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“Time inconsistency”: the Phillips curve example, an analysis for intermediate macroeconomics

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This paper provides the algebra and a panel diagram to attempt to examine the so-called inflation-unemployment (or Phillips curve, or aggregate supply) example, the most popular example in the literature when introducing the concept of “time inconsistency” or “dynamic inconsistency.” The resulting panel diagram, along with the derivations presented in the appendices, is used to analyze the different possible outcomes, depending on the scenarios–rule or pre-commitment, cheating, and equilibrium–and find out whether there is indeed “time inconsistency” or “dynamic inconsistency” in the said example.

JEL classification: E31, E52, E61
Keywords: Philips curve, aggregate supply, time inconsistency, dynamic inconsistency, short-run optimal policy, long-run optimal policy, rational expectations, rules versus discretion

1. Introduction

Inconsistency between “the [1]optimal plan of future behaviour chosen as of a given time” and “the optimizing future behaviour of the individual” arises because *either* preferences change over time (Strotz’s problem of “intertemporal tussle” due to non-exponential discounting of future felicities [1956]) *or* constraints change over time (Kydland and Prescott’s “inconsistency of optimal plans” [1977]). With either changing preferences or changing constraints, plans at different dates will not necessarily coincide and therefore are “inconsistent,” but all are optimal (although one is optimal while the others are suboptimal) because each plan is derived from maximization of an objective function reflecting preferences subject to some constraint/s.

[1] (1) can also be rewritten as $\pi_t^e = (1/\alpha)\bar{y} + \pi_t - (1/\alpha)\bar{y}$, and its analogue is the consumer's budget line. Since here the constraint is upward sloping, $d\pi/dy$ is (-) “price” of y/”price of π = the (-) “price” of y in terms of $\pi = (1/\alpha)/1$.

The issue of "time inconsistency" is a not a mere intellectual curiosity. For instance, "time inconsistency" in the tradition of Strotz (revived by Phelps and Pollak [1968] and made popular by Laibson [1997]) can help explain a wide range of economic phenomena: consumption/saving/growth, self-regulation/control, job search, choice of retirement, information acquisition, investment in human capital, procrastination, addiction, etc. On the other hand, Kydland and Prescott's idea [1977] helps explain, in the case of monetary policy, why "the pursuit of the first best tends to push the economy away from the second best of a rule with low inflation, and toward the third best of discretionary policy with high inflation" [Barro 1986:28]. It can also explain "hold-up" problems that may lead to underinvestment and other sub-optimality results.

The focus of the paper shall be on Kydland and Prescott's idea of "inconsistency," also using the inflation-unemployment (or Phillips curve or the aggregate supply) as an example. In the macro literature on the topic, one can find inconsistent/confusing terminologies as well as looseness in the use of the word "equilibrium." (For the record, Kydland and Prescott [1977] did not use the phrase "time inconsistency." Calvo [1978] used "time inconsistency," while Fischer [1980] used "dynamic inconsistency.") Using equations and graphs, the paper will clarify if indeed there is "time inconsistency" or dynamic inconsistency in the said example.

The paper is organized as follows. Section 2 presents the Phillips curve example and examines the different possible outcomes. Specifically, section 2.1 describes the components of the model. In this Phillips curve example, the supply side of the economy is summarized by the Phillips curve or the aggregate supply curve–short run and long run–while demand side is summarized by some monetary policy. There are two agents in the economy, the private sector (or the wage setters) and the policymaker, whose objective/s and preferences as described by their respective utility functions. The private sector sets nominal wages on the basis of its expected inflation rate, and their expectations are formed rationally. The utility of the private sector is maximized when its expected inflation rate equals the actual inflation rate chosen by the policymaker. The rational expectations equilibrium (*REE*) line of private sector coincides with the vertical Phillips curve or vertical AS. The policymaker, on the other hand, has control over inflation, and the choice of the inflation rate depends on (is independent of) the private sector's expected inflation rate when his/her preferences are quadratic (quasi-linear, as shown in the next section). The rule for the choice of the inflation rate is summarized by the so-called optimal policy (*OP*) line. Section 2.2 provides two alternative graphical derivations of optimal policy line. The derivations of the equation for the *OP* line using alternative methods—equating the slopes, substitution method, equating marginal cost and marginal benefit, and Lagrangian method—are presented in Appendix A.

Section 2.3 clarifies the distinction among short-run optimal points, *REE*, and the long-run optimal point. Section 2.4 presents a three-panel diagram and uses it to graphically illustrate the possible outcomes–rule or commitment, cheating, and equilibrium. (The derivations are presented in Appendix B.) In the literature, the rule solution is labeled as time inconsistent while the equilibrium solution is labeled as time consistent. However, the terms "time inconsistency" or "dynamic inconsistency" are a misnomer because there is actually no inconsistency in the so called "time inconsistency" or "dynamic inconsistency" problem. Choices are different because the policymaker faces different constraints depending on the private sectors on inflationary expectations, as shown in the panel diagrams. Different optimization problems yield different solutions or, graphically, different points of intersections.

Section 3 replaces the quadratic utility function of the policymaker with a quasi-linear function. Another possible outcome—over-expected case—is added, and the four possible outcomes are interpreted using simple game theory. That the terms "time inconsistency" or "dynamic inconsistency" are misnomers is even more visible when the possible outcomes are analyzed using simple game theory. Section 4 briefly discusses the distinction between rules and discretion. Section 5 concludes the paper.

2. The inflation-output example when the policymaker's utility function is quadratic

2.1. The model

The model is based on Kydland and Prescott [1977], Barro and Gordon [1983a, 1983b], and Blanchard and Fischer [1989]. In this setup, the economy is described by the following: short-run and long-run Phillips curves or aggregate supply curves, private sector's objective and expectations formation, and policymaker's objectives and preferences.

Phillips curve or aggregate supply. Output in the short run obeys the expectational Phillips curve

$$y_t = \bar{y} + \alpha(\pi_t - \pi_t^e), \qquad \alpha > 0 \tag{1}$$

which can be rewritten as

,

$$\pi_t = \underbrace{\left(\pi_t^e - \frac{1}{\alpha}\bar{y}\right)}_{\text{v intercept}} + \underbrace{\left(\frac{1}{\alpha}\right)}_{slope} y_t \tag{1'}$$

where y is output (in natural logarithm), $\bar{y}$ is the full-employment level or the natural rate of output (in natural logarithm), π is the actual inflation rate, π^e is the expected inflation rate as embodied in predetermined nominal wages, α is a parameter measuring the sensitivity of output to inflation surprise ($\pi - \pi^e$), and subscript t is time or period.

Equation (1) or (1’), the short-run Phillips curve (*SRPC*) or the short-run aggregate supply (*SRAS*) curve, is one of the constraints faced by the policymaker. It is a relationship between output and inflation which assumes sticky-price adjustment and/or sticky-wage adjustment. Equation (1) shows that output is above (equal to; below) the full-employment level whenever the actual inflation rate is higher than (equal to; lower than) the expected inflation rate.

In a $y - \pi$ diagram, the slope of the *SRAS* indicates that the policymaker has to accept ($1/\alpha$) unit/s of inflation to gain a unit increase in output.[2] Since a *SRAS* is drawn for a given π^e, an increase in π^e from π^T to π_E increases the v. intercept and shifts the *SRAS* upward and to the left (see (1’) and Figure 1).[3]

Output will equal its full-employment level when there are no inflationary surprises. In the long run, on the average, wage setters (the private sector) will neither underpredict nor overpredict the inflation rate, i.e., $\pi^e = \pi$, and output is given by

$$y = \bar{y}, \qquad (2)$$

i.e., output is equal to its full-employment level. Note, however, that if even in the short run the private sector is able to exactly predict the inflation rate, i.e., $\pi_t^e = \pi_t$, then

$$y_t = \bar{y}, \qquad (2)$$

output is equal to its full-employment level even in the short run. The long-run Phillips curve (*LRPC*) or the long-run aggregate supply (*LRAS*) curve (2) is vertical at $\bar{y}$. The intersection of a given *SRAS* with the *LRAS* shows the π^e associated with it (see Figure 1).

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[2] As α increases (decreases), the v. intercept increases, the slope decreases, and the SRAS pivots around the point where it intersects the LRAS; this means that the SRAS becomes flatter (steeper).

[3] The policymaker’s objective function may either be a utility function or a loss function. Two forms of the policymaker’s objective function are used in the literature: quasi-linear and quadratic. See Walsh [2010: 271-273].

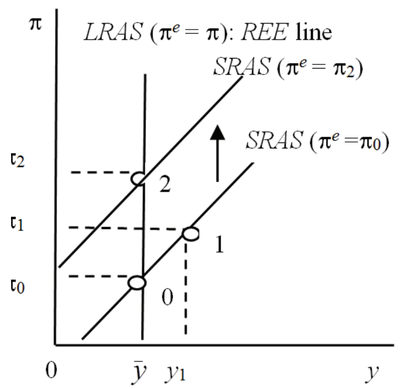

FIGURE 1. Short-run versus long-run aggregate supply

The private sector's expectations. It is assumed that the private sector forms its expectations rationally and that its utility function is given by

$$U_t^P = -(\pi_t - \pi_t^e)^2 \tag{3}$$

where U_t^P is the private sector's utility. Equation (3) shows that deviations of π from π^e, whether positive or negative, negatively affect the utility of the private sector, $U^P = 0$ only $\pi^e = \pi$, and $U^P < 0$ when $\pi^e \neq \pi$.

The private sector chooses the expected inflation rate π^e so as to maximize equation (3), and its utility is maximized when

$$\partial U^P / \partial \pi = -2(\pi - \pi^e) = 0 \quad \Leftrightarrow \quad \pi^e = \pi \tag{3.1}$$

where

$$\pi_t^e \equiv E_{t-1}\pi_t \tag{3.2}$$

Equation (3.1) is the rational expectations equilibrium (*REE*) or perfect foresight equilibrium because there is no uncertainty in the model, condition for the private sector which is derived from the private sector's utility maximization while equation (3.2), that the expected inflation rate at time *t*, π_t^e, is the rational expectation of π_t based on all the available information as of the end of *t*-1 or just the beginning of time *t*, $E_{t-1}\pi_t$, is the rational expectations (*RE*) assumption.

Even if the private sector forms expectations rationally, it may overpredict or underpredict the inflation rate but in the long run, on the average, $\pi^e = \pi$; if in the short run it fully anticipates the inflation rate, then $\pi_t^e = \pi_t$.

In a $\pi^e = \pi$ diagram, the *REE* line is the $\pi^e = \pi$ line or the 45° line; in a $y - \pi$ diagram, the *REE* line coincides with the vertical *LRAS*.

The policymaker's utility function. The objective function of the policymaker is given by

$$U_t = -a(y_t - y^T)^2 - b(\pi_t - \pi^T)^2, \quad 0 < a,b < \infty, y^T > \bar{y} > 0, \pi^T > 0 \tag{3}$$

where U is the utility of the policymaker, y^T is the target output level, π^T is the target inflation rate, a is the weight on output deviations, b is the weight on inflation deviations which is a measure of the degree of the policymaker's inflation aversion, and the other variables are as defined before.[4]

The policymaker has output and inflation targets, and his/her utility is negatively affected by deviations from these targets. The utility function is quadratic in the deviation of output from the target level ($y - y^T$) and the deviation of inflation rate from the target rate ($\pi - \pi^T$), implying that the policymaker has a symmetrical attitude to positive and negative deviations.

Distortions/imperfections in labor and product markets (due to taxes, monopoly unions, or monopolistic competition) make the full-employment or natural rate of employment of output too low. Thus, output target is higher than the full-employment level of output, i.e., $y^T > \bar{y}$.

It is assumed that the policymaker manipulates or has control over the inflation rate π and that it is set after the private sector sets nominal wages based on their expectations about the inflation rate π^e.

The preferences of the policymaker, as summarized by his/her utility function, can be illustrated graphically using contours (see Figure 1). $U = 0$ only at the bliss point where $(y, \pi) = (y^T, \pi^T)$, and $U < 0$ when $y \neq y^T$ and $\pi \neq \pi^T$.[5] The farther is the indifference curve

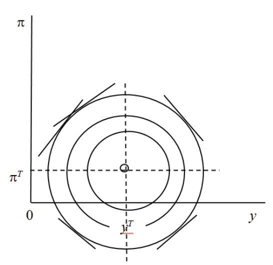

FIGURE 2. Preferences of the policymaker Maker

from the bliss point, the lower the utility level.

[4] The policymaker's objective function may either be a utility function or a loss function. Two forms of the policymaker's objective function are used in the literature: quasi-linear and quadratic. See Walsh [2010:271-273].

[5] Note also that a change in y^T and/or π^T will shift the entire indifference map.

The shape of the contours depends on parameters a and b.[6] In the region where $(\pi - \pi^T) > 0$ and $(y - y^T) < 0$, a relatively steep (flat) indifference curve indicates that the policymaker is willing to accept a larger (smaller) increase in inflation for a given increase in output.[7]

Setting the total differential of U equal to zero

$$dU = 0 = -2a(y - y^T)dy - 2b(\pi - \pi^T)d\pi, \tag{4.1}$$

gives the slope of the indifference ellipse

$$\left(\frac{d\pi}{dy}\right)_{dU=0} = -\frac{a(y-y^T)}{b(\pi-\pi^T)} \tag{4.1}$$

The indifference curves are positively sloped either when $(\pi - \pi^T) > 0$ and $(y - y^T) < 0$ or $(\pi - \pi^T) < 0$ and $(y - y^T) > 0$ and become steeper (flatter) as the weight of output deviation a increases (decreases) and/or the measure policymaker's inflation aversion b decreases (increases).

The absolute value of the slope of the indifference curve $|d\pi / dy|_{IC}$, called the marginal rate of 'substitution' of π *for y*, $MRS_{\pi \text{ for } y}$, indicates the addition to (reduction in) π that the policymaker is *willing* to accept in order to have one more (less) unit of y and yet the he/she remains on the same utility level. The $MRS_{\pi \text{ for } y}$ declines (increases) as y increases when $(y - y^T) < 0$ (> 0).

2.2. Deriving the policymaker's short-run optimal policy (OP) line $a/b = 1$

The policymaker chooses output and inflation to maximize his/her utility function (4) subject to the constraint given by the *SRAS* (1):

$$\max_{y,\pi} \left\{U = -a(y-y^T)^2 - b(\pi-\pi^T)^2\right\}.$$

$$s.t. \quad y = \bar{y} + \alpha(\pi - \pi^e)$$

The solution to this problem yields the condition for optimal policy (OP). The OP line can be illustrated in either the $y - \pi$ diagram or the $\pi^e - \pi$ diagram.[8]

[6] Each indifference curve is a concentric circle, a circle with (y^T, π^T) at its center when the policymaker is equally concerned about output and inflation from targets, i.e., when $a/b = 1$. On the hand, each indifference curve is a concentric oval, an ellipsoid with a vertical (horizontal) orientation and (y^T, π^T) at its center when the policymaker attaches more (less) weight on deviations in inflation rate than on deviations in output, i.e., when $a/b > 1$ (<1); in this case, each indifference is relatively steep (flat). See Carlin and Soskice [2006:143].

[7] The slope of the indifference curve depends on the region. In the region to the NW of the bliss point, $(\pi - \pi^T) > 0$ and $(y - y^T) < 0$; the other regions are to the SE, SW, and NE of the bliss point.

[8] The analogue of the SRAS (1) or (1') rewritten as $\pi^e - (1/\alpha)\,\bar{y} = -(1/\alpha)y + \pi$ is the budget line the analogue of the downward sloping OP line in $y - \pi$ diagram is the income-consumption curve while the analogue of the upward sloping OP line in $\pi^e - \pi$ diagram is the Engel curve.

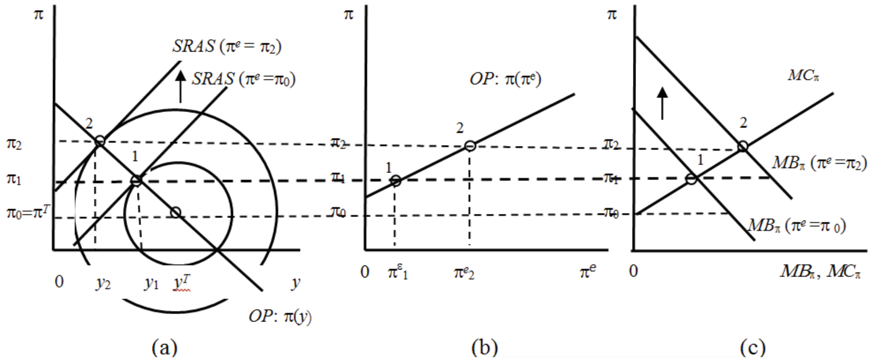

FIGURE 3. Alternative derivations of the OP Line

The alternative ways to get the equation for the *OP* line follow: the method of equating slopes, the method of equating marginal benefit and marginal cost, the substitution method, and the Lagrangian method (see Appendix).

Method of equating slopes. Equating the slope of the *SRAS* (1’) and the slope of the indifference curve (4.2)

$$\underbrace{-\frac{a(y-y^T)}{b(\pi-\pi^T)}}_{\left(\frac{d\pi}{dy}\right)_{IC}} = \underbrace{\frac{1}{\alpha}}_{\left(\frac{d\pi}{dy}\right)_{PC}} \tag{5}$$

yields

$$y-y^T = \frac{b}{a\alpha}(\pi-\pi^T) \quad \Leftrightarrow \quad \pi = \underbrace{\pi^T + \frac{a\alpha}{b}y^T}_{\text{v intercept}>0} \underbrace{-\frac{a\alpha}{b}}_{\text{slope}<0} y \tag{6.1}$$

which is the equation for *OP* line in the $y - \pi$ diagram. This *OP* line is a locus of y,π combinations for which the policymaker’s objective function (4) is maximized subject to the *SRAS* constraint (1).[9] As shown in Figure 1, it is a locus of all optimal y,π combinations, i.e., a locus of points of tangency between the indifference curve of the policymaker and the *SRAS*.[10]

[9] The $\pi(y)$ line as the OP line is also called the “social expansion path” [Heijdra and van der Ploeg 2002] or the MR line [Carlin and Soskice 2006].

[10] The OP line in either diagram (i) becomes flatter (steeper) as: the measure of inflation aversion increases (decreases), the weight on output deviations a decreases (increases), and the sensitivity of output to inflation surprise decreases (increases) and (ii) shifts up as y^T increases and/or π^T increases.

To see why a point on the *OP* line, such as point 1, is an optimal point, consider points 0 and 1 on the *SRAS* constraint drawn given that $\pi^e = \pi_0$ (Figure 3a). Starting at point 0, the policymaker can increase his/her utility by increasing π from π_0 to π_1, i.e., by moving to point 1 which lies on a higher indifference.

The *OP* line - $\pi(y)$ line - can also be derived graphically. Assume that initially the point of tangency is point 1. An increase in π^e from π_0 to π_2 shifts the *SRAS* to the left (i.e., output decreases at each π), and the new point of tangency is at point 2, where the optimal y is lower and the optimal π is higher; the policymaker's utility level is lower since y is an economic good, while π is an economic bad. Connecting points of tangencies 1 and 2 yields the downward sloping $\pi(y)$ line as the *OP* line in the $y - \pi$ diagram (see Figure 3a).

Since the optimal y decreases and the optimal π increases as π^e increases, there is a positive relationship between π and optimal π. This means that $\pi(\pi^e)$ line as the *OP* line in the $\pi^e - \pi$ is upward sloping.

Note that both points 1 and 2 are optimal points, but point 2 is sub-optimal as compared to point 1.

The $\pi(y)$ equation (6.1) as the equation for the *OP* line can be converted into $\pi(\pi^e)$ equation. Combining (6.1) and (1)

$$\pi = \pi^T + \frac{a\alpha}{b} y^T - \frac{a\alpha}{b}\overbrace{(\bar{y} + \alpha(\pi - \pi^e))}^{=y,\text{ using (1)}} \quad ,$$

and rearranging/simplifying

$$\pi(1 + \frac{a\alpha^2}{b}) = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) + \frac{a\alpha^2}{b}\pi^e$$

yields the equation for the *OP* line in $\pi^e - \pi$ diagram

$$\pi = \underbrace{\frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})}_{\text{v intercept}>0} + \underbrace{\frac{a\alpha^2}{b + a\alpha^2}}_{0<\text{slope}<1}\pi^e. \tag{6.2}$$

In the $\pi^e - \pi$ diagram, the *OP* line is upward sloping but flatter than the 45° line (see Figure 3b).[11]

[11] The OP line (the term used by Blanchard and Fischer [1989]) in the $\pi^e = \pi$ diagram, like the OP line in the $y - \pi$ diagram, (i) becomes flatter (steeper) as b increases (decreases), a decreases (increases), and α decreases (increases) and (ii) shifts up as y^T increases and/or π^T increases.

The OP line in either diagram becomes flatter (steeper) as: the measure of inflation aversion b increases (decreases), the weight on output deviations a decreases (increases), and the sensitivity of output to inflation surprise α decreases (increases).

Note also that, in either a $y - \pi$ diagram or a $\pi^e = \pi$ diagram, an increase in y^T and/or an increase in π^T will increase the v. intercept of the OP line, i.e., it will shift up the OP line.

Method of equating marginal benefit and marginal cost. The $\pi(\pi^e)$ equation (6.2) can also be derived by equating the MB_π equation and the MC_π equation (see Appendix)[12],

$$MB_\pi = -2a\alpha(\alpha(\pi - \pi^e) - (y^T - \bar{y})) \Leftrightarrow \pi = \underbrace{\pi^e + \frac{1}{\alpha}(y^T - \bar{y})}_{\text{v intercept}} \underbrace{- \frac{1}{2a\alpha^2}}_{slope} MB_\pi, \quad (7.1)$$

$$MC_\pi = 2b(\pi - \pi^T) \Leftrightarrow \pi = \underbrace{\pi^T}_{\text{v.intercept}} + \underbrace{\frac{1}{2b}}_{slope} MC_\pi, \quad (7.2)$$

and $(\partial U/\partial y)(\partial y/\partial\pi) = MB_\pi$ = marginal utility, or marginal benefit of higher inflation through higher output while $\partial U/\partial\pi$ = (-) MC_π = marginal utility of higher inflation, or the negative of the marginal cost of higher inflation (Obstfeld and Rogoff [1996: 637]).

The $\pi(\pi^e)$ line can also be derived using the MB_π,MC_π diagram (Figure 3c). The intersection between a specific MB_π curve and the MC_π curve determines a specific optimal inflation rate. Assume that the initial intersection is at point 1. As π^e increases, the upward sloping MC_π curve stays the same but the downward sloping MB_π curve shifts to the right (i.e., the marginal benefit of higher inflation through output decreases at each π) and, at the initial optimal π, $MB_\pi > MC_\pi$; thus, the optimal π must increase to restore equality between MB_π and MC_π, now at point 2. Thus, there is a positive relationship between π^e and π along the *OP* line. (see Figure 3b).

2.3. Short-run optimal points, REE, and the long-run optimal point

At the outset, before discussing the possible solutions/outcomes, a distinction must be made among short-run optimal points, rational expectations equilibrium, and the long-run optimal point. Short-run optimal points are points on the *OP* line while *REE* points are points on the *REE* line or the *LRAS*.

The point of tangency between the *LRAS* and an indifference curve is the long-run optimal point; if, even at time *t*, $\pi_t^e = \pi_t$, and such tangency occurs, then the long-run optimal point is attained even at time *t*.

In Kydland and Prescott’s [1977] terminologies, the long-run optimal point is the point of “optimal equilibrium” while the point of intersection between the OP line and the *REE* line is the point of “consistent equilibrium.” In the literature, other terminologies are used to refer to these points, as the discussion below will show.

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[12] The MB$_\pi$ curve will shift as y^T and, more importantly, as π^e changes and will become flatter as *a* increases and/or α increases. The MC$_\pi$ curve will shift only when π^T changes and will become flatter as *b* increases.

2.4. Possible outcomes

Figure 4 illustrates the possible outcomes using three alternative diagrams. The possible outcomes (solutions)—represented by points *R* for rule or commitment, *C* for cheating, and *E* for equilibrium—arise from different scenarios (mathematical problems).

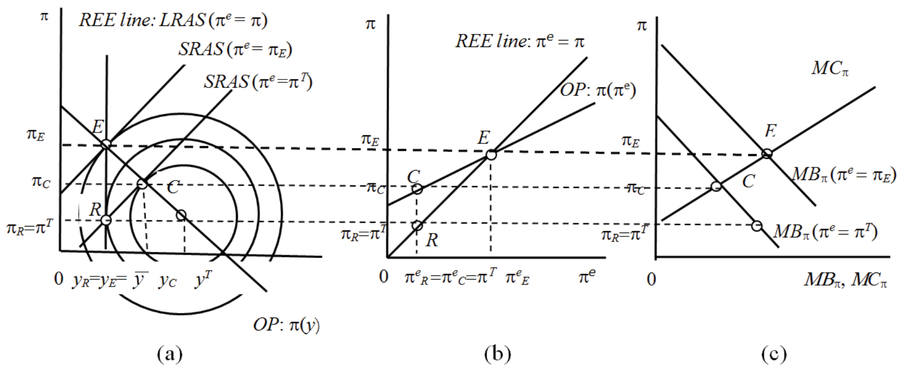

FIGURE 4. Possible outcomes illustrated using alternative diagrams

Rule or commitment.[13] In Figure 4a, consider only the *LRAS* or the *REE* line and the indifference map. Suppose that the policymaker can commit to an inflation rate, π^T, and sets $\pi = \pi^T$, and that the private sector also expects that the inflation rate would be π^T. In this scenario, using the equation for the *SRAS* (1), output under commitment is $y_R = \bar{y}$.

Point *R* represents the solution to maximizing (4) s.t. the vertical *LRAS* (2), since $y = \bar{y} + \alpha(\pi - \pi^e)$, $\pi - \pi^e$, $\pi^e = E\pi$, $= \pi^T$. In Figure 4a, point *R* is *the* point of long-run optimal equilibrium because it is *the* point of tangency between an indifference curve and the *LRAS* or the *REE* line. This is Kydland and Prescott's [1977] point of "optimal equilibrium."

Note that here there is no short-run optimization involved, as the policymaker chooses which is not on the *OP* line. In particular, the solution is given by a point, *R*, that lies *on* the *REE* line but *off* the *OP* line, as shown in Figures 4a and 49(b). Since the *OP* line is a locus of points of intersection between MB_π curves and the MC_π curve and point *R* is not on the OP line, it follows that $MB_\pi \neq MC_\pi$; specifically, with $\pi = \pi^T$, $MB_\pi > MC_\pi$, as shown in Figure 4c. Thus, the "commitment" solution is consistent with *REE* but not optimal in the short run.

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[13] The rule or commitment solution is also referred to as "equilibrium under the Taylor rule" [Sorensen and Whitta-Jacobsen 2010:661] or commitment or "optimal policy under precommitment" [Walsh 2010:281].

In a game without commitment, the "commitment" solution cannot be an equilibrium [Fudenberg and Tirole 1991:76].

Cheating.[14] Here the private sector mistakenly believes that the inflation rate would be $\pi^e = \pi^T$, i.e.,

$$\pi^e = \pi_C^e = \pi^T, \tag{8.1}$$

but the policymaker will choose π based on the $\pi(\pi^e)$ equation (A4.2), the equation for the *OP* line. Substituting (B2.1) into (A4.2) yields the solution for the inflation rate under cheating

$$\pi_C = \pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y}) \tag{8.2}$$

Substituting (B2.1) and (B2.2) into (A2) gives the solution for output under cheating,

$$y_C = \bar{y} + \frac{a\alpha^2}{b + a\alpha^2}(y^T - \bar{y}) \tag{8.3}$$

In this cheating scenario, once the private sector has committed itself to $\pi^e = \pi^T$, the policymaker will choose the $\pi = \pi_C$ to maximize its utility. The cheating solution, point *C*, is optimal since it is an intersection between the MB_π curve and the MC_π curve (see Figure 4c) and a point on the *OP* line (see Figures 4a and 4b). Point *C,* however, is a point *on* the *OP* line but *off* the *REE* line, i.e., the cheating solution is optimal but not consistent with *RE*. Thus, point *C* cannot be the equilibrium of the model where the private sector forms expectations rationally.

Equilibrium.[15] In a model where the problem is given by

$$\max_{y,\pi} \left\{ U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \right\} \;\; s.t. \;\; y = \bar{y} + \alpha(\pi - \pi^e)$$

and

$$\max U^P = -(\pi - \pi^e)^2,$$

[14] The cheating solution is also referred to as "cheating outcome with surprise inflation" [Sorensen and Whitta-Jacobsen 2010:661] or cheating solution [Heijdra and van der Ploeg 2002:241] or "fooling" solution [Blanchard and Fischer 1989].

[15] This equilibrium solution, Kydland and Prescott's [1977] "consistent equilibrium," is also referred to as "time-consistent rational expectations equilibrium" [Sorensen and Whitta-Jacobsen 2010:662] or equilibrium under discretion [Walsh 2010:279-280], solution under discretion [Heijdra and van der Ploeg 2002:239], or "consistent" solution [Kydland and Prescott 1977].

or equivalently

$$\left.\begin{array}{l} \max\limits_{y,\pi} \left\{U = -a(y - y^T)^2 - b(\pi - \pi^T)^2\right\} \\ s.t. \quad y = \bar{y} + \alpha(\pi - \pi^e) \\ \qquad \pi = \pi^e \\ \qquad \pi^e \equiv E\pi \end{array}\right\} , \tag{P3}$$

the equilibrium solution is given by point E, the intersection between the *OP* line and the *REE* line in the $\pi^e - \pi$ diagram, or the intersection between the *OP* line $\pi(y)$ and the *REE* line or the *LRAS* in $y - \pi$ diagram, or the equality between MB_π and MC_π (see Figures 4a, 4b, and 4c).

The inflation rate chosen optimally by the policymaker is a function of inflationary expectations. Since expectations are rational, and since the private sector or the wage setters know that inflation rate will be set based on equation (6.2), the equilibrium expected inflation rate must satisfy

$$\begin{aligned} \pi^e &= E\pi \\ &= E\left[\frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2}\pi^e\right] \end{aligned}$$

Using equation (6.2) for π and noting that $E\pi^T = \pi^T$, $E\bar{y} = \bar{y}$, and $E(\pi^e) = E(E\pi) = E\pi = \pi^e$,

$$\pi^e = E\underbrace{\left[\frac{b}{(b + a\alpha^2)}\pi^T + \frac{a\alpha}{(b + a^2)}(y^T - \bar{y}) + \frac{a\alpha^2}{(b + a\alpha^2)}\pi^e\right]}_{\text{using (6.2) for } \pi}$$

$$\pi^e\left(1 - \frac{a\alpha^2}{b + a\alpha^2}\right) = \frac{b}{(b + a\alpha^2)}\pi^T + \frac{a\alpha}{(b + a\alpha^2)}(y^T - \bar{y})$$

$$\pi^e = \frac{b + a\alpha^2}{b} \cdot \left[\frac{b}{(b + a\alpha^2)}\pi^T + \frac{a\alpha}{(b + a\alpha^2)}(y^T - \bar{y})\right]$$

yields the equilibrium expected inflation rate

$$\pi_E^e = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}). \tag{9.1}$$

Substitution of (9.1) into (6.2) yields the equilibrium inflation rate

$$\pi_E = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}), \tag{9.2}$$

which is the same as the equilibrium expected inflation rate (9.1). The equilibrium actual inflation rate relative to the target inflation rate is higher: (i) the greater is the wedge between the target output level and the full employment level; (ii) the greater is *a*/*b*, the weight the policymaker put on output stabilization relative to inflation stabilization; and (iii) the greater the value of α.

Using equations (1), (9.1), and (9.2), the equilibrium level of output is

$$y_E = \bar{y}. \tag{9.3}$$

Comparing the different possible outcomes. Comparing the results

$$\overbrace{\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})}^{\pi_E} > \overbrace{\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})}^{\pi_C} > \overbrace{\pi^T}^{\pi_R} > 0, \tag{10.1}$$

$$\overbrace{\bar{y} + \left(\frac{a\alpha^2}{b + a\alpha^2}\right)(y^T - \bar{y})}^{y_C} > \overbrace{\bar{y}}^{y_E = y_R} > 0, \tag{10.2}$$

$$0 < \overbrace{-a\left(\frac{b + a\alpha^2}{b}\right)(y^T - \bar{y})^2}^{U_E} < \overbrace{-a(y^T - \bar{y})^2}^{U_R} < \overbrace{-a\left(\frac{b}{b + a\alpha^2}\right)(y^T - \bar{y})^2}^{U_C}, \tag{10.3}$$

$$\overbrace{0}^{U_E^P = U_R^P} < \overbrace{-\left(\frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})\right)^2}^{U_C^P}, \tag{10.4}$$

i.e., in terms of the utility of the policymaker, the third-best outcome is the equilibrium outcome (point *E*), the second-best outcome is the "rules" or "commitment" outcome (point *R*), and the first-best outcome is the cheating outcome (point *C*), as shown in equation (10.3) and Figure 4.

The inflation bias, given by

$$\begin{aligned} \text{inflation bias} &\equiv \pi_E - \pi_R \\ &= \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) - \pi^T \\ &= \frac{a\alpha}{b}(y^T - \bar{y}) \end{aligned}$$

arises in this model because the policymaker has an incentive to cheat

$$incentive\ to\ cheat \equiv U_C - U_R = -a(b/(b + a\alpha^2))(y^T - \bar{y})^2 - (-a)(\bar{y} - y^T)^2$$
$$= (-ab + a(b + a\alpha^2))/(b + a\alpha^2)(y^T - \bar{y})^2 > 0$$
$$= (a\alpha^2)/(b + \alpha^2)(y^T - \bar{y})^2 > 0$$

due to his/her inability to precommit to its target inflation rate once the private sector have formed its expectations. The incentive (or temptation) to cheat is larger, and thus the amount of inflation bias will be larger, the larger is α, the larger is α, the smaller is b, and the larger is $(y^T - \bar{y})$.

There will be inflation that is higher than π^T as a result of the optimizing behavior of the policymaker and the maximizing behavior of the private sector. Had the policymaker chosen to stick to the rule ($\pi = \pi_R = \pi^T$) and had the private sector expected this ($\pi^e = \pi^e_R = \pi^T$), the private sector would have the same payoff ($U^P_R = U^P_E = 0$), but the policymaker would have a higher payoff ($U_R > U_E$). Why would the policymaker choose to have a higher inflation rate? It cannot credibly commit itself to what is the optimal plan of inflation because this plan is not time consistent.

3. The inflation-output example when the policymaker's utility function is quasi-linear

The model is the same as in the previous section, but now equation (4) is replaced by

$$U_t = a(y_t - y^T) - b(\pi_t - \pi^T)^2, \qquad a, b > 0,\ y^T, \pi^T > 0, \tag{11}$$

i.e., it assumed now that the utility function of the policymaker is quasi-linear.[16] The policymaker likes to have output that is higher than the full employment level and, as in the quadratic case, dislikes inflation deviations, whether positive or negative, from the target/desired level.

Using equations (11) and (1'), either the method of equating slopes which give

$$\underbrace{\frac{a}{2b(\pi - \pi^T)}}_{\left(\frac{d\pi}{dy}\right)_{IC}} = \underbrace{\frac{1}{\alpha}}_{\left(\frac{d\pi}{dy}\right)_{E}}, \tag{12}$$

[16] It does not matter for decisions whether U is linear in $y - y^T$, or $y - \bar{y}$, or y.

or the substitution method which give

$$\frac{\partial U}{\partial \pi}=0: \qquad \underbrace{a\alpha}_{MB_\pi=\frac{\partial U}{\partial y}\cdot\frac{\partial y}{\partial \pi}} + \underbrace{-2b(\pi-\pi^T)}_{(-)MC_\pi=\frac{\partial U}{\partial \pi}}=0\,, \tag{13}$$

yields the equation for *OP* line:

$$\pi=\pi^T+(1/2)\,(a/b)\alpha\,. \tag{14}$$

In this case, the optimal π is independent of y and therefore π^e (via 1). This means that the *OP* line is a horizontal line either in the $y-\pi$ diagram (see Figure 5a) or the $\pi^e-\pi$ diagram (Figure 5b).

Thus, equation (11) has a special property. No matter what π^e (or y via 1) is, or where the position of the *SRAS* is, the policymaker always maximizes his/her utility by generating the same π.

Notice also that in the $MB_\pi, MC_\pi-\pi$ diagram (see Figure 5c and using equation 13)

$$MB_\pi=a\alpha\,, \tag{13.1}$$

$$\pi=\underbrace{\pi^T}_{\text{v.intercept}}+\underbrace{\frac{1}{2b}}_{slope}MC\,. \tag{13.2}$$

Equating (13.1) and (13.2) also yields the optimal π (12).

To derive graphically the OP line in the $y-\pi$ diagram (Figure 5a), consider an increase in π. As π increases, the SRAS shifts to the left and, the optimal y is higher but the optimal π remains the same; the points of tangencies yield the horizontal OP line. What this implies in the $\pi^e-\pi$ diagram (Figure 5b) is that the OP line is also a horizontal line since π increases, but the optimal π remains the same. In MB_π, $MC_\pi-\pi$ diagram (Figure 5c), as π increases, the MC_π curve does not shift as in the quadratic case, but the MB_π curve also does not shift unlike in the quadratic case, and therefore the optimal π, given by their point of intersection also remains the same. This is why both points C and E lie on the intersection point of the MB_π and MC_π curves.

In summary, in a quadratic (quasi-linear) case: (i) the *OP* line is downward sloping (horizontal) in the $y-\pi$ diagram and upward sloping (also horizontal) in the $\pi^e-\pi$ diagram, and (ii) the MB_π curve is downward sloping (vertical) in the MB_π, $MC_\pi-\pi$ diagram and shifts (does not shift) as π^e changes.

The procedure for deriving the solution values for π, y, U, and U^P under different possible outcomes is the same as that presented in the previous section. We can therefore simply summarize the results as follows:

$$\underbrace{\pi_C = \pi_E = \pi_O}_{\pi^T + (1/2)(a/b)\alpha} > \underbrace{\pi_R}_{\pi^T}, \tag{14.1}$$

$$\underbrace{y_C}_{\bar{y}+(1/2)(a/b)\alpha^2} > \underbrace{y_E = y_R}_{\bar{y}} > \underbrace{y_O}_{\bar{y}-(1/2)(a/b)\alpha^2}, \tag{14.2}$$

$$\underbrace{-a(\bar{y}-y^T)^2 - ((a/b)\alpha^2(\bar{y}-y^T)) - b((1/2)(a/b)\alpha)^2}_{U_O} < \underbrace{-a(y^T-\bar{y}) - b((1/2)(a/b)\alpha)^2}_{U_E} < \underbrace{-a(y^T-\bar{y})}_{U_R} < \underbrace{-a(y^T-\bar{y}) + (1/2)(a/b)a\alpha(1/2)^2}_{U_C} \tag{14.3}$$

$$\underbrace{U_O^P}_{-(-(1/2)(a/b)\alpha)^2} = \underbrace{U_C^P}_{-(1/2)(a/b)\alpha)^2} < \underbrace{U_E^P}_{0} = \underbrace{U_R^P}_{0}, \tag{14.4}$$

These results are qualitatively similar to those in the quadratic case. Note that another outcome scenario, $\pi^e > \pi$ where $\pi_0^e = \pi^1 + (1/2)(a/b)\alpha$, $\pi_0 = \pi^1$, and subscript O stands for over-expected. Point O does not lie in either the *OP* line or the *REE* line (see Figure 5). Finally, using (14.3)

$$\text{incentive to cheat} \equiv U_C - U_R = (1/2)(a/b)a\alpha^2(1/2) > 0$$

and, using (14.1)

$$\text{inflation bias} \equiv \pi_E - \pi_R = (1/2)(a/b)\alpha > 0$$

The incentive to cheat is larger and, thus, the amount of inflation bias will be larger, the larger is α, the larger is a, and the smaller is b as in the quadratic case but are no longer affected by $(y^T - \bar{y})$ unlike in the quadratic case.

The four possible outcomes—points R, C, O and E in Figure 5—are also analyzed using Table 1 below along with equations (14.3) and (14.4). Essentially, it is a game between policymaker who chooses the inflation rate and the utility the private sector which chooses the expected inflation rate, where the payoff of each depends on the action of the other. It is assumed that both the policymaker and the private sector have common knowledge and common rationality.

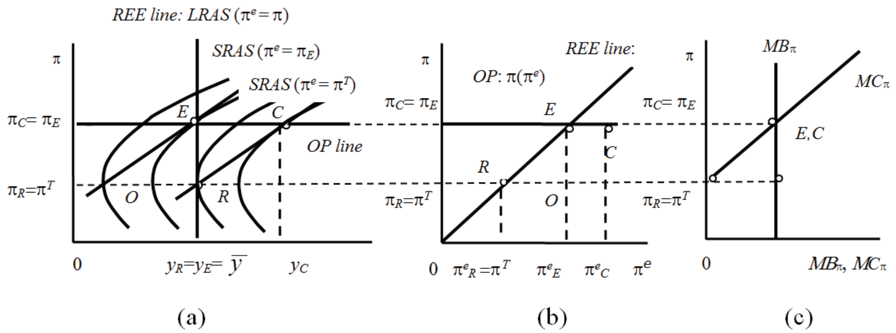

FIGURE 5. Possible outcomes, quasi-linear case

TABLE 1

		Policymaker	
		$\pi = \pi^T$	$\pi = \pi^T + (1/2)(a/b)\alpha$
Private sector	$\pi^e = \pi^T$	$U^P_R > U_R$	$U^P_C > U_C$
	$\pi^e = \pi^T + (1/2)(a/b)\alpha$	$U^P_O > U_O$	$U^P_E > U_E$

This is a game with sequential moves, with the private sector moving first, but since the equilibrium solution is derived by backward induction, it is as if the policymaker is the one moving first. The policymaker would reason that if the private sector chooses $\pi^e = \pi^T$, then he/she should choose $\pi = \pi^T + (1/2)(a/b)\alpha$ since $U_C > U_R$ but if the private sector chooses $\pi^e = \pi^T + (1/2)(a/b)\alpha$, then he/she should choose $\pi = \pi^T + (1/2)(a/b)\alpha$ since $U_E > U_O$. So no matter what the private sector expects, the policymaker would choose $\pi = \pi^T + (1/2)(a/b)\alpha$. Knowing this, the private sector would choose $\pi = \pi^T + (1/2)(a/b)\alpha$ since $U^P_E > U^P_C$.

4. Rules versus discretion in the inflation-output example

In this inflation-output example, the policymaker, whether under rules or discretion, attempts to optimize; specifically, he/she maximizes his/her objective function subject to the *LRAS* in the case of rules and subject to the *SRAS* in the case of discretion. Whether the policymaker’s preferences are quadratic or quasi-linear, policy conducted according to a rule (chosen to pertain to a large number of periods) would lead to a $\pi = \pi_R = \pi^T$ for all *t*. In contrast, policy choices made in a discretionary period-by-period manner would lead, for all *t*, to $\pi = \pi_E = \pi^T + (1/2)(a/b)\alpha$ in the quasi-linear case and $\pi = \pi_E = \pi^T + (1/2)(a/b)\alpha(y^T - \bar{y})$.

According to McCallum [1989], “the rules versus discretion distinction centers on the process by which the π value is determined, not what those values turn out to be”. Specifically, under discretion, the policymaker chooses each period *t*'s value of π, on the basis of period *t*’s optimization; this means that the policymaker optimizes in period *t* so as to choose period *t*’s (single period's) π. Thus, a discretionary policy is conducted on a period-by-period basis, and therefore the

π choices in the different periods are independent from each other. By contrast, in the case of a rule, the policymaker optimizes so as to choose a π rule to be applicable for a large number of periods, not just the current period, a π which he/she merely implements in each period. In short, "rule-type policymaking involves implementation in each period of a formula designed to apply to periods in general, while discretionary policymaking involves freshly made decisions in each period" [McCallum 1989].

5. Conclusion

This paper has attempted to provide the algebra and a panel diagram for the "time inconsistency: the Phillips curve example," the most popular example in the literature when introducing the concept. Specifically, the three kinds of diagrams used in the literature—the IC-LRAS-SAS in $y - \pi$ diagram, the $\pi^e = \pi$ line and the *OP* line in the $\pi^e - \pi$ diagram, and the MB_π, $MC_\pi - \pi$ diagram—are placed as panels in a single diagram.

The resulting panel diagram is used to analyze the different possible outcomes, depending on the scenarios: rule or pre-commitment, cheating, and equilibrium. It is shown that: (i) the rule or pre-commitment solution is the long-run optimal point (a point of tangency between the *LRAS* and an indifference curve) and a *REE* point, but not a short-run optimal point since it is off the *OP* line (not a point of tangency between *SRAS* and indifference curve and an outcome where $MB_\pi > MC_\pi$; (ii) the cheating solution is a point on the *OP* line (a tangency between another *SRAS* and another indifference curve and thus a $MB_\pi = MC_\pi$ point) but *off* the *REE* line, i.e., it is short-run optimal but rational expectations equilibrium does not hold; and (iii) the equilibrium solution is a point on the *OP* line (a tangency between still another SRAS and sill another indifference curve and thus another $MB_\pi = MC_\pi$ point) and on the *REE* line.

In the literature, the rule solution is labeled as time inconsistent while the equilibrium solution is labeled as time consistent. The terms "time inconsistency" or "dynamic inconsistency" are a misnomer because there is actually no inconsistency in the so called "time inconsistency" or "dynamic inconsistency" problem. Choices are different because the policymaker faces different constraints depending on the private sector's on inflationary expectations, as shown in the panel diagrams. Different optimization problems yield different solutions or, graphically, different points of intersections. That the terms "time inconsistency" or "dynamic inconsistency" are a misnomer is even more visible when the possible outcomes are analyzed using simple game theory concepts.

Finally, the terms "time inconsistency" or "dynamic inconsistency" in this Phillips curve example are misleading because the model is a one-period model and therefore a static model.

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Appendix A

Derivation of the OP line

The policymaker chooses output and inflation to maximize

$$U_t = -a(y_t - y^T)^2 - b(\pi_t - \pi^T)^2, \qquad 0 < a, b < \infty, y^T > \bar{y} > 0, \pi^T > 0 \quad \text{(A1)}$$

subject to

$$y_t = \bar{y} + \alpha(\pi_t - \pi_t^e) \qquad \alpha > 0 \ . \quad \text{(A2)}$$

The solution to this problem yields the condition for optimal policy (*OP*). The *OP* line can be illustrated in either the $y - \pi$ diagram or the $\pi^e = \pi$ diagram.

The alternative ways to get the equation for the *OP* line follow: the method of equating slopes, the method of equating marginal benefit and marginal cost, the substitution method, and the Lagrangian method, as shown below.

Method of equating slopes. The slope of the SAS can easily be seen by rewriting (A2) as

$$\pi_t = \underbrace{\left(\pi_t^e - \frac{1}{\alpha}\bar{y}\right)}_{\text{v intercept}} + \underbrace{\left(\frac{1}{\alpha}\right)}_{slope} y_t \ . \quad \text{(A7)}$$

The slope of the indifference ellipse is found by setting the total differential of U (A1) equal to zero

$$dU = 0 = -2a(y - y^T)dy - 2b(\pi - \pi^T)d\pi,$$

$$\Rightarrow \left(\frac{d\pi}{dy}\right)_{dU=0} = -\frac{a(y - y^T)}{b(\pi - \pi^T)}. \quad \text{(A3)}$$

Equating the slope of the *SRAS* (from (A7) and the slope of the indifference curve (A3)

$$\underbrace{-\frac{a(y - y^T)}{b(\pi - \pi^T)}}_{\left(\frac{d\pi}{dy}\right)_{IC}} = \underbrace{\frac{1}{\alpha}}_{\left(\frac{d\pi}{dy}\right)_{\varepsilon}},$$

yields

$$y - y^T = \frac{b}{a\alpha}(\pi - \pi^T) \Leftrightarrow \pi = \underbrace{\pi^T + \frac{a\alpha}{b}y^T}_{\text{v intercept} > 0} \underbrace{- \frac{a\alpha}{b}}_{\text{slope} < 0} y \quad \text{(A4.1)}$$

which is the equation for *OP* line in the $y - \pi$ diagram.

The $\pi(y)$ equation (A4.1) as the equation for the *OP* line can be converted into $\pi(\pi^e)$ equation. Combining (A4.1) and (A2)

$$\pi = \pi^T + \frac{a\alpha}{b} y^T - \frac{a\alpha}{b}(\overbrace{\bar{y} + \alpha(\pi - \pi^e)}^{=y,\ \text{using (1)}}) ,$$

and rearranging/simplifying

$$\pi(1 + \frac{a\alpha^2}{b}) = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) + \frac{a\alpha^2}{b}\pi^e$$

yields the equation for the *OP* line in $\pi^e - \pi$ diagram

$$\pi = \underbrace{\frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})}_{\text{v intercept}>0} + \underbrace{\frac{a\alpha^2}{b + a\alpha^2}}_{0<\text{slope}<1}\pi^e \tag{A4.2}$$

.

Lagrangian method. The Lagrangian of the problem is

$$L = -a(y - y^T)^2 - b(\pi - \pi^T)^2 + \lambda[\bar{y} + \alpha(\pi - \pi^e) - y], \tag{A5}$$

and the first-order necessary conditions are

$$\frac{\partial L}{\partial \pi} = -2b(\pi - \pi^T) + \lambda\alpha = 0 , \tag{A5.1}$$

$$\frac{\partial L}{\partial y} = -2a(y - y^T) - \lambda = 0 , \tag{A5.2}$$

$$\frac{\partial L}{\partial \lambda} = \bar{y} + \alpha(\pi - \pi^e) - y = 0 , \tag{A5.3}$$

where λ is the Lagrange multiplier. Plugging in the λ implied by (A5.2) into (A5.1)

$$y - y^T = \frac{b}{a\alpha}(\pi - \pi^T) \quad \Leftrightarrow \quad \pi = \underbrace{\pi^T + \frac{a\alpha}{b}y^T}_{\text{v intercept}} \underbrace{- \frac{a\alpha}{b}}_{\text{slope}} y ,$$

also yields (A4.1), the equation for the *OP* line in the $y - \pi$ diagram. (A5.3) recovers the constraint (A2).

Again, the $\pi(y)$ equation (A4.1) as the equation for the *OP* line can be converted into $\pi(\pi^e)$ equation (A4.2). Substituting (A1.3) or (A2) for y in the $\pi(y)$ equation (A4.1) yields the $\pi(\pi^e)$ equation (A4.2).

Substitution method. In this method, the constrained maximization problem is converted into an unconstrained maximization problem. Using (A2) to substitute out for y in (A1), the policymaker's problem becomes

$$\max_{\pi} U = -a((\overbrace{\bar{y} + \alpha(\pi - \pi^e)}^{=y,\text{ using (1)}}) - y^T)^2 - b(\pi - \pi^T)^2$$

and the condition for maximization is

$$\frac{\partial U}{\partial \pi} = 0: \quad \underbrace{-2a\alpha(\alpha(\pi - \pi^e) - (y^T - \bar{y})}_{=\frac{\partial U}{\partial y}\cdot\frac{\partial y}{\partial \pi}} + \underbrace{-2b(\pi - \pi^T)}_{=\frac{\partial U}{\partial \pi}} = 0, \tag{A6}$$

where

$$\frac{\partial U}{\partial y}\cdot\frac{\partial y}{\partial \pi} \equiv MB_\pi = -2a\alpha(\alpha(\pi - \pi^e) - (y^T - \bar{y}) \quad \Leftrightarrow$$

$$\pi = \underbrace{\pi^e + \frac{1}{\alpha}(y^T - \bar{y})}_{\text{v intercept}} \underbrace{-\frac{1}{2a\alpha^2}}_{slope} MB_\pi, \tag{A6.1}$$

$$\frac{\partial U}{\partial \pi} \equiv (-)MC_\pi = -2b(\pi - \pi^T) \Leftrightarrow \pi = \underbrace{\pi^T}_{\text{v.intercept}} + \underbrace{\frac{1}{2b}}_{slope} MC_\pi \tag{A6.2}$$

and $(\partial U/\partial y)(\partial y/\partial \pi) = MB_\pi$ = marginal utility, or marginal benefit of higher inflation through higher output while $\partial U/\partial \pi = (\text{-})MC_\pi$ = marginal utility of higher inflation, or the negative of the marginal cost of higher inflation.

Simplifying (A6)

$$2a\alpha[-\alpha(\pi - \pi^e) + (y^T - \bar{y})] = 2b(\pi - \pi^T)$$

$$-(a/b)\alpha^2(\pi - \pi^e) + (a/b)\alpha(y^T - \bar{y}) = (\pi - \pi^T)$$

$$\pi(1 + (a/b)\alpha^2) = \pi^T + (a/b)\alpha(y^T - \bar{y}) + (a/b)\alpha^2\pi^e$$

$$\pi\left(\frac{b + a\alpha^2}{b}\right) = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) + \frac{a\alpha^2}{b}\pi^e$$

$$\pi = \frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})\frac{a\alpha^2}{b + a\alpha^2}\pi^e,$$

also yields the $\pi(\pi^e)$ equation (A4.2).

The $\pi(\pi^e)$ equation (A4.2) as the equation for the *OP* line can also be converted into $\pi(y)$ equation (A4.1). Using the *SRAS* (A2) to substitute for π^e in (A4.2)

$$\pi = \frac{b}{b + a\alpha^2}ð^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2}\overbrace{[\pi - (1/\alpha)(y - \bar{y})]}^{\pi^e \text{ from (A2)}}$$

and simplifying

$$\pi\left(1-\frac{a\alpha^2}{b+a\alpha^2}\right)=\frac{b}{b+a\alpha^2}\pi^T+\frac{a\alpha}{b+a\alpha^2}y^T-\frac{a\alpha}{b+a\alpha^2}y$$

$$\pi\left(\frac{b}{b+a\alpha^2}\right)=\frac{b}{b+a\alpha^2}\pi^T+\frac{a\alpha}{b+a\alpha^2}(y^T-y)$$

$$\pi=\pi^T+\frac{a\alpha}{b}(y^T-y)$$

yields the $\pi(y)$ equation (A4.1).

Equating marginal benefit and marginal cost. Notice from (A3) that the $\pi(\pi^e)$ equation (A4.2) can be derived by equating the MB_π equation (A6.1) and the MC_π equation (A6.2):

$$\underbrace{-2a\alpha(\alpha(\pi-\pi^e)-(y^T-\bar{y})}_{MB_\pi}=\underbrace{2b(\pi-\pi^T)}_{MB_\pi}$$

$$\pi=\frac{b}{b+a\alpha^2}\pi^T+\frac{a\alpha}{b+a\alpha^2}(y^T-\bar{y})\frac{a\alpha^2}{b+a\alpha^2}\pi^e.$$

The private sector's REE line and the vertical LRAS. The private sector chooses the expected inflation rate π^e so as to maximize

$$U_t^P=-(\pi_t-\pi_t^e)^2, \tag{A7}$$

and its utility is maximized when

$$\partial U^P/\partial\pi=-2(\pi-\pi^e)=0 \Leftrightarrow \pi^e=\pi \tag{A7.1}$$

where

$$\pi_t^e\equiv E_{t-1}\pi_t. \tag{A7.1}$$

(3.1) is the rational expectations equilibrium (*REE*), or perfect foresight equilibrium while (A7.1) is the *REE* assumption. In a $\pi^e-\pi$ diagram, the *REE* line is the $\pi^e=\pi$ line or the 45° line; in a $y-\pi$ diagram, the *REE* line is the same as the vertical *LRAS*.

Appendix B

Solutions for π, y, U, and U^P under the different scenarios

The solution for the optimal values of π and y are derived using, respectively, (A4.2) and (A2). The value of U is derived using (A1) while the value U^P is derived from (A7).

Rule or commitment (*R*). In this scenario

$$\pi^e = \pi_R^e = \pi^T \ , \quad \text{(B1.1)}$$

$$\pi = \pi_C = \pi^T . \quad \text{(B1.2)}$$

which, when substituted into the *SRAS* (A2), yields the output under commitment:

$$y_R = \bar{y} + \alpha(\overbrace{\pi^T}^{\pi=\pi_R} - \overbrace{\pi^T}^{\pi^e=\pi_R^e}) = \bar{y} \cdot \quad \text{(B1.3)}$$

The utility of the policymaker, using (A1), (B1.1), and (B1.3), is

$$U_R = -a(\overbrace{\bar{y}}^{y=y_R} - y^T)^2 - b(\overbrace{\pi^T}^{\pi=\pi_R} - \pi^T)^2 = -a(\bar{y} - y^T)^2, \quad \text{(B1.4)}$$

while the utility of the private sector, using (A7), (B1.1), and (B1.2), is

$$U_R^P = -(\overbrace{\pi^T}^{\pi=\pi_R} - \overbrace{\pi^T}^{\pi^e=\pi_R^e})^2 = 0 \ \cdot \quad \text{(B1.5)}$$

Note that in this scenario, the problem is reduced to maximizing (A1) subject to the vertical *LRAS* (A7.1)

$$\left.\begin{array}{l} \max\limits_{\pi} \left\{U = -a(y - y^T)^2 - b(\pi - \pi^T)^2\right\} \\ s.t. \quad y = \bar{y} \end{array}\right\}, \quad \text{(P1)}$$

which yields the optimal choice $\pi = \pi^T$

Cheating (*C*). In this scenario, the problem is given by

$$\left.\begin{array}{l} \max\limits_{y,\pi} \left\{U = -a(y - y^T)^2 - b(\pi - \pi^T)^2\right\} \\ s.t. \quad y = \bar{y} + \alpha(\pi - \pi^e) \\ \qquad \pi^e = \pi^T \end{array}\right\}. \quad \text{(P2)}$$

This is so because the private sector's expectation is given by

$$\pi^e = \pi_C^e = \pi^T, \tag{B2.1}$$

but the policymaker will choose π based on the π(π^e^) equation (A4.2), the equation for the OP line. Substituting (B2.1) into (A4.2) yields the solution for the inflation rate under cheating

$$\begin{aligned}\pi_C &= \frac{b}{b+a\alpha^2}\pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T-\bar{y}) + \frac{a\alpha^2}{b+a\alpha^2}\overbrace{(\pi^T)}^{\pi^e=\pi_C^e} \\ &= \pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T-\bar{y})\end{aligned} \tag{B2.2}$$

Substituting (B2.1) and (B2.2) into (A2) gives the solution for output under cheating

$$\begin{aligned}y_C &= \bar{y} + \alpha\left(\overbrace{\left(\pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T-\bar{y})\right)}^{\pi=\pi_C} - \overbrace{\pi^T}^{\pi^e=\pi_C^e}\right) \\ &= \bar{y} + \frac{a\alpha^2}{b+a\alpha^2}(y^T-\bar{y})\end{aligned} \tag{B2.3}$$

Using (A1), (B2.2) for π, and (B2.3) for y, the utility of the policymaker under cheating is

$$\begin{aligned}U_C &= -a\left(\overbrace{\left(\bar{y} + \frac{a\alpha^2}{b+a\alpha^2}(y^T-\bar{y})\right)}^{y=y_C} - y^T\right)^2 - b\left(\overbrace{\left(\pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T-\bar{y})\right)}^{\pi=\pi_C} - \overbrace{\pi^T}^{\pi^e=\pi_C^e}\right)^2 \\ &= -a\left(\left(\frac{a\alpha^2}{b+a\alpha^2}-1\right)(y^T-\bar{y}\right)^2 - b\left(\left(\frac{a\alpha}{b+a\alpha^2}\right)(y^T-\bar{y})\right)^2 \\ &= -\left[a\left(\frac{-b}{b+a\alpha^2}\right)^2 + b\left(\frac{a\alpha}{b+a\alpha^2}\right)^2\right](y^T-\bar{y})^2 \\ &= -\left(\frac{b^2 + b(a\alpha)^2}{(b+a\alpha^2)^2}\right)(y^T-\bar{y})^2 \\ &= -a\left(\frac{b(b+a\alpha^2)}{(b+a\alpha^2)^2}\right)(y^T-\bar{y})^2\end{aligned}$$

and thus

$$U_C = -a\left(\frac{b}{b+a\alpha^2}\right)(y^T - \bar{y})^2 . \tag{B2.4}$$

And, using (A7), (B2.1), and (B2.2), the utility of the private sector is given by

$$\begin{aligned} U_C^P &= -(\overbrace{\pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T - \bar{y})}^{\pi=\pi_C} - \overbrace{\pi^T}^{\pi^e=\pi_C^e})^2 \\ &= -\left(\frac{a\alpha}{b+a\alpha^2}(y^T - \bar{y})\right)^2 < 0 \end{aligned} \tag{B2.5}$$

Equilibrium (*E*). In this scenario, the problem is given by

$$\left.\begin{aligned} &\max_{y,\pi} \left\{U = -a(y - y^T)^2 - b(\pi - \pi^T)^2\right\} \\ &s.t. \quad y = \bar{y} + \alpha(\pi - \pi^e) \\ &\qquad \pi = \pi^e \\ &\qquad \pi^e \equiv E\pi \end{aligned}\right\} , \tag{P3}$$

or, equivalently, the policymaker's problem is

$$\max_{y,\pi} \left\{U = -a(y - y^T)^2 - b(\pi - \pi^T)^2\right\} \quad s.t. \; y = \bar{y} + \alpha(\pi - \pi^e) \tag{P3.1}$$

and the private sector's problem is

$$\max U^P = -(\pi - \pi^e)^2 \text{ where } \pi^e = E\pi . \tag{P3.2}$$

The inflation rate chosen optimally by the policymaker is a function of inflationary expectations. Since expectations are rational, and since the private sector knows that inflation rate will be set based on (A4.2), the equilibrium expected inflation rate must satisfy

$$\begin{aligned} \pi^e &= E\pi \\ &= E\left[\frac{b}{b+a\alpha^2}\pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T - \bar{y}) + \frac{a\alpha^2}{b+a\alpha^2}\pi^e\right] \end{aligned}$$

Using (A4.2) for π and noting that $E\pi^T = \pi^T$, $E\bar{y} = \bar{y}$, and $E(\pi^e) = E(E\pi) = E\pi = \pi^e$,

$$\pi^e = \underbrace{E\left[\frac{b}{(b+a\alpha^2)}\pi^T + \frac{a\alpha}{(b+a^2)}(y^T - \bar{y}) + \frac{a\alpha^2}{(b+a\alpha^2)}\pi^e\right]}_{\text{using (6.2) for } \pi}$$

$$\pi^e\left(1 - \frac{a\alpha^2}{b+a\alpha^2}\right) = \frac{b}{(b+a\alpha^2)}\pi^T + \frac{a\alpha}{(b+a\alpha^2)}(y^T - \bar{y})$$

$$\pi^e = \frac{b+a\alpha^2}{b}.\left[\frac{b}{(b+a\alpha^2)}\pi^T + \frac{a\alpha}{(b+a\alpha^2)}(y^T - \bar{y})\right]$$

yields the equilibrium expected inflation rate

$$\pi_E^e = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}). \tag{B3.1}$$

Substitution of (B3.1) into (A4.2)

$$\pi_E = \frac{b}{b+a\alpha^2}\pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T - \bar{y}) + \frac{a\alpha^2}{b+a\alpha^2}\underbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})\right)}_{=\pi^e = \pi_E^e}$$

$$= \left(\frac{b+a\alpha^2}{b+a\alpha^2}\right)\pi^T + \left(\frac{b a\alpha +}{b(b+a\alpha^2)} + \frac{a\alpha^2(a\alpha)}{b(b+a\alpha^2)}\right)(y^T - \bar{y})$$

$$= \pi^T + \left(\frac{a\alpha(b+a\alpha^2)}{b(b+a\alpha^2)}\right)(y^T - \bar{y})$$

yields the equilibrium inflation rate

$$\pi_E = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}). \tag{B3.2}$$

which is the same as the equilibrium expected inflation rate (B3.1).

Using (A2), (B3.1), and (B3.2)

$$y_E = \bar{y} + \alpha\left(\overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})\right)}^{\pi = \pi_E} - \overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})\right)}^{\pi^e = \pi_E^e}\right)$$

the equilibrium level of output is

$$y_E = \bar{y}. \tag{B3.3}$$

Using (A1), (B3.3), (B3.2), and (B3.1)

$$U = -a(\overbrace{\bar{y}}^{y=y_E} - y^T)^2 - b\left(\overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})\right)}^{\pi=\pi_E} - \overbrace{\pi^T}^{\pi^e=\pi^e_E}\right)^2$$

$$= -(a + \frac{b(a\alpha)^2}{b^2})(\bar{y} - y^T)^2$$

$$= -\frac{ab^2 + b(a\alpha)^2}{b^2}(\bar{y} - y^T)^2$$

$$= -ab\left(\frac{b + a\alpha^2}{b^2}\right)(\bar{y} - y^T)^2$$

$$= -ab\left(\frac{b + a\alpha^2}{b^2}\right)(y^T - \bar{y})^2$$

and thus the utility of the policymaker is

$$U_E = -a\left(\frac{b + a\alpha^2}{b}\right)(y^T - \bar{y})^2 \, . \qquad \text{(B3.4)}$$

Using (A1), (B3.1), and (B3.2), the utility of the private sector is

$$U^P_E = -\left(\overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})\right)}^{\pi=\pi_E} - \overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})\right)}^{\pi^e=\pi^e_E}\right)^2 = 0 \, . \qquad \text{(B3.5)}$$

Comparing the different possible inflation-output outcomes. Comparing the results

$$\overbrace{\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})}^{\pi_E} > \overbrace{\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})}^{\pi_C} > \overbrace{\pi^T}^{\pi_R} > 0 \, , \qquad \text{(B4.1)}$$

$$\overbrace{\bar{y} + \left(\frac{a\alpha^2}{b + a\alpha^2}\right)(y^T - \bar{y})}^{y_C} > \overbrace{\bar{y}}^{y_E=y_R} > 0 \, , \qquad \text{(B4.2)}$$

$$0 < \overbrace{-a\left(\frac{b + a\alpha^2}{b}\right)(y^T - \bar{y})^2}^{U_E} < \overbrace{-a(y^T - \bar{y})^2}^{U_R} < \overbrace{-a\left(\frac{b}{b + a\alpha^2}\right)(y^T - \bar{y})^2}^{U_C} \, , \qquad \text{(B4.3)}$$

$$\overbrace{0}^{U_E^P = U_R^P} < \overbrace{-\left(\frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})\right)^2}^{U_C^P}. \tag{B4.4}$$

Inflation bias and incentive to cheat. The inflation bias

$$\begin{aligned} \text{inflation bias} &\equiv \pi_E - \pi_R \\ &= \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) - \pi^T \\ &= \frac{a\alpha}{b}(y^T - \bar{y}) \end{aligned} \tag{B5.1}$$

arises in this model because the policymaker has an incentive to cheat

$$\begin{aligned} \textit{incentive to cheat} \equiv U_C - U_R &= -a(b/(b + a\alpha^2))(y^T - \bar{y})^2 - (-a)(\bar{y} - y^T)^2 \\ &= (-ab + a(b + a\alpha^2))/(b + a\alpha^2)\,(y^T - \bar{y})^2 > 0 \\ &= (a\alpha^2)/(b + \alpha^2)(y^T - \bar{y})^2 > 0 \end{aligned} \tag{B5.2}$$