THE RELATIONSHIP BETWEEN TRUE YIELD AND BOOK-YIELD: EFFECTS OF DEPRECIATION METHOD*

BY

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The rate of return on the assets of an economic unit is a useful measure of investment performance only if it can be compared with some standard of acceptability. This standard varies with the purpose of the investment evaluation being undertaken, and the three criteria with which book-yield is often compared are as follows:

1) A minimum acceptable rate of return on investment proposals is a useful standard, for example, in planning and in exercising managerial control over the performance of investments already started.

2) Returns achieved on past investments are helpful in testing for improvements in overall managerial efficiency.

3) Yields earned by other economic units at the current point in time is of primary importance in checking, say, on how well the firm is keeping abreast of its competitors.

In the simple case of an investment project whose actual cash inflow (and therefore, true yield) turned out to be exactly the same as forecasted, it is only natural to suppose the rate of return on undepreciated assets to be the same as the rate originally expected, and to be the same regardless of the time at which the calculation is done. Furthermore, it would not be unreasonable to suppose such a rate of return figure to be the same as the rate of return on an identical investment project undertaken by some other economic unit.

Yet these suppositions will probably prove unfounded if rates of return are computed on the basis of income and book value figures reported for this investment project. To begin with, different firms may employ different methods of depreciation, destroying any possible direct comparability in reported figures. Also, with respect to com-

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parability within the project itself, book-yield will not even be necessarily the same at all points in time unless the firm carefully matches depreciation method and the investment's cash flows.

For these reasons, this study takes the criterion of full comparability as being of central importance in analyzing investment performance, accepting as ideal the method of calculating income and depreciation that results in constant book-yield throughout the project's life.[1]

Following a commonly made assumption, time is considered to be a continuous variable for purposes of analysis. By so doing, interest is assumed to be compounded continuously, rather than at discrete intervals of time.[2] No analytical effect is lost for the type of analysis done here; instead, we gain through this assumption the advantage of being able to use the calculus. An assumption of discrete time achieves the same results but at the cost of dealing with unwieldy summations of series. Furthermore, book-yield actually embraces a range of closely related ratios, having been calculated in practice by dividing profits by beginning, ending, or average book value, and for each case, by using either gross or net book value figures.[3] In a sense, therefore, by assuming continuous time, we simply use another way of calculating book-yield.

It is further assumed that the effects of accrual accounting compensate over time, in such a way as to make revenue and expense figures the same as if transactions are recorded on a cash basis. With this and the assumption that current assets other than cash less current liabilities remains the same through time, the terms "funds" and "cash," and "funds flow" and "cash flow" are used interchangeably.

II. 1. Cash Flow and Book-Yield in a Single Investment Project. An investment operation undertaken by an economic unit may be described as a four-stage conversion process, whereby (1) initial investment funds are converted into fixed factors of production, (2) variable factors of production are purchased and combined with the fixed factor input (over the life of the investment operation), which are thereby converted

[1] Essentially the same position is suggested in J. R. Hicks, *Value and Capital* (2nd ed.; Oxford: The Clarendon Press, 1946), chap. 14; Ralph C. Jones, *Effects of Price Level Changes on Business Income, Capital, and Taxes* (New York: American Accounting Association, 1956), p. 112; Ezra Solomon, *The Theory of Financial Management* (New York: Columbia University Press, 1963), pp. 125-31.

[2] See, for example, Paul A. Samuelson, "Some Aspects of the Pure Theory of Capital," *Quarterly Journal of Economics*, LI, No. 3 (May, 1937), n. 2, p. 470. He points out that there is a continuously compounded interest rate (also known as the "force of interest"), r, which exactly corresponds to a discretely compounded rate, i, such that

$$r = \log_e (1 + i).$$

[3] National Association of Accountants, *Return on Investment as a Guide to Managerial Decisions* (Research Report No. 35, 1959).

into physical output of a form more directly usable by those served by the economic unit, (3) the physical output is sold and the owners of the variable factors are paid, thus converting physical output into a cash inflow (which may theoretically be positive or negative in whole or in part, depending on the relative sizes of sales revenue and costs of production and marketing), and (4) at the termination of the investment operation, the final conversion of fixed factor input back to funds is made when they are sold for scrap or otherwise disposed of.

We shall first review in this section the relationship of cash flows to the internal rate of return. Recalling the criterion of comparability, we shall then show that (1) book-yield equals true yield when the book-yield constancy criterion is satisfied, and (2) with a given cash flow, there is only one way of calculating income, depreciation and book value that will satisfy the book-yield constancy criterion adopted here. We shall conclude this section by considering the behavior of book-yield over project life in the specific case where the cash inflow stream is constant and when straight-line depreciation is used.

Let us begin with a single investment project undertaken at a given point in time, T, *for* $O \leq T$, and which involves an initial cash outlay of $I^{\circ}(T)$. This amount represents the cost of acquiring and installing fixed factors of production, which we shall assume to consist entirely of depreciable assets such as plant and equipment. Two simplifications are made here which, however, involve no loss of generality: (1) Strictly speaking, $I^{\circ}(T)$ should include non-depreciable assets such as cash tied up in the form of fixed cash balances, accounts receivable, and other current assets needed to carry out the sales and production activities called for by investment operations. We shall assume that these are zero.[4] (2) It is also assumed that all fixed input acquisitions are made at one point in time, rather than over some length of time.

The subsequent application of labor, raw materials and other input factors results in physical output over some length of time, n. This length of time denotes some positive real number and is defined as the productive life of the investment project. If physical output at time $T + t$ (where $O \leq t \leq n$) can be sold to generate total sales revenue of $S^0(t)$[5] and if the cost of labor, raw materials and other inputs not

[4] Both scrap values and current asset balances may be included by raising $I^{\circ}(T)$ and the cash inflow at the end of the investment's life (or over some length of time in the course of which residual values are recovered).

[5] I.e., $S^0(T+t-T)$. While there is no need at this point to specify time invariance in any of the variables, this notation is used here in conformity with what is used later. By the assumption of continuous time, all the variables mentioned here are of course to be properly interpreted as rates. Thus, $S^0(t)$ is the sales revenue rate at time $T + t$. To avoid a possible source of confusion, however, (with "rate of return") we shall simply refer to sales revenue rate as sales revenue, and so on.

It might also be noted that there is a possibility of receiving sales revenue even before time T. But since this is unusual, we shall limit our analysis to the situation where $o \leq t \leq n$.

to be capitalized is $L^0(t)$, then at the point in time $T + t$, a cash in flow of $R^0(t)$, which is equal to $S^0(t) - L^0(t)$, accrues to the financier of the investment project. As used here, cash inflow has to be specifiec in time pattern as well as in magnitude. The former is defined as some function of time, $R^0(t)$, whose values for $O \leq t \leq n$ are used to denote the distribution of cash inflow over project life, and the latter is definec as the simple integral of $R^0(t)$ over project life.[6]

To simplify that analysis, we shall make the further assumptior that regardless of the size of the investment expenditure, each dolla invested generates the same cash inflow pattern and volume of physica output over time ,and consequently, the same time pattern and mag nitude of cash inflow. A unit of investment, therefore, is defined a: $I(T) = 1$; in addition, we are able to express sales revenue, the cos of variable input, and cash inflow, in terms of their respective uni flows (denoted by the unsuperscripted letters and defined as the flow per dollar of initial investment) and the related investment outlay:

$$S^0(t) = I^\circ(T)\ S(t)$$
$$L^0(t) = I^\circ(T)\ L(t)$$
$$R^0(t) = I^\circ(T)\ R(t).$$

Adopting a simple definition of the internal rate of return or true yield as the rate r which equates the investment outlay to the present value of its generated cash inflow stream over a length of time, n we have the basic cash flow relationship for a single investment out lay made at time T[7]:

$$(2.1) \qquad I^\circ(T) = I^\circ(T) \int_T^{T+n} R(s\text{-}T)\ e^{-r(s-T)}\ ds.$$

This may be further simplified by letting $t = s\text{-}T$ and dividing both sides of the equation by $I^\circ(T)$:

$$(2.2) \qquad I(T) = 1 = \int_O^n R(t)\ e^{-rt}\ dt, \qquad R(t) \geq O.$$

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[6] While the distinction is not clear-cut, it is made to emphasize the fact that two investments generating, say, a total of \$100 over their entire lives (i.e., the cash inflows have the same magnitude), would have different true yields if, say the bulk of the \$100 is received early in life in one project and late in life in the other.

[7] A more general definition of the internal rate of return is the rate which equates to zero all cash flows related to a given investment project evaluated at any point in time. In other words, if investment outlays happen to be spread over a length of time and cash inflow is received over still another length of time, the internal rate of return equates all these cash flows (which are negative and positive, respectively) to zero, regardless of the point in time selected as the equating moment.

In the equivalent case where time is a discrete variable, we have

$$(2.1a) \qquad I^\circ_T = I^\circ_T \sum_{s=T+1}^{T+n} R_{s-T} \left(1 + \frac{r}{b}\right)^{-b(s-T)}$$

This is of course a standard expression in compound interest calculations and in engineering economy analyses,[8] and simply repeats the assumption made earlier, whereby each dollar invested generates a characteristic cash inflow stream that is independent of the size of the investment outlay.

Having reviewed the generators of cash inflow and the relationship of cash inflow to the internal rate of return, let us turn to the problem of how to divide cash inflow into income and depreciation (or equivalently, how to measure book value) under the constraint of constancy in book-yield over the life of the investment project.

The cash inflow generated by a unit investment in plant and equipment consists of the two components, income, denoted by $E(t)$ and depreciation, denoted by $D(t)$ for $o \leq t \leq n$. Defining, for convenience in subsequent analysis, that depreciation $D(t)$ is a non-positive number for every t, unit cash flow at time t measured from the time of investment $I(T)$ may be stated as:

$$R(t) = E(t) - D(t). \tag{2.3}$$

Making the additional definitions of book value at time t, $V(t)$, as the undepreciated portion of the asset's initial cost, we have

$$V(t) = -\int_t^n D(s)\,ds \tag{2.4}$$

from which it follows that

$$\frac{dV(t)}{dt} = D(t). \tag{2.5}$$

Inasmuch as total depreciation over the life of the investment project started at time T is equal to the initial outlay, which we have defined to be fully depreciable,[9]

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where b = the number of compounding periods per year
r = the force of interest, or the nominal rate which, when compounded continuously, yields an effective annual rate i; i.e., $r = i$ if $b = 1$.

The term $(1+\frac{r}{b})^{-b(s-T)}$ approaches $e^{-r(s-T)}$ as b approaches infinity.

See Eugene L. Grant and W. Grant Ireson, *Principles of Engineering Economy* (4th ed.; New York: The Ronald Press Company, 1960), appendix B. They tabulate values of r and i, some of which are as follow: (for $b = \infty$)

i	r
5%	4.879016%
10	9.531018
15	13.976194
20	18.232156.

[8] *Ibid.*, chap. 3.

[9] Although we shall have occasion to qualify this statement later on in connection with the "ideal" method of calculating depreciation. *Infra*, n. 10, p. 16.

Book value and earnings may also be stated in terms of the full amount of investment outlay, in the same manner mentioned earlier, with respect to the other flow variables:

$E^0(t) = I^0(T)\ E(t)$ and $V^0(t) = I^0(T)\ V(t)$.

$$V(o) = -\int_{o}^{n} D(t)\ dt = I(T) = 1. \tag{2.6}$$

Using the notation established above, book-yield at time t, which we shall denote as $y(t)$, is expressed as:

$$y(t) = \frac{R(t) + D(t)}{V(t)} = \frac{E(t)}{V(t)}, \quad o \leq t < n. \tag{2.7}$$

We shall assume that the function $D(t)$ is such that $V(t) = o$ if and only if $t = n$.

Constancy in book-yield over the life of the investment project means of course that its first derivative is zero:

$$\frac{dy(t)}{dt} = \frac{[V(t)\frac{dE(t)}{dt} - E(t)\frac{dV(t)}{dt}]}{[V(t)]\ 2} = o \qquad o \leq t < n.$$

Simplifying the above, we obtain the condition for constancy in book-yield, whereby the ratio of the respective rates of change in income and book value equals book-yield:

$$\frac{\frac{dE(t)}{dt}}{\frac{dV(t)}{dt}} = \frac{E(t)}{D(t)} = y \quad o \leq t < n,$$

or,

$$\frac{dE(t)}{dt} = y\ D(t), \quad o \leq t < n. \tag{2.8}$$

Let us now show that the book-yield over project life is constant if and only if the book value at every point in time t is equal to the discounted value of all subsequent receipts, using the internal rate of return as the discount factor, such that

$$V(t) = \int_{t}^{n} R(s)\ e^{-r(s-t)}\ ds. \tag{2.9}$$

First we shall show that the condition is sufficient. Using equation 2.5, we obtain depreciation at time t for this definition of book value:

$$D(t) = -R(t) + r \int_{t}^{n} R(s)\ e^{-r(s-t)}\ ds.^{10} \tag{2.10}$$

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[10] Early discussions of this method may be found in J. S. Taylor, "A Statistical Theory of Depreciation," *Journal of the American Statistical Association*, XVIII.

Transposing terms and simplifying, the above is seen to be the same as $rV(t) = E(t)$ and therefore, from equation 2.7, we have the book yield $y = \frac{E(t)}{V(t)}$ equal to r which is constant. The sufficiency condition is proved.

Now let us show that the condition is necessary. Rewriting equations 2.3 and 2.5, we have the differential equation

$$-yV(t) + \frac{dV(t)}{dt} = -R(t),$$

which has a general solution factor, $c.e^{yt}$, c constant. Equation 2.9 provides us with a particular solution.

Hence, we have the solution for $V(t)$:

$$V(t) = c\ e^{yt} + \int_t^n R(s)\ e^{y\cdot(s-t)}\ ds.$$

However, we have specified that $V(n) = o$, $V(o) = 1$, and that $R(s) \geq o$ for all s, $o \leq s \leq n$. This implies that $c = o$ *and* $\int_o^n R(s)\ e^{-ys}\ ds = 1$.

By the uniqueness of the Laplace transform, we see that $y = r$ when y is constant over the life of the investment project, and that for this to be true, book value has to be expressed as an equation 2.9.

No. 144 (December, 1923), 1010-23; Harold Hotelling, "A General Mathematical Theory of Depreciation," *Journal of the American Statistical Association*, XX, No. 150 (September, 1925), 340-53; Gabriel A. D. Preinreich, "Annual Survey of Economic Theory: The Theory of Depreciation," *Econometrica*, VI, No. 3 (July, 1938), 219-41. It should be recalled that the definition of true yield makes the accumulated value of all cash flows equal to zero at any point in time within the project's productive life. Thus, at any point in time t, for $o \leq t \leq n$,

$$\int_t^n R(s)\ e^{-r(s-t)}\ ds = \int_o^t R(s)\ e^{rs}\ ds + e^{rt}$$

We are then able to express book value, depreciation and earnings in terms of historical data rather than expected future cash inflow, although of course implicit estimates are made of future cash inflow when r is used as the accumulating factor. Manipulating, then, the respective equations and letting

$$J_t = \int_o^t R(s)\ e^{-rs}\ ds,$$

we obtain

(2.9a) $V(t) = e^{rt}\ (1-J_t)$

(2.10a) $D(t) = r\ e^{rt}\ (1-J_t)-R(t)$

from which we obtain an expression for profits,

(2.10b) $E(t) = r\ e^{rt}\ (1-J_t)$.

Furthermore, it should be noted that $D(t)$, using this definition, includes all investment outlays. Had there been any residual values, i.e., current assets and scrap values of fixed assets, $D(t)$ would have included such values, considering them as recovered, or depreciated, over the life of the project, so that $V(n)$ is zero.

Book-yield, in other words, varies over the life of the investment project if a depreciation method other than that indicated by equation 2.10 is used. As an example, let us go through the process of deriving the book-yield of a single investment project whose expected (and, we shall assume, fully attained) cash inflow is a constant stream over time and for which straight-line depreciation is used in the company's books of account. It is worthwhile doing so because of the assumption commonly made in capital budgeting studies of a constant cash inflow stream, and of the common use of straight-line depreciation in ongoing investments.

Consider, then, an investment made at time zero with a total outlay of one dollar, which generates a constant cash inflow stream for the length of time given by n, and for which straight-line depreciation is used.[11] Assuming no residual values (as before), we use equation 2.2 to compute the constant R for a given true yield r, as:

$$R = \frac{1}{\int_0^n e^{-rt}\,dt} = \frac{r\,e^{rn}}{e^{rn} - 1} \quad [12] \qquad o < t \leq n$$

By straight-line depreciation, the constant depreciation expense D will be $-\frac{1}{n}$. Book value at any point in time will be the undepreciated portion of the investment outlay (using equation 2.4), which is $-\int_t^n \frac{1}{n}\,ds$, or $1 - \frac{t}{n}$. Book-yield, therefore, at time t for this particular investment project, is derived by using equation 2.7, whereby $y(t) = \frac{R+D}{V(t)}$:

$$y(t) = \left[\frac{r\,e^{rn}}{e^{rn} - 1} - \frac{1}{n}\right]\left[\frac{n}{n-t}\right] = \left[\frac{n\,r\,e^{rn}}{e^{rn} - 1} - 1\right]\left[\frac{1}{n-t}\right]$$

The book-yield of this single investment project, therefore, changes with t, rising as t rises. Taking limits, we see that

$$\lim_{t \to o} y(t) = \frac{r\,e^{rn}}{e^{rn} - 1} - \frac{1}{n} = \frac{R + D}{V(O)}$$

and $$\lim_{T \to n} y(t) = \frac{R + D}{V(n)} = \infty$$

[11] This particular analysis is fairly well-known, having been done by, among others, National Association of Accountants, *op. cit.;* Ezra Solomon, "Return on Investment: The Relation of Book Yield to True Yield," Society of Petroleum Engineers of AIME, Paper Number SPE655, 1963; Yuji Ijiri and Alexander A. Robicheck, "Accounting Measurement and Rates of Return," Working Paper No. 30; Graduate School of Business, Stanford University, 1965. (Mimeographed.)

[12] This is the continuous-compounding equivalent of the so-called "capital recovery factor" in compound interest mathematics, which is in this case equivalent to

$$\frac{i\,(1 + i)^n}{(1 + i)^{n-1}}$$

Reported profit is of course constant over the life of this investment project, while book value falls over project life. Early in project life, in other words, book-yield is close to the ratio of reported profit and the initial investment cost. As the project becomes older, the ratio increases as book value approaches zero. Only at one point, as it happens, is $y(t)$ equal to r, that is, when

$$t = \frac{e^{rn} - nr - 1}{r(e^{rn} - 1)},$$

which is obtained by setting $y(t)$ equal to r and solving for t.

It has been found, in fact, that the depreciation method appropriate for a constant cash inflow stream is such that

$$D(t) = -\frac{r\, e^{rt}}{e^{rn} - 1}, \quad {}^{13}$$

which rises over time at the rate r (as may be seen by obtaining $dD(t)/dt$). Likewise, it has been found that the straight-line depreciation is appropriate for a certain cash inflow stream that falls linearly over project life, such that

$$R(t) = \frac{1 + r(n - t)^{14}}{n}.$$

Such variations in book-yield over the life of the investment project may be misleading if not allowed for in evaluating past investments' performance; if an investment proposal was accepted because it was expected to earn a true yield r, its performance should not be judged as being below or above standard on the basis of the side effects, as it were, of the depreciation method used. Similarly, such increases in book-yield should not be taken to mean improvements in managerial performance over time.

The avoidance of this interpretation problem is often cited as one advantage of using gross, rather than net, book value as the denominator of the book-yield formula.[14] Of course this method eliminates the effect of the depreciation method used and if cash inflow is constant, book-yield is constant, but is still not the same as r. It would not, furthermore, eliminate much of the interpretation problem with respect to projects which do not generate a constant cash inflow.

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[13] Hotelling, *op. cit.*, anticipated this type of analysis. More recently, other authors have referred to various aspects of it. For example, Jones, *op. cit.*, pp. 112-13, voices some impressions regarding the correspondence between certain depreciation methods and cash inflow. See also Hector R. Anton, "Depreciation, Cost Allocation and Investment Decisions," *Accounting Research* (April, 1956), pp. 117-34, cited by Ijiri and Robicheck, *op. cit.*, p. 15; Robert S. Carlson, "Measuring Period Profitability: Book Yield Versus True Yield" (unpublished Ph.D. dissertation, Stanford University, 1964), pp. 23-25.

[14] For example, see National Association of Accountants, *op cit.*

Under certain assumptions that will be discussed below, this problem is also less serious in the multiple-investment case when an equilibrium investment pattern is achieved; the book-yield for the firm as a whole will then remain constant over time, though still not necessarily equal to true yield.

II. 2. Cash Flow and Book-Yield in a Multiple Investment Situation. We examined in the previous section the basic relationship of cash flow, depreciation, income, book value, the internal rate of return, and book-yield, within the context of a single investment project. Let us now extend these relationships to an economic unit, or a firm, taken as a whole, which is defined as a collection of mutually exclusive investment projects undertaken by the firm over time. In making this definition, we are assuming that the cash flows related to the set of all investment projects exhaust the cash flow of the firm. That is to say, all revenues earned and all costs incurred, are identified with individual projects. Similarly, the book value of the firm is defined to equal the aggregate book value of individual investment projects.[15]

Adding the assumption, to those made in the previous section, that unit cash inflow is time invariant, i.e., that a dollar invested generates the same cash inflow stream, $R(t)$, for $o \leq t \leq n$, regardless of the time at which the investment outlay is made. This assumption allows a direct expression of the cash inflow of the firm as a whole, $R^*(T)$,[16] in terms of the underlying unit cash inflow. $R^*(T)$, then, consists of all cash inflows generated by past investments which are still operating at time T. Thus, we have, letting $I^*(T) =$ investment at T,

$$(2.11) \qquad R^*(T) = \int_{T-n}^{T} I^*(s)\, R(T-s)\, ds.$$

By the same token, total earnings, $E^*(T)$, depreciation, $D^*(T)$, and book value, $V^*(T)$, of the firm may be expressed in terms of their corresponding unit values:

$$(2.12) \qquad E^*(T) = \int_{T-n}^{T} I^*(s)\, E(T-s)\, ds.$$

$$(2.13) \qquad D^*(T) = \int_{T-n}^{T} I^*(s)\, D(T-s)\, ds.$$

[15] Under our earlier assumption of zero residual values, this assumption poses no problem with respect to current assets. But in a case where current assets are admitted to the analysis, the definition of the firm which we are now making requires also the identification of current assets among investment projects.

[16] Note that R, which denotes the cash inflow of an individual project, is a function of t, with $o \leq t \leq n$, since R is time invariant. However, R^* and other flow variables applicable to the firm as a whole are functions of T, with $O \leq T$. Time $T = O$ may be conveniently considered as the time of incorporation of the firm.

$$(2.14) \qquad V^*(T) = e^{rT} \int_{T-n}^{T} I^*(s)\, ds \int_{T}^{s+n} D(u\text{-}s)\, e^{-ru}\, du.$$

Furthermore, since unit profit at the point in time, t, measured from the time of investment, is expressed as:

$$E(t) = y(t)\ V(t),$$

the firm's earnings may be alternatively expressed as:

$$(2.15) \qquad E^*(T) = \int_{T-n}^{T} y(T\text{-}s)\ I^*(s)\ V(T\text{-}s)\ \text{ds}.$$

The firm's book-yield at time T, denoted by $y^*(T)$, is defined in a manner analogous to project book-yield, that is, as the ratio of the firm's income to its book value at time T, such that:

$$(2.16) \qquad y^*(T) = \frac{E^*(T)}{V^*(T)} \quad ,$$

which may be stated alternatively as follows (using equation 2.4):

$$(2.17) \qquad y^*(T) = \frac{\int_{T-n}^{T} y(T\text{-}s)\ I^*(s)\ ds \int_{T}^{s+n} D(u\text{-}s)\ du}{\int_{T-n}^{T} I^*(s)\ ds \int_{T}^{s+n} D(u\text{-}s)\ du}.$$

In other words, the book-yield of the firm at time T is the weighted average at T of project book-yields, $y(t)$, using the related book values, $V(s+t)$, of investments made at points in time denoted by s, where $T\text{-}n \leq s \leq T$.

Let us now consider the effect on the firm's book-yield of using, and not using, the definition of book value and depreciation indicated in equations 2.9 and 2.10, respectively.

If book value and depreciation are so defined (i.e., equations 2.9 and 2.10), then we see directly by substituting for $E(t)$ and $D(t)$ in equations 2.12 and 2.14, respectively, and calculating the firm's book-yield in equation 2.16, that y^* is constant and that, as to be expected, it is equal to true yield on investment projects.

But now, suppose that the firm depreciates its investment outlays in a manner different from equation 2.9. As we found in the previous section, this practice results in a project book-yield that is a function of t, which makes $y^*(T)$ dependent on the age composition at T of the firm's investment projects. If, therefore, project book-yield starts low and rises over the life of an investment project (such as our example in section **11.1**), firms which consist of relatively new investments will have a relatively low book-yield. This is of course the direct result of the greater proportion of young (and hence, low-yield) investments.

Only in the case where the firm grows at an exponential rate, g, for $-\infty < g < \infty$, such that $I^*(T) = I^*(O)\ e^{gT}$, would the age composition of investments be unchanged and therefore would be the only instance where book-yield of the firm will remain constant through time. In other instances where investment outlays take some other form, the weighting scheme would vary with T, and hence, so would y^*.[17]

The proof is straightforward. If company book-yield is constant over T, then.

$$\frac{dy^*(t)}{dT} = \frac{d}{dT}\left[-\frac{E^*(T)}{V^*(T)}\right] = O \qquad \begin{matrix} T > O \\ V^*(T) > O \end{matrix}$$

or, carrying the indicated operations through

$$\frac{dE^*(T)}{dT} = y^*\ D^*\ (T).$$

This gives us the equation

$$\frac{d}{dT} \int_n^o I^*(T\text{-}u)\ E(u)\ du = y^* \int_n^o I^*(T\text{-}u)\ D(u)\ du,$$

which is obtained by letting $u = T\text{-}s$ and changing the integration variables accordingly in equations 2.12 and 2.13. This is a homogeneous differential equation of the form $\alpha I^*(T)' + \beta I^*(T) = O$, which has the solution $I(T) = I(O)e^{gt}$, where $I(O)$ and g are constants of the solution. We shall interpret $I(O)$ as the amount invested at the base point in time zero and g as the continuous growth rate in investment outlays.

We shall consider in the next section the important case where firms grow at some constant rate g. As we have demonstrated above, such a growth pattern in investment causes the firm's book-yield to be constant over T; a different investment pattern would cause book-yield to vary over time. An interesting and significant case of transient fluctuation in investment outlays results directly from the internal financing of investments, which is described in Appendix A. We shall, however, not analyze specifically the implications of these other investment patterns.

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[17] As we shall see below, a constant y^* indicates only that the weighted average of project book-yields, $y(t)$, does not change over T. It does not necessarily mean that y^* equals r (unlike the analogous result found for project book-yields).

II. 3. The Book-Yield of Growing Firms: An Example.
For firms which grow at g, equations 2.13 and 2.14 may be written as follows for $n \leq T$:

$$R^{*}(T) = \int_{T-n}^{T} I^{*}(O)\ e^{gs}\ R(T\text{-}s)\ ds$$

$$D^{*}(T) = \int_{T-n}^{T} I^{*}(O)\ e^{gs}\ D(T\text{-}s)\ ds$$

$$V^{*}(T) = -\int_{T-n}^{T} I^{*}(O)\ e^{gs}\ ds \int_{T}^{s+n} D(u\text{-}s)\ du.$$

Simplifying the above expressions by letting $w = T\text{-}s$ and $V = u\text{-}s$ and changing the integration variables accordingly, we obtain the total flows at T of all investments made during the preceding n length of time:

$$(2.11a) \qquad R^{*}(T) = I^{*}(O)\ e^{gT} \int_{n}^{o} e^{\text{-}gw}\ R(w)\ dw$$

$$(2.13a) \qquad D^{*}(T) = I^{*}(O)\ e^{gT} \int_{n}^{o} e^{\text{-}gw}\ D(w)\ dw$$

$$(2.14a) \qquad V^{*}(T) = -\ I(O)\ e^{gT} \int_{n}^{o} e^{\text{-}gw}\ dw \int_{w}^{n} D(v)\ dv.$$

Substituting in equation 2.16, $I(O)\ e^{gt}$ drops out. We are also able to reverse the bounds of integration (for the sake of convenience later on) without altering the ratio:

$$(2.18) \qquad y^{*} = \frac{\int_{o}^{n} \left\{ R(w) + D(w) \right\} e^{\text{-}gw}\ dw}{-\int_{o}^{n} e^{\text{-}gw}\ dw \int_{o}^{n} D(v)\ dv}.$$

The book-yield of the firm, therefore, is a function of n and is independent of T and of the value of $I(O)$ when investment outlays grow at the rate g.

For $g = o$, equation 2.18 is:

$$y^{*} = \frac{\int_{o}^{n} R(w) + D(w)\ dw}{-\int_{o}^{n} dw \int_{w}^{n} D(v)\ dv}$$

That is to say, book-yield for non-growing firms is the ratio of the absolute sum of earnings generated by each investment project and the total of its book value over n. Noting that the denominator falls faster the faster is depreciation, and that the denominator is greater (for a given internal rate of return) the more delayed cash inflow is, we can make the following conclusions with respect to the firm's constant book-yield: (1) y^* rises as n approaches infinity, (2) y^* is relatively higher for projects which generate a rising or postponed cash inflow stream, and (3) y^* increases the faster is depreciation. This situation was studied by Ezra Solomon[18] who examined the effects on y^* of various characteristics of $R(w)$ and $D(w)$.

The denominator of equation 2.18 may be further simplified by changing the order of integration such that

$$\int_o^n -D(v)\, dv \int_o^v e^{-gw}\, dw.$$

If $g \neq o$ and we integrate over w,

$$\int_o^n -D(v)\, \frac{1-e^{-gv}}{g}\, dv.$$

$$= \left\{ \frac{1}{g} \right\} \left\{ \int_o^n -D(v)\, dv + \int_o^n D(v)\, e^{-gv} \right\} dv$$

Since $\int_o^n - D(v) = 1$, we can restate equation 2:18 as follows:

$$y^* = g \left\{ \frac{\int_o^n R(w)\, e^{-gw}\, dw + \int_o^n D(w)\, e^{-gw}\, dw}{1 + \int_o^n D(v)\, e^{-gv}\, dv} \right\},$$ [19]

[18] Solomon, "Return. . . ," *op. cit.*, mentions the following components of $R(w)$ and $D(w)$:

1) Length of project life
2) Accounting policy with respect to capitalization and depreciation of investment outlays
3) Timing of cash inflow relative to cash outlays
   a. Length of the construction period
   b. Lag between project completion and the beginning of the cash inflow stream
   c. Time shape of the project's cash inflow stream.

[19] Carlson, *op cit.*, pp. 55, 124-26, arrives at an expression similar to this, but from a different point of view. It should be noted that from this equation, the behavior of book-yield with respect to various parameters may be studied exclusively from project cash flows: the discounted value of the earnings and book value streams are obtained (discounting back to $t = o$ or forward to $t = n$), using the growth rate as the discount factor. Dividing these accumulated values results in the firm's book-yield.

which we shall rewrite, letting $H = \int_0^n R(w)\, e^{-gw}\, dw$ and $K = \int_0^n D(w)\, e^{-gw}$:

$$(2.19) \qquad y^{\circ} = g\,\frac{(H+K)}{(1+K)}.$$

When $g = r$, H becomes one (from equation 2.2) and thus y° equals the growth rate also. This, it will be noted, is irrespective of the depreciation method used and of the characteristics of the cash inflow stream, including time lags, residual values, and all other factors which affect $R(t)$, and the depreciation method used.

Suppose now that $g \neq r$. Considering for this purpose both n and g as independent variables, i.e., $y^{\circ}\ (g, n)$, we can take partial derivatives of equation 2.19, in order to show, for a given true yield, cash inflow pattern and depreciation method, how book-yield is affected by the length of productive life of individual investments and how book-yield varies with the rate of growth of the firm.[20]

Taking the derivative of $y^{\circ}\ (g, n)$ with respect to g, we obtain

$$\frac{\partial y^{\circ}\ (g,n)}{\partial g} = g\ \frac{[1+K]\ [-gH-gK] - [H+K]\ [-gk]}{[1+K]^2} + \frac{[H+K]}{[1+K]}$$

$$= \frac{[1+H]\ K + [1-g^2]\ [H+K^2]}{[1+K]^2}$$

Similarly,

$$\frac{\partial y^{\circ}(g,n)}{\partial n} = g\ \frac{[1+K]\ [R(n)e^{-gn} + D(n)e^{-gn}] - [H+K]\ [D(n)e^{-gn}]}{[1+K]^2}$$

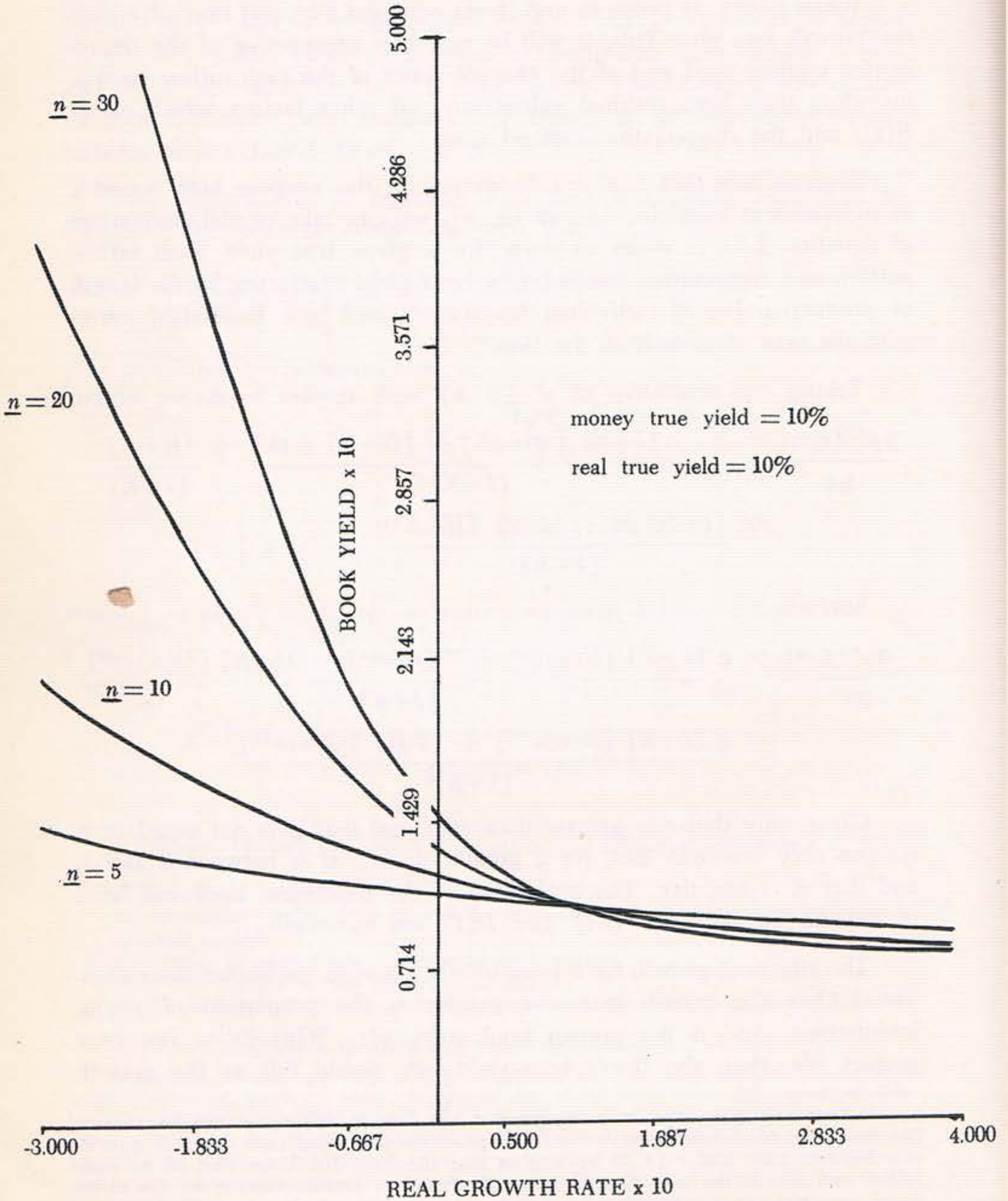
$$= g\ \frac{[1+K]\ [R(n)e^{-gn}] + [1-H]\ [D(n)e^{-gn}].}{[1+K]^2}$$

Given only that r is greater than zero and that g is not equal to r, we can only conclude that, for g greater than r, H is between 0 and 1, and that K is negative. The evaluation of the expression itself will have to depend on whatever $R(t)$ and $D(t)$ are specified.

The effects of growth have been briefly noted in the earlier discussion. The higher the growth rate, the greater is the proportion of young investments. And if the project book-yield, $y(t)$, happens to rise over project life, then the firm's book-yield, y°, would fall as the growth

[20] A growth rate equal to r implies that the firm is making no net investment, i.e., cash flow to and from the firm's funds providers just equal each other. A growth rate between zero and r $(r > o)$ implies that the firm distributes part of its cash inflow and retains the rest. Growth rates of greater or smaller than r, on the other hand, represent net investment and net disinvestment, respectively.

CHART 1. Book-yield as a function of project life and real growth rate, with a constant real true yield. (Level cash inflows, straight-line depreciation.)



rate increases. This situation of rising project book-yields (and hence, falling company book-yield when the growth rate increases) exists in connection with the more commonly-used straight-line and accelerated depreciation methods, and level or rising cash inflow. Rising project book-yield also results in conjunction with falling project cash inflow as long as these do not, at any point, fall below those indicated by equation 2.10 as the cash inflow $R(t)$ which produces a $D(t)$ equivalent to these standard depreciation methods.

For the same reasons, the effect on company book-yield of changes in individual projects' life depends on the influence such changes have on project cash inflow and on project depreciation. If it happens that the rate of change in both cash inflow and depreciation cause project book-yields to rise over its life, then the same conclusion made above with respect to falling company book-yield would also hold.

In our specific example of constant cash inflow and straight-line depreciation, which is illustrated in Chart 1, y^* approaches minus infinity as g approaches minus infinity and approaches zero as g approaches infinity. This is the direct result of the earlier conclusions made with respect to project book-yield. And for g greater than r, the longer is project life, the lower would be the firm's book-yield. Conversely, for g smaller than r, the longer is project life, the greater would be the firm's book-yield.

The behavior of company book-yield has some significant implications regarding the evaluation of investment performance.

To begin with, we have found that the firm's book-yield remains steady through T under the assumption of a constant growth rate. It goes without saying that if true yield were known, it would surely be used for investment evaluation. The problem is, when true yield is unknown, in what direction and to what extent to adjust book-yield in order to make it a more useful figure; in short, how to interpret book-yield.

Firstly, suppose that it is desired to evaluate changes in managerial efficiency over time. Assuming that the firm's true yield never changes and that both project life and growth rate are constant (these same assumptions are used in the succeeding paragraphs unless otherwise specified), we would be able to conclude from the constant book-yield figure that managerial efficiency has remained stable. This would be correct. But if investments fluctuate in some fashion, book-yield would no longer be stable and the conclusion regarding unchanging managerial efficiency may possibly not be made.

Secondly, even when book-yields are steady, conclusions may be misleading when book-yield is used to compare the absolute level of investment performance among different economic units (i.e., among firms, among divisions of the same firm, and so on). For example, observation of book-yield may show that one firm has a rate of return of 14% and the other, of 11%. If it happens that project lives in the first firm are twice as long as project lives in the second, a conclusion based on book-yield alone would definitely be wrong if the two firms are experiencing the same growth rate and are enjoying the same true yield.

Thirdly, it would be misleading, for the same reasons mentioned in the second item above, to compare actual performance with expectations at the time the investment proposal was evaluated and accepted. The standard of comparison, a time-adjusted figure usually, would not be useful in evaluating the absolute level of investment performance indicated by book-yield.

Fourth. Consider two firms, with the same true yield and with the same length of project lives. But suppose that the first firm retains more of its cash inflow than the second. Solely because of a difference in growth rate, book-yield in this situation would be lower in the first firm than in the second, leading possibly to the conclusion that investment opportunities in the former are limited compared with those of the latter, and are indeed so low that the yield on old capital is even dragged down.

Finally, the relationship between true yield and book-yield is of direct concern in the regulation of public utilities, which is based on the concept of an "adequate" or "fair" rate of return on investors' capital. If the fair rate objective is based on cost of capital, interest rates, or is in some fashion a criterion based on the time value of money, then allowing the same rate on the regulated firm's net book value may well result (particularly in the light of long lives and low growth rate that one ordinarily expects of public utilities) in an allowed return to investors that is far less than the one initially considered to be "fair."