

DECREASING ABSOLUTE RISK AVERSION: A CURIOSUM

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Of the two measures of risk aversion due to Arrow (1970) and Pratt (1964), the "index of absolute risk aversion" falls out more naturally from the concavity of the utility function as the odds you tack on the probability of winning before a consumer of convex preferences will accept an actuarially fair gamble. For the sake of generating testable hypotheses, the behavior of this index has been subject to further specification.

A specification due to Arrow which itself has found increasing currency involves the index's behavior with respect to the decision maker's original wealth position. Arrow's famous assumption is that the measure of absolute risk aversion decreases with increasing wealth. He proceeded to show that, with this specification, the greater the wealth the greater is the proportion of wealth in risky assets. Arrow, of course, believed this to be a reasonable, if not too profound, an outcome but which if true would lend evidence to the assumption.

First, let us just reflect on what the assumption of decreasing absolute risk aversion, DARA, entails. The Arrow-Pratt measure of absolute risk aversion is defined as

$$R_A = - \frac{U_{yy}(Y)}{U_y(Y)}$$

where U_{yy} and U_y are the first and second derivatives, respectively, of the utility function with respect to income, Y . A decreasing absolute risk aversion means that

$$\frac{dR_A}{dY} = \frac{U_y U_{yyy} - U_{yy} U_{yyy}}{U_y^2} < 0$$

For this to be true, it is necessary that $U_{yyy} > 0$ (as it is understood that $U_y > 0$ and $U_{yy} < 0$). Thus the rate at which marginal utility diminishes should fall for the assumption to hold. Note that the latter is not a sufficient condition for DARA. It is however, around this condition that we will construct an apparent curiosity.

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Risk in Production

Consider a firm producing good x in a purely competitive setting both in the product and the input markets. The production process is subject to some random error, i.e.,

$$(1) \quad \hat{x} = \hat{f}(K, L)$$

where K and L are capital and labor inputs, respectively. For simplicity, we assume \hat{f} to be normally distributed. The firm is a maximizer of some sort — it maximizes the expected utility of profit, $\hat{\pi}$, the latter being defined as

$$(2) \quad \hat{\pi} = p\hat{f}(K, L) - wL - rK$$

where P , w , r are price of the product X , the prices of labor and capital, respectively. The concave utility of profit is then $V(\hat{\pi})$ and we are interested in $EV(\hat{\pi})$ which is

$$(3) \quad EV(\hat{\pi}) = EV(p\hat{f}(K, L) - wL - rK)$$

The Taylor series expansion of $V(\hat{\pi})$ around the mean, $\bar{\pi}$ of $\hat{\pi}$ is

$$(4) \quad V(\hat{\pi}) = V(\bar{\pi}) + V'(\bar{\pi})(\pi - \bar{\pi}) + \frac{V''(\bar{\pi})(\pi - \bar{\pi})^2}{2} + \sum_{j=3}^n \frac{V^j(\bar{\pi})(\pi - \bar{\pi})^j}{j!}$$

where $V' = \partial V / \partial \pi$ and $V'' = \partial^2 V / \partial \pi^2$.

The expected value of eq. (4) is

$$(5) \quad EV(\hat{\pi}) = V(\bar{\pi}) + \frac{V''(\bar{\pi})\sigma_{\pi}^2}{2}$$

as moments above the second disappear under expectation due to the normality assumption. Note further that

$$(6) \quad \bar{\pi} = p\bar{f}(K, L) - wL - rK$$

$$\sigma_{\pi}^2 = P^2 [E(f)^2 - (Ef)^2]$$

Substituting these into (5), the objective function of the firm becomes

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$$(7) \quad EV(\hat{\pi}) = V(\bar{p}f)(K, L) - \omega L - rK + \frac{V''(\bar{\pi})}{2} p^2 [E(f)^2 - (Ef)^2]$$

The firm now employs K and L so as to maximize $EV(\hat{\pi})$. Taking the partial with respect to L and setting to zero we get:

$$\begin{aligned} \frac{\partial EV(\hat{\pi})}{\partial L} &= V'[p\bar{f}_L - \omega] + \frac{\sigma_\pi^2}{2} V''[p\bar{f}_L - \omega] + \\ &\quad \frac{V''}{2} \frac{\partial}{\partial L} \{P^2 [E(f^2) - (Ef)^2]\} = 0 \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial}{\partial L} P^2 [E(f^2) - (Ef)^2] &= \{P^2 E(2ff_L) - 2(EfEf_L)\} \\ &= 2P^2 [E(ff_L) - (Ef)(Ef_L)] \\ &= 2P^2 \text{cov}(ff_L) \end{aligned}$$

from the fact that the derivative of the expectation is equal to the expectation of the derivative under fairly general conditions (Cramer, 1963). We thus get:

$$\begin{aligned} (P\bar{f}_L - \omega) \left[\frac{V' + V'''\sigma_\pi^2}{2} \right] &= -\frac{V''}{V''} P^2 \text{cov}(ff_L) \\ (i) \quad (P\bar{f}_L - \omega) &= -\frac{P^2 \text{cov}(ff_L)}{\frac{(V' + V'''\sigma_\pi^2)}{2}} \end{aligned}$$

In the case of K , we get:

$$\begin{aligned} \frac{\partial EV(\hat{\pi})}{\partial K} &= (P\bar{f}_K - r) \left[V' + \frac{V'''\sigma_\pi^2}{2} + V'' p^2 \text{cov}(ff_K) \right] = 0 \\ (ii) \quad (P\bar{f}_K - r) &= -\frac{V''}{\frac{(V' + V'''\sigma_\pi^2)}{2}} P^2 \text{cov}(ff_K) \end{aligned}$$

A Variance-Based Measure of Risk Aversion

Consider the equations (i) and (ii). To the expression

$$R_V = - \frac{V''}{\frac{V' + V''' \sigma_\pi^2}{2}}$$

we give the name "variance-based measure of risk aversion." Note that the denominator is really the expected marginal utility of the mean of profit and is expected to be positive whenever the problem is of interest. $V'' < 0$ from the concavity of V . The measure R_V has the following properties:

(a) $R_V \geq 0$. This is true since the denominator is expected to be positive.

(b) $R_V = R_A$ if $\sigma_\pi^2 = 0$

(c) If the absolute risk aversion measure R_A is nonincreasing with respect to profit, i.e., $(\partial R_A / \partial \pi) \leq 0$, then $R_V \leq R_A$. This is so since $V''' > 0$ and the denominator of R_V is strictly greater than that of R_A .

(d) If the absolute risk aversion is nonincreasing,

$$R_V \rightarrow 0 \text{ as } \sigma_\pi^2 \rightarrow \infty.$$

(e) If $V''' < 0$, which means that absolute risk aversion cannot be decreasing, R_V rises as σ_π^2 rises.

Properties (a) and (e) are reasonable. Property (d) presents a problem. For R_V to make some sense, absolute risk aversion should be increasing, i.e., $V''' \leq 0$. If the latter is not true, i.e., $V''' > 0$, then we get seemingly paradoxical results from (i) and (ii).

A Curiosity

Suppose $V''' > 0$. Then from (i) and (ii), we note that as $\sigma_\pi^2 \rightarrow \infty$

$$(8) \quad P_{\bar{L}} - w \rightarrow 0 \quad P_{\bar{K}} - r \rightarrow 0$$

Two cases stand out. We consider only the case of capital K :

(a) $\text{cov}(ff_K) > 0$: when high production levels are observed to occur together with high capital productivity, then $\text{cov}(f, f_K) > 0$. Suppose we start from a finite σ_π^2 . The firm chooses K so as to equate

$$(9) \quad P\bar{f}_K = - \frac{V''}{V' + \frac{V'''}{2} \sigma_\pi^2} P^2 \text{cov}(f, f_K) + r$$

The firm is acting conservatively, hiring less capital than a risk neutral firm ($V''' = 0$). As $\sigma_\pi^2 \rightarrow \infty$, the first right hand side expression becomes vanishingly small and the firm raises its hiring of K to a level warranted by a risk neutral individual. Thus, with $\text{cov}(ff_K) > 0$, the firm is sensitive to small risks but is less so to large ones. This is standing the Friedman-Savage hypothesis on its head.

(b) $\text{Cov}(f, f_K) < 0$: Suppose high production levels are observed to occur together with low capital productivity. Suppose we start with a finite σ_π^2 and $|R_v P^2 \text{cov} f, f_K| < r$. The first right-hand side expression of (9) is negative and the firm is acting aggressively, hiring more K than a risk neutral firm would. The intuitive reasoning is simple: The firm is trying to induce high f by deliberately lowering f_K . As σ_π^2 rises, the firm abandons this enterprise progressively by hiring more and more K until at the limit, a risk neutral solution is reached. Again, increasing risk induces a risk neutral behavior!

The contention here is that decreasing absolute risk aversion makes for paradoxical behavior in this simple model. The crux of the problem is that the denominator of R_v , $(V' + \frac{V'''}{2} \sigma_\pi^2)$ is as we

noted the expected marginal utility of profit to the firm. A $V''' > 0$ means that the marginal utility of expected profit rises indefinitely as σ_π^2 rises indefinitely. This is and of itself intuitively difficult to accept.

A Non-Curiosity with $V''' < 0$

To start out, if $V''' < 0$, then the expected marginal utility of expected profit decreases with rising risk as our intuition would readily have it. Applying this assumption to the present problem, we observe that for a given finite variance and positive $\text{cov}(f, f_K)$, the

firm is acting conservatively hiring less K than warranted by a risk-neutral firm. As σ_π^2 rises, R_V also rises and the degree of conservatism rises further. At some sizeable σ_π^2 , the denominator of R_V approaches zero and the firm stops hiring K . Thus with $V''' < 0$, higher and higher risk induces greater conservatism.

On the other hand, if $cov(f, f_K) < 0$, for some finite σ_π^2 and $|R_V P^2 cov(f, f_K)| < r$, the firm is acting aggressively trying to induce high f by keeping f_K low. The greater the variance σ_π^2 , the more vigorously the firm pursues this task until $f_K = 0$. One way or the other, $V''' < 0$ generates a more consistent story for this type of firm.

A Possible Source of Noncomparability

It must be noted that in the Arrow-Pratt measure of absolute risk aversion, u'' and u' are partials of the original utility function and not partials of the expected utility. On the other hand,

$$R_V = \left(- \frac{V''}{V' + V''' \sigma_\pi^2} \right) \frac{1}{2}$$

where although V'' , V' and V''' are partials of the original utility function, the denominator is the expected marginal utility of the expected profit. In other words, R_V is really a composite of the original partials and a partial under expectation. From (4), it is clear that the marginal utility of profit is a considerably longer expression than the denominator of R_V .

This status of R_V is however another matter and should not detract from the difficulties we meet along the decreasing absolute risk aversion path which hinges on $V''' > 0$. If the latter is questionable, then we may have to look beyond "decreasing absolute risk aversion" and the Arrow model for explanation. Encarnación's approach (1983) looks even better as it also predicts money balances to be inferior, a finding made by Montes (1982). Note that Arrow's model on the contrary predicts the noninferiority of money balances.

Summary

"Decreasing absolute risk aversion," an assumption very much in currency, stands or falls with the positiveness of the third derivative with respect to income. We saw that the positiveness of the third derivative generates firm behavior which is paradoxical. Fur-

thermore, with normality, the expected marginal utility of income will rise with variance — a rather uncomfortable consequence. Alternatively, the negativeness of the third derivative allows the generation of more reasonable behaviors.

REFERENCES

- Arrow, K. (1971), "The Theory of Risk Aversion," in *Essays in the Theory of Risk Bearing* (Amsterdam: North-Holland Publishing Company), pp. 90-120.
- Cramer, H. (1963), *Mathematical Methods of Statistics*, Princeton University Press, pp. 67-68.
- Encarnación, José, Jr. (1983), "Portfolio Choice and Risk," University of the Philippines, School of Economics, Discussion Paper No. 83-07 (May).
- Montes, Manuel (1982), "Are Money Balances an Inferior Asset?: Rational Expectations and the Household Demand for Money," Third National Convention in Statistics, PICC (December 13-14).
- Pratt, J.W. (1964), "Risk Aversion in the Small and in the Large," *Econometrica* (January — April).