B$ some i. Let $A = (a_1, a_2, a_3, \ldots a_k, d, \ldots, d)$ and $B = (b_1, b_2, b_3, \ldots b_k, d, \ldots, d)$, $d \geq 0$, a_i, $b_i > d$, so that $A, B \varepsilon \Omega$. Note that we have renumbered the coordinates so that all k elements in A unequal to corresponding elements in B are in front. Let $a_i > b_i\ \forall i = 1 \ldots k$. Thus APB. By (A3a), $\exists$ an ε^*-neighborhood $N(B, \varepsilon^*)$ of B, $\varepsilon^* >$ o, such that $APB' \varepsilon N(B, \varepsilon^*)$.

Let $B^\circ = (b_1, b_2, \ldots, b_k, d + \gamma, d, \ldots, d)$. For small enough γ, $B^\circ \varepsilon\ N(B, \varepsilon^*)$. Consider the n vector, $\lambda = (\lambda_1, \ldots, \lambda_k, \lambda_i = 1)$, $i = k + 1 \ldots n$; and $\delta = (\delta_1, \ldots, \delta_k, \delta_i =$ o$)$, $i = k + 1 \ldots n$. Find the affine transforms of A & B°, $(\lambda A + \delta)$ and $(\lambda B^\circ + \delta)$ while letting $\lambda_j \to$ o, $j = 1 \ldots k$ and $\delta_i = d$, $j = 1 \ldots k$. Consider the element $D = (d, d_i, d)$ of Ω. For small enough λ_j, $(\lambda A + \delta)$ is as close to D as is desired. Let $N(D, \varepsilon^*)$ be the ε neighborhood around D. Clearly, $(\lambda A + \delta)\ \varepsilon\ N(D, \varepsilon)$ for λ_j small enough. Consider now the element of Ω, $B'' = (b_i'' = d, b''_{k+1} = d + \gamma, b''_{k+j} = d)$, $\forall_i = 1 \ldots k$, $j = 2$, in other words all elements in B'' equals d except the $k + 1^{\text{th}}$ element equalling $d + \gamma$, $\gamma > 0$. Clearly by (A1), $B''RD$. Now $(\lambda B^\circ + \delta) \geq B''$ all i and $(\lambda B^\circ + \delta) > B''$ for $i = k + 1$. Thus by assumption $(\lambda B^\circ + \delta)\ P\ B''\ R\ D$ or $(\lambda B^\circ + \delta)\ P\ D$. Thus $(\lambda B^\circ + \delta)\ P\ (\lambda A + \delta)$, contradicting (A2).

(b) Let $A = (a_1 \ldots a_3, a_{e+1} \ldots a_k, d \ldots d)$ and $B = (b_1 \ldots be, b_{e+1} \ldots b_k, d \ldots d)$; $a_i, b_i \geq d, d \geq$ o, so that $A, B \varepsilon \Omega$. Let $a_i > b_i$, $i = 1 \ldots e$; $a_i < b_i$, $i = e + 1 \ldots k$. Thus $A > B$ some i but $A < B$ some i, let APB. Consider the vector $\lambda = (\lambda_1 \ldots \lambda_e, \lambda_j = 1)$ $\forall j > e$, and let $\delta = (\delta_1 \ldots \delta_e, \delta_j = \text{o})$ $\forall j > e$. Find $(\lambda A + \delta)$ and $(\lambda B + \delta)$. Let $\lambda_i \rightarrow$ o, $\forall i = 1 \ldots e$, and let $\delta_i = d$, $i = 1 \ldots e$. Consider the set $D = (d, d, \ldots, d) \varepsilon \Omega$. For $\lambda_i >$ o, $(\lambda A + \delta)$ R D by (A1). Consider the set $B° = (b_i° = d, b_{e+1} \ldots b_k, d, \ldots d)$ $\forall i = 1 \ldots e$. $B°$ has all elements equal to d except $b_{e+1} \ldots b_k$. By (A1), $B°$ R D_i and, by transitivity of ordering WIAC, $(\lambda A + \delta)$ R $B°$. But for small enough λ_i, $B° \varepsilon N(\lambda B + \delta, \varepsilon^*)$ and by (A2), $(\lambda A + \delta)$ P $(\lambda B + \delta)$. Thus $(\lambda A + \delta)$ P $B°$, a contradiction. Q.E.D.

The difference between Propositions 2 and 3 must be emphasized. Proposition 3 is valid even if the domain of definition is restricted to $\Omega = R^n_{>0}$. It is therefore a far stronger result.

Continuity, though an interesting mathematical animal, is no more than a very useful technical tool in economics. It has no inherent intuitive economic meaning. The Paretian and the Transformation Invariance axioms are on the other hand invested with economic significance. Continuity is dispensable and that is what we are going to exploit. The question we want answered is: Do we attain consistency if we drop continuity? The answer is yes.

Proposition 4: The lexicographic choice rule, L, satisfies both the set of (A1′) and (A2) and the set of (A1) and (A2′).

Proof: If $A \geq B$, all i, $A > B$, some i, ALB and $B\hat{L}A$. If ALB, (cA) L (cB) and $(cA + d)$ L $(cB + d)$. If $A \geq B$, ALB.

Since a particular choice rule satisfies the axioms, consistency is attained. Q.E.D.

Summary

We have shown that Strong Paretoness, Proportional Transformation Invariance and Continuity, together, form an inconsistent system over $\Omega = R^n_{\geq 0}$ and no choice rule can represent the system. Likewise, we showed that Weak Paretoness, Affine Transformation Invariance and Lower Semi-Continuity, together, allow only a trivial choice rule. On the other hand, dropping the continuity assumption makes lexicographic ordering a candidate as a choice rule. Restricting the domain of definition to a strictly positive space allows a representation that satisfies Strong Paretoness, proportional Transformation Invariance and Continuity. The main results here, generated without the Osborne Lemma, differ from those that can be generated from the Lemma in that those involve nonrepresentability by a function while our results zero in on the consistency of the systems themselves.

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