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INSTABILITY OF EQUILIBRIUM GROWTH (ENDOGENOUS OR NOT)

José Encarnación, Jr.*

If the solution to a dynamic optimization problem is interpreted as an equilibrium growth path, then the Harrod instability proposition applies not only to the neoclassical growth model but also to the more recent endogenous growth version.

1. Introduction

The Harrod "knife-edge" proposition says that equilibrium growth—Harrod called it warranted growth—is unstable. At one time it was thought that Solow (1956) and the neoclassical growth model reversed the Harrod proposition. Actually, that model avoids the stability question—cf. Hahn (1987)—because it lacks equilibrium growth in the sense of Harrod. However, an equilibrium growth path could be defined in that model by the solution to a dynamic optimization problem. But Kurz (1968) has shown that that equilibrium path is also unstable. Surveys of the recent literature on endogenous growth theory—see Helpman (1992), Hammond and Rodriguez-Clare (1993), Barro and Sala-i-Martin (1995)—are silent on the stability question. This is surprising since the model in Romer (1986), which initiated this literature, is explicitly based on dynamic optimization, and one would expect that endogenous growth equilibrium would be similarly unstable.

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Section 2 is a quick review of the Harrod thesis. Section 3 argues that there is no equilibrium growth in the neoclassical model. In Section 4 the solution to an optimization problem is interpreted as an equilibrium growth path, which is unstable. Section 5 shows instability in the basic model of endogenous growth, and Section 6 is a concluding remark.

2. Harrod

There are various formulations of the Harrod thesis; see e.g. Encarnación (1965) and the references there cited. A simple one might be as follows. Assume that aggregate real output Y requires the amount of capital $K = vY$ ($v \doteq$ const), and desired investment $\dot{K}^d = v\dot{Y}$ so that firms' decisions regarding $\dot{Y}$ imply a corresponding $\dot{K}^d$ and vice versa. Warranted or equilibrium growth (EG) is defined by $\dot{K}^d = \dot{K}$ where $\dot{K} = sY$ (s = const), i.e. $v\dot{Y} = sY$ or $\dot{Y}/Y = s/v$. Since $\dot{K}/K = s/v$, Y and K will grow at the same percentage rate along the EG path. Consumption being given by $(1 - s)Y$, output demanded $Y^d = \dot{K}^d + (1 - s)Y$ while $Y = \dot{K} + (1 - s)Y$.

Make the reasonable assumption that if $Y^d = Y$ at any given time, then the growth rate $\dot{Y}/Y$ will be maintained, but if $Y^d < Y$ (or $Y^d > Y$), the growth rate will be reduced (or raised). Noting that $\dot{K}^d < \dot{K}$, $Y^d < Y$, and $\dot{Y}/Y < s/v$ are equivalent statements, EG instability is immediate.

3. The Neoclassical Model

Let $Y = F(K, L)$ where L is the labor force, assuming that full employment is maintained. F is homogenous of degree one, so putting $k = K/L$, one can write $Y/L = f(k)$ in per worker terms. The neoclassical model assumes $f'(0) = \infty$, $f' > 0$ and $f'' < 0$. To simplify the discussion, we put $\dot{L}/L = 0$ but there is a capital depreciation rate $\beta > 0$ so that

(1) $\dot{k} = f(k) - c - \beta k$

where $c = (1 - s)f$ is consumption per worker. In contrast to Harrod, v is variable, full employment holds, and there is no investment function to determine desired investment $\dot{k}^d$ per worker.

Plotting $sf(k)$ and βk in a diagram with k on the horizontal axis, suppose there is a stationary point $(k^0,\ sf(k^0))$ where $\dot{k} = 0$. Solow (1956) showed that k will converge to k^0 but Harrod's question of EG stability was not addressed at all simply because there is no $I \doteq S$ (or investment = saving) equilibrium condition in the neoclassical model. (i) If $\dot{k}^d = sf - \beta k$ is considered to be the investment function, then $\dot{k}^d = \dot{k}$ identically in which case the possibility of disequilibrium growth is ruled out. But equilibrium growth and stability are empty concepts without such a possibility, for the question of stability (which asks whether the economy will return to equilibrium) can be raised only if there is a disequilibrium. (ii) It might be thought that equation (1) can be interpreted as an $I = S$ equilibrium condition. However, that would make the *ex post* $\dot{k} + \beta k$ the investment function, which is *ex ante*. Again, that would rule out the possibility of disequilibrium. (iii) Finally, it seems to be the usual understanding that the full employment assumption in the neoclassical model takes the place of an $I = S$ equilibrium condition to determine the model. But this only means that the model would be overdetermined by the addition of an $I = S$ condition since firms do not make investment decisions for the purpose of maintaining full employment. To summarize, there is growth in the model, but not equilibrium growth.

4. Optimization

In order to define an EG path in the neoclassical model, one could take the view that the representative (i.e. average) consumer, who owns the representative firm, has the problem of maximizing

$$(2)\quad \int_0^\infty u(c(t))e^{-\delta t}dt$$

subject to (1) given $k(0)$; δ is the discount rate and the utility function u satisfies $u'(0) = \infty$, $u' > 0$ and $u'' < 0$. The solution to the problem gives the optimal growth path which, being optimal, can be interpreted as an EG path. Kurz (1968) has shown that if the production function is Cobb-Douglas and u is a constant elasticity function, then the saving fraction s is constant along the optimal path. In what follows we will confine the discussion to the likely case $k(0) < k^0$ assuming a stationary state k^0.

Formulating the problem in terms of optimal control, the current value Hamiltonian $H = u(c) + \lambda(f(k) - c - \beta k)$ must be maximized by choice of c, so $\partial H/\partial c = u'(c) - \lambda = 0$ or

(3) $u'(c) = \lambda.$

λ has a natural interpretation as the inputed value (in utility terms) of a unit of $\dot{k}$. A necessary condition for the optimal path is that $\dot{\lambda} = \delta\lambda - \lambda\partial H/\partial k$ or

(4) $\dot{\lambda} = \lambda(\delta + \beta - f'(k))$

which can be written

(4') $\dot{\lambda}/\lambda + \partial\dot{k}/\partial k = \delta.$

To see the rationale for (4'), suppose an additional unit of k which raises $\dot{k}$ by the amount $\partial\dot{k}/\partial k$. Since λ is the (utility) value of a unit of k, $\dot{\lambda}/\lambda$ is its percentage increase (decrease if $\dot{\lambda} < 0$) per unit time. Writing R for the left-hand side of (4'), R is therefore the percentage increase in the value of k one unit time later. On the other hand, c consumed now can be thought of as having a value 100 δ percent higher from the viewpoint of a unit time later. Clearly, if $R > \delta$, one should have more $\dot{k}$ and less c, and if $R < \delta$, there should be more c and less $\dot{k}$. In short, $R = \delta$ is necessary for intertemporal efficiency whether one is on the optimal path or off it.

In the phase plane with k on the horizontal axis and λ on the vertical, every point lies on exactly one trajectory or path and each path satisfies (1) and (4). Suppose there is a stationary point

(k^0, λ^0) where $\dot{k} = 0$ and $\dot{\lambda} = 0$. ($\dot{\lambda} = 0$ where $f'(k) = \delta + \beta$, and $\dot{k} = 0$ where $c = f(k) - \beta k$. Thus, from (3), $\lambda = u'(f - \beta k)$ defines the $\dot{k} = 0$ locus.) It is a fact that this stationary point is a saddle point; also, the optimal path leads to the saddle point asymptotically and it is the only path that does so. This means that any neighboring path will diverge from the optimal or EG path, which is therefore unstable. Notice that increasing returns are not involved.

It is true that in an optimal control problem, c can be chosen to put a centrally directed economy on the optimal path given the current value of k, but this is not the case in the present context of decentralized competitive agents where equilibrium growth is merely interpreted as the solution to an optimization problem. If the economy happens to be on the EG path, it will stay there. But if some adventitious event puts the economy on a neighboring path, individual agents will stay on that new path and will not return to the EG path.

This brings us to the observation that the EG path can be called a saddlepoint path (Kurz, 1968) or, to use Harrod's description, a "knife-edge". Consider any point (k^*, λ^*) on the EG path. A Taylor linear approximation to (1) and (4) in the neighborhood of (k^*, λ^*) is given by

(1a) $\dot{k} = (k - k^*)\partial(f - c - \beta k)/\partial k + (\lambda - \lambda^*)\partial(f - c - \beta k)/\partial\lambda$

(4a) $\dot{\lambda} = (k - k^*)\partial\lambda(\delta + \beta - f')/\partial k + (\lambda - \lambda^*)\partial\lambda(\delta + \beta - f')/\partial\lambda$

where the derivatives are evaluated at (k^*, λ^*). That is,

(1b) $\dot{k} = Ak + B\lambda + const$

(4b) $\dot{\lambda} = Ck - D\lambda + const$

where $A = f' - \beta > 0$, $B = -1/u''(c^*) > 0$, $C = -\lambda^* f'' > 0$, and $D = f' - \delta - \beta > 0$. (In B, $c^* = f(k^*) - \beta k^* - k^*$.)

The characteristic equation of the system (1b) and (4b) is

$$\begin{vmatrix} A-\zeta & B \\ C & -D-\zeta \end{vmatrix} = 0$$

so the characteristic roots are $\zeta = (A - D \pm ((D - A)^2 + AD + BC)^{1/2})/2$, i.e., the roots are real and opposite in sign, which property characterizes the usual saddlepoint. This makes it appropriate to describe the EG path itself as a saddlepoint path, showing more directly its instability.

5. The Basic Model of Endogenous Growth

In "the basic model" (Romer, 1986, p. 1034) of endogenous growth with increasing returns to knowledge (or, alternatively, knowledge and physical capital in fixed proportions), it is assumed that there is a large number N of competitive firms and the representative firm has a production function $F(k, K, x)$ where k is its firm-specific stock of knowledge, K is aggregate knowledge defined as $K = Nk$, and x denotes other inputs specific to the firm. (We are following Romer's notation so some symbols now have meanings different from those in the preceding sections, but that should not cause any confusion.) F is concave and homogeneous of degree one in k and x given K, and convex in K. Suppressing the fixed x, write $f(k, K) = F(k, K, x)$.

The firm's k, which does not depreciate, can be increased by the representative consumer (who owns the representative firm) by foregoing consumption so that the firm can invest in research:

(5) $\dot{k}/k = g((f - c)/k)$.

It is assumed that $g' > 0$, $g'' < 0$, g is bounded from below by $g(0) = 0$ and from above by $\alpha < \delta$. The objective is to maximize (2) subject to (5) given $k(0)$ and the path $K(t)$, $t \geq 0$. A normalization puts $g'(0) = 1$ to fix the unit of k since c and k are measured in different units.

The current-value Hamiltonian $H = u(c) + \lambda kg$, so $\partial H/\partial c = u' - \lambda g' = 0$ or

(6) $u' = \lambda g'$.

A necessary condition for the optimal path is that $\dot{\lambda} = \delta\lambda - \partial H/\partial k$ or

(7) $\dot{\lambda} = \delta\lambda - \partial\dot{k}/\partial k$

$= \lambda(\delta - g - Jg')$

where $J = f_k - (f - c)/k$ and f_k denotes the partial of $f(k, K)$ with respect to its first argument. Letting $k^*(t)$, $\lambda^*(t)$, $t \geq 0$, denote a solution to the optimization problem, an equilibrium path requires the given $K(t)$ to be such that $K(t) = Nk^*(t)$. Romer (1986) gives sufficient conditions for such a path where k and c grow without bound. There is no stationary point, but as in Section 4, we can pick any point (k^*, λ^*) on the EG path (provided k^* is not too small) for the purpose of showing instability.

First we note that $J > 0$ if k is not too small, for the following reason. The investment $f - c$ increases k by the amount $\dot{k}$ which increases f by the amount $f_k\dot{k} = f_k gk$. In effect there is an investment-output relationship such that $(f - c)/k$ gives $f_k g$. Since f_k is increasing with k and K along the EG path but g, which has an upper bound $\alpha < \delta$, can increase only fractionally, f_k must increase more than $(f - c)/k$. Thus $J > 0$ for k large enough.

A Taylor linear approximation to (5) and (7) in the neighborhood of (k^*, λ^*) is given by

(5a) $\dot{k} = (k - k^*)\partial kg/\partial k + (\lambda - \lambda^*)\partial kg/\partial\lambda$

(7a) $\dot{\lambda} = (k - k^*)\partial\lambda(\delta - g - Jg')/\partial k + (\lambda - \lambda^*)\partial\lambda(\delta - g - Jg')/\partial\lambda$

where the derivatives are evaluated at (k^*, λ^*). Straightforward calculations (see the Appendix) give

(5b) $\dot{k} = Ak + B\lambda + const$

(7b) $\dot{\lambda} = Ck - D\lambda + const$

where $A = g + Jg' > 0$
$B = -g'g'/(u'' + \lambda^* g''/k^*) > 0$
$C = -\lambda^*(J^2 g''/k^* + g' f_{kk}) > 0$
$D = -\lambda^* Jg'g''/(k^* u'' + \lambda^* g'') + (g + Jg') > 0.$

($D > 0$ since $\lambda^* Jg'g''/(k^* u'' + \lambda^* g'') = Jg'/(k^* u''/\lambda^* g'' + 1) < Jg'$.)

Seeing that the signs of the coefficients A, B, C and D in (5b) and (7b) are the same as those in (1b) and (4b), the last paragraph of Section 4 can be repeated here. Thus the EG path in the basic model of endogenous growth is unstable.

A simple extension of the model that includes the accumulation of physical capital separately can also be shown to be unstable, and it would be reasonable to conjecture that more complicated models would be similarly unstable in the absence of built-in stabilizers.

6. Concluding Remark

We conclude that if equilibrium growth is formulated in the usual way, the Harrod instability proposition has more generality than has been thought. We have seen that equilibrium growth paths, whether endogenous or not, are unstable. The implications for positive theory are clear, since unstable equilibria are rarely to be seen. Finally, increasing returns are not necessary for instability, and of course they are not sufficient given the possibility of built-in stabilizers.

Appendix

From (6) one has

(A1) $u''(c)\partial c/\partial\lambda = g' - (\lambda g''/k)\partial c/\partial\lambda$

so that, writing $Q = ku'' + \lambda g''$.

(A2) $\partial c/\partial\lambda = kg'/Q$

which is used to get

(A3) $\partial g/\partial\lambda = -g'g'/Q$

(A4) $\partial g'/\partial\lambda = -g'g''/Q$

(A5) $\partial J/\partial\lambda = g'/Q$.

Also, one has

(A6) $\partial g/\partial k = Jg'/k$

(A7) $\partial g'/\partial \mathrm{k} = Jg''/k$

(A8) $\partial J/\partial k = f_{kk} - J/k$.

Using (A3)-(A8) in (5a) and (7a), evaluating all derivatives at (k^*, λ^*), then gives (5b) and (7b).

References

Barro, R.J. and X. Sala-i-Martin (1995), *Economic Growth,* New York: McGraw-Hill.

Encarnación, J. (1965), "On Instability in the Sense of Harrod," *Economica,* 32: 330-337.

Hahn, F.H. (1987), "Hahn Problem," in *The New Palgrave Dictionary of Economics,* ed. J. Eatwell, M. Milgate and P. Newman, London: Macmillan.

Hammond, P.J. and A. Rodriguez-Clare (1993), "On Endogenizing Competitive Growth," *Scandinavian Journal of Economics,* 95: 391-425.

Harrod, R.F. (1939), "An Essay in Dynamic Theory," *Economic Journal,* 49: 14-33.

Helpman, E. (1992), "Endogenous Macroeconomic Growth Theory," *European Economic Review,* 36: 237-267.

Kurz, M. (1968), "The General Instability of a Class of Competitive Growth Processes," *Review of Economic Studies,* 35: 155-174.

Romer, P.M. (1986), "Increasing Returns and Long-run Growth," *Journal of Political Economy,* 94: 1002-1037.

Solow, R.M. (1956), "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics,* 70: 65-94.