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Common Belief Revisited

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Abstract

This study presents how selection of equilibrium in a game with many equilibria can be made possible when the common knowledge assumption (CKA) is replaced by the notion of common belief. Essentially, this idea of pinning down an equilibrium by weakening the CKA is the central feature of the *global game* approach which introduces a natural perturbation on games with complete information. We argue that since common belief is another form of departure from the CKA, it can also obtain the results attained by the global game framework in terms of selecting an equilibrium. We provide here necessary and sufficient conditions.

Following the program of weakening the CKA, we weaken the notion of common belief further to provide a less stringent and a more natural way of *believing* an event. We call this belief process as iterated quasi-common p-belief which is a generalization to many players of a two-person iterated p-belief. It is shown that this converges with the standard notion of common p-belief at a sufficiently large number of players. Moreover, the *agreeing to disagree* result in the case of beliefs (Monderer & Samet, 1989 and Neeman, 1996a) can also be given a generalized form, parameterized by the number of players.

Keywords: common p-belief; common knowledge assumption; global games
JEL Classification: D83; C70

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1. Introduction.

Common knowledge among players on certain features of the game is an essential yet a very stringent assumption that often leads to many Nash equilibria. In a strict sense of a term, common knowledge requires a hierarchical "knowing" of an event or payoff structure of the game *i.e.* not only that everyone should know the event, but that everyone should also know that everyone knows it, and that everyone should know that everyone knows that everyone knows it, and so on, *ad infinitum*[1]. It is therefore on this rigorousness that many economists maintain that common knowledge assumption (CKA) can never really be sustained in practice. As Wilson (1987) would put it, "only by weakening the CKA will the theory approximate reality".

However, abandoning the CKA has far better consequences than just making a theory more realistic. In the field of equilibrium selection, Carlsson and van Damme (1993) showed that by introducing a small amount of noise on player's payoff, any initial multiple equilibria in a game contracts into a unique rationalizable equilibrium. Their approach even generalizes the Harsanyi-Selten's criterion for selection in the sense that players always choose the risk-dominant equilibrium even when the other equilibrium is payoff-dominant. This kind of mechanism, termed as the "global game approach" has particularly found its application on explaining why crisis phenomenon can become a stable outcome[2].

In this paper, we show that the success achieved by the global game method for equilibrium selection can also be attained by another form of departure from the CKA known as *common belief.* Monderer and Samet (1989), in an attempt to weaken the CKA[3], ventured on the idea that if players do not fully know and therefore, only believe an event is true with probability of at least p (or simply say p-believe), the hierarchical structure of information that is similar to common knowledge can still be maintained. It must be noted here that while common knowledge about the

[1] This definition, confounding as it may seem, was given a simpler mathematical formulation by Aumann (1976). This formulation was later applied by Monderer and Samet (1989) on common belief. Geanakoplos (1994) and Fagin, Halpern et.al. (1999) provide an excellent discussion on common knowledge.

[2] See Morris and Shin (2003) for a survey of global game application on crisis structures.

[3] The idea of a weaker form of common knowledge first came about from Rubinstein's (1989) notion of almost common knowledge. He showed, however, in a modernized version of "coordination-attack problem" that optimizing agents cannot behave as if they are in a common knowledge environment and therefore cannot achieve the typical Nash equilibrium of the game.

event is no longer at its fullest, the common knowledge on the rationality of players is preserved. An event, say E, is said to be commonly p-believed if everybody p-believes E, everybody p-believes that everybody p-believes E, and so on. They established that players can still achieve an equilibrium similar to a game played under a common knowledge environment for as long as they behave as ε-optimizers. Thus, if a game has multiple equilibria under complete information, these equilibria will continue to be stable within a certain neighborhood under the common belief setup. Our approach therefore is a refinement of their result since it attempts to reduce this multiplicity into a unique equilibrium criterion.

In this study, we refrain from using "noisy" private signals and focus only on the information partition that each agent received from the very start. As we assume here a common prior among agents, differences of the probability estimates (posteriors) on a certain event are attributed only to the differences in the information sets that each agent may have. Thus, the inability of every agent to fully pin point the true state of the economy is due to the realization that there are other "states" in her information set that are indistinguishable from the true one. As a result, any event that proceeds from the occurrence of the true state can not be fully ascertained and will not generate a definitive action from any player unless one's information partition is entirely contained in that event. In the end, the uncertainty of the realization of a certain event or outcome is tantamount to the uncertainty that players may have in making an action that favors such outcome. Nonetheless, we show in this paper that despite this uncertainty, players can be made to coordinate their actions provided that they all have a relatively strong common belief on a certain outcome.

Moreover, in line with the program of weakening the CKA by using common belief structures[4], we further weaken the notion of common belief itself to provide an even less stringent and a more natural way of *believing* an event. Rather than having a collective notion of belief on E (as if people gather together, share their beliefs with one another, and declare that they all p-believe E), we argue that what comes more natural is an iterated process where each one simply refers her belief with the rest of the people's belief – which is generally taken together as one. The words like "I

[4]In the literature, there arise a number of ways of defining belief process to approximate common knowledge, but the idea of common p-belief is the one generally used as a standard. For the comparison of common p-belief, iterated p-belief , and weak common p-belief, see Morris (1999). For common repeated p-belief and joint common repeated p-belief, see Monderer and Samet (1996).

believe that others believe that it is so" or "I believe that others believe in what I believe" are examples of this kind of reasoning which considers "others" as if another individual. This manner of reasoning allows us to introduce the notion of iterated quasi-common p-belief where an agent i p-believes an event E, i p-believes that others p-believe E, i p-believes that others p-believe that i p-believes E, and so on. Despite being a weaker form than common p-belief in general, it is shown that if everyone follows this iterative binary approach of belief (between one's own and that of the rest), it could approximate the notion of common p-belief as the number of individuals involved becomes larger.

The rest of the paper is organized as follows. Section 2 presents our leading example, illustrating how in a common belief environment a unique equilibrium can be selected. We provide necessary and sufficient conditions for this selection in Section 4 while a formal framework is presented beforehand in Section 3. Sections 5 and 6 deals with a general characterization of common belief and the agreeing to disagree result through the use of iterated quasi-common p-belief. Section 7 concludes.

2. Example.

Suppose at a certain time a finite number of investors are all contemplating to choose between two possible actions: to attack or not to attack the currency. The symmetric payoff to any agent i with respect to other agents' actions is common knowledge and is depicted below:

		Others	
		Attack	Not attack
Player *i*	Attack	3, 3	0, π
	Not attack	π, 0	π, π

Notice that if $\pi > 3$, then i knows that all the others will choose "not attack" and so it pays him to do the same action; whereas if $\pi < 0$, then everyone will end up attacking the currency. Only when $\pi \in [0, 3]$ that the outcome is not definitive since either everyone will attack or everyone will not attack. For example, if $\pi = 1$, even if the the equilibrium $(3, 3)$ is payoff-dominant to equilibrium $(1, 1)$, nothing prohibits the agents from settling at $(1,1)$ if everyone knows that everyone will choose "not attack". But when will everyone know that? Only when everyone knows that [everyone knows that everyone will choose "not attack"], and so on. This infinite hierarchy of knowledge is actually the heart of common knowledge which allows some sub-optimal choices to even become Nash equilibria.

Now we show that when we weaken the common knowledge assumption to common p-belief, a unique cut-off equilibrium for all π within the interval $[0,3]$ can be achieved.

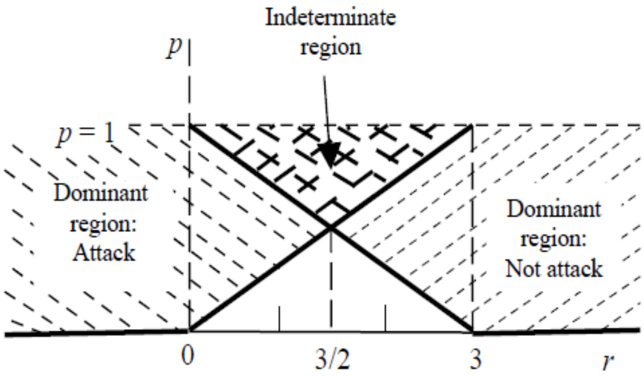
First, we show that the prediction of common p-belief on dominant regions is consistent with the scenario under common knowledge environment. Suppose the event "$\pi > 3$" is common p-belief. Then i believes that "not attack" will be chosen by all other agents with probability at least p. Since "$\pi > 3$" is commonly p-believed by all, i will also choose "not attack" with probability at least p. But will i really end up choosing "not attack" for a sufficiently high p? Observe that for i to choose otherwise *i.e.* "attack", it must be that the expected net payoff from attacking is greater than zero or

$$3(1-p) + 0 \cdot p > \pi \Leftrightarrow p < \frac{3-\pi}{3}.$$

Since $\pi > 3$ at the event "$\pi > 3$", the above inequality cannot hold for a sufficiently high p and so i will always choose "not attack". A similar argument follows for i to always choose "attack" whenever the event "$\pi < 0$" is common p-belief. At "$\pi < 0$", and for i not to attack, the net expected payoff for doing so must be positive, which in turn cannot hold in the following inequality since p needs to become very low, *i.e.*

$$\pi > 3p + 0 \cdot (1-p) \Leftrightarrow p < \frac{\pi}{3}.$$

Suppose the event "$\pi \in [0,3]$" is common p-belief, then players may either think that everyone will attack or everyone will not attack. However, the indeterminacy of outcome that each i have within this region is not complete and is limited only to the intersection of conditions for attacking ($p \geq \frac{\pi}{3}$) and not attacking ($p \geq \frac{3-\pi}{3}$). In the figure below, we see that for $\pi \in [0, \frac{3}{2})$ and $p \in [0,1]$, every i *knows* that the condition for attacking has more chances of being followed than that of not attacking. This makes "attack" to be the best-response action for any agent i. In contrast, for $\pi \in (\frac{3}{2}, 3]$, "not attack" maintains the best-response for each i and so only when $\pi = \frac{3}{2}$ that players will be indifferent between the two actions.



In the succeeding two sections, this observation on how a unique equilibrium is chosen from a game with multiple equilibria will be given a more general treatment. Our analysis will focus on information partitions rather than on the usage of noisy signals, typical of the global game approach. We will also depart from the normal one-dimensional payoff structure by simply incorporating payoffs into "states" that affect the action of agents. Finally, notice the use of symmetric binary interaction between player i and the rest of the players in the above example. This will be given a more detailed analysis in Sections 5 and 6.

3. Beliefs and Information Structure.

Consider a probability space $(\Theta, \Sigma, \varphi)$, where Θ is a countable payoff state space, Σ is a $\sigma-$ algebra of events in Θ, and φ is a probability measure assigned on the elements of Σ. For each agent $i \in I = \{1, 2, 3, ..., n\}$, let Ψ_i, interpreted as the information available to i, be a finite partition of Θ into measurable sets. For any true state $\theta \in \Theta$, denote the subset of Ψ_i that contains θ as $\Psi_i(\theta)$, which also represents the set to which i knows θ could belong whenever it occurs. Write ζ_i for the $\sigma-$ algebra generated by Ψ_i, i.e. ζ_i contains all the unions of the elements of Ψ_i. For event $E \in \Sigma$ and probability $p \in [0, 1]$, we say that "at θ, i believes E with probability of at least p" (or simply "i p-believes E at θ") if the probability of E given that θ has occurred is at least p for agent i. Denoting $B_i^p E$ as the event that i p-believes E at θ, we formally write $B_i^p E = \{\theta \mid \varphi(E \mid \Psi_i(\theta)) \geq p\}$. Furthermore, the intersection of p-beliefs of agents on E is denoted by $B_*^p E = \bigcap_{i \in I} B_i^p E$. The last two statements are made clearer by the following example.

Let $I = \{1, 2\}$ and $\Theta = \{a, b, c, d, e\}$ where each state in Θ has equal prior probability of 1/5. Suppose $\Psi_1 = \{\{a, b\}, \{c, d, e\}\}$ and $\Psi_2 = \{\{a, c, d\}, \{b\}, \{e\}\}$. Consider the event $E = \{c, d\}$ at $\theta = c$ and with $p \geq 0.6$. Then,

$B_1^{0.6}E = \left\{\theta \;\middle|\varphi\left(\{c, d\}\,|\{c, d, e\}\right) = \frac{2}{3}\right\} = \{c, d, e\}$
$B_2^{0.6}E = \left\{\theta \;\middle|\varphi\left(\{c, d\}\,|\{a, c, d\}\right) = \frac{2}{3}\right\} = \{a, c, d\}$, and
$B_*^{0.6}E = \{c, d\}$.

Now consider a binary action game where each agent i chooses either an action α or its alternative $\sim \alpha$. Pick two disjoint events E^{α}, $E^{\sim\alpha} \in \Sigma$ such that $E^{\alpha} \cup E^{\sim\alpha} = \Theta$[5], where E^{α} is the set of all payoff states where every agent will choose α and $E^{\sim\alpha}$ is the set where every agent will choose the alternative. Thus, at a given θ, i surely picks α ($\sim \alpha$) whenever $\Psi_i(\theta) \subseteq E^{\alpha}$ (resp. $\Psi_i(\theta) \subseteq E^{\sim\alpha}$). If some payoff states in $\Psi_i(\theta)$ are not in E^{α}, then i will only have a certain probability of at least $p \in [0, 1]$ that everyone will be choosing α and this is denoted by $B_i^p E^{\alpha}$. A similar interpretation is given to $B_i^p E^{\sim\alpha}$.

The following are straightforward properties of belief operator $B : \Sigma \to \Sigma$ and are known in the literature of common belief (see Monderer and Samet, 1989 &1996 and Morris, 1999).

P1: If $E \in \zeta_i$, then $B_i^p E = E$
P2: $B_i^p(B_i^p E) = B_i^p E$
P3: (Monotonicity) If $E \subseteq F$, then $B_i^p E \subseteq B_i^p F$
P4: If (E_m) is a decreasing sequence of events, then $B_i^p \left(\bigcap\limits_m E_m\right) = \bigcap\limits_m B_i^p(E_m)$
P5: $B_i^p E \Rightarrow \varphi\left(E\,|B_i^p E\right) \geq p$
P6: If $s \geq p$, then $B_i^s E \subseteq B_i^p E$

Definition 1. An event E is common p-belief at θ if $\theta \in C^p E$, where

$$C^p E \equiv B_*^p E \cap B_*^p B_*^p E \cap B_*^p B_*^p B_*^p E \cap ... \equiv \bigcap_{k \geq 1} \left(B_*^p\right)^k E.$$

[5]This condition is without loss of generality eventhough $E^{\alpha} \cup E^{\sim\alpha} \subseteq \Theta$ would have been more lenient. Nothing substantial hinges on this difference and is made only for the purpose of simplifying our presentation.

This definition provides a graphic illustration of common p-belief. Nevertheless, Monderer and Samet (1989) argued that it would be difficult to identify common p-belief by knowing all the infinite hierarchical conditions this definition requires. Thus, following Aumann's simpler characterization of common knowledge, they provided a similar concise description on common p-belief without losing its original meaning through the use of the notion of evident p-belief.

Definition 2. An event F is evident p-belief if it is a fixed point of B_i^p for all i, i.e. $F \subseteq B_i^p F$, for all i.

Proposition 1. (Monderer and Samet, 1989). Event E is common p-belief at θ if and only if there exists an evident p-belief event F such that $\theta \in F$ and $F \subseteq B_i^p E$, for all i.

Remark. In "redefining" common p-belief, Monderer and Samet showed in their proof that $C^p E$ is an evident p-belief event (i.e. $C^p E \subseteq B_i^p(C^p E)$, for all i) and E is a common p-belief at θ with $\theta \in C^p E$ and $C^p E \subseteq B_i^p E$, for all i.

4. Equilibrium Selection.

The system to which an equilibrium is chosen under a common belief environment mimics that of the global game approach since it leads the game to a unique cut-off equilibrium between two dominant regions. In the previous section, we show that there are occasions in which a player chooses an action regardless of the decision of others. We show later in Proposition 3 that the existence of such dominant action can lead to a dominant event where everyone else ends up doing the same action. However, in the intermediate case where all players only p-believe an event, where $p < 1$, any action favoring the realization of that event cannot be made certain. While it is true that players, in principle, cannot still make a decisive action even when the event is commonly p-believed, we nonetheless show that there is a criterion in which players can further refine their information. This criterion rests on the principle of duality such that even if players commonly p-believe an event E^α with $p < 1$, each of them knows that at every level of the hierarchy of beliefs, everyone believes $E^{\sim\alpha}$ with probability of at most $1-p$. In other words, there exists a form of common knowledge on how each one compares one's belief on one event as against its alternative. The awareness of this information is what, in turn, allows for the equilibrium selection in this type of incomplete information game.

Furthermore, our approach in this presentation is quite general in the sense that it does not give any reference or condition about the type of payoff structure of the game. It simply assumes that payoffs are included in the type of state that influences the decision of a player, such that if a certain state is one that makes everyone choose α, then the payoff that it offers to each one is higher than any other state that makes them choose $\sim \alpha$. We start now our presentation by characterizing first the existence of dominant events and then proceed by introducing the notion of duality and belief deduction; tools which will be used for generating conditions for selection.

Proposition 2. (Dominant Events) If E^{α} is common p-belief at θ and there exists a $k \in I$ whose $\Psi_k(\theta) \subseteq E^{\alpha}$, then all players will choose α.

Proof:
Pick a $k \in I$ where $\Psi_k(\theta) \subseteq E^{\alpha}$. Then $B_k^1 E^{\alpha} = \{\theta \mid \varphi\left(E^{\alpha} \mid \Psi_k(\theta)\right) = 1\} = \Psi_k(\theta)$. Since E^{α} is common p-belief at θ then we have $\theta \in C^p E^{\alpha} \subseteq B_k^1 E^{\alpha} \subseteq E^{\alpha}$ which implies that even if θ is not certainly identified, every i knows that θ is in E^{α} and so everyone will choose α. *q.e.d.*

Definition 3. Denote $B_i^{\overline{1-p}} E$ as the event that i believes E by at most $1-p$ (or that i $(\overline{1-p})$-believes E). Furthermore, denote $C^{\overline{1-p}} E$ whenever event E is common $(\overline{1-p})$-belief at θ where $\theta \in C^{\overline{1-p}} E \equiv \bigcap_{k \geq 1} \bigcap_{i \in I} \left(B_i^{\overline{1-p}}\right)^k E$.

Lemma 1. (Duality)
i) $B_i^p E^{\alpha} = B_i^{\overline{1-p}} E^{\sim\alpha}$ and
ii) $C^p E^{\alpha} = C^{\overline{1-p}} E^{\sim\alpha}$

Proof:

$$
\begin{aligned}
\text{i) } B_i^p E^\alpha &= \{\theta \mid \varphi\left(E^\alpha \left| \Psi_i(\theta)\right.\right) \geq p\} \\
&= \{\theta \mid \varphi\left(\Theta \backslash E^\alpha \left| \Psi_i(\theta)\right.\right) \leq 1-p\} \\
&= \{\theta \mid \varphi\left(E^{\sim\alpha} \left| \Psi_i(\theta)\right.\right) \leq 1-p\} \\
&= B_i^{\overline{1-p}} E^{\sim\alpha} \\
\text{ii) } C^p E^\alpha &= \left\{\theta \mid \ \varphi\left(E^\alpha \left| \bigcap_{k\geq 1} \left(B_*^p\right)^k E^\alpha \right.\right) \geq p\right\} \text{ by definition and P5} \\
&= \left\{\theta \mid \ \varphi\left(\Theta \backslash E^\alpha \left| \bigcap_{k\geq 1} \left(B_*^p\right)^k E^\alpha \right.\right) \leq 1-p\right\} \\
&= \left\{\theta \mid \ \varphi\left(E^{\sim\alpha} \left| \bigcap_{k\geq 1} \left(B_*^{\overline{1-p}}\right)^k E^{\sim\alpha} \right.\right) \leq 1-p\right\} \text{ by (i) above} \\
&= C^{\overline{1-p}} E^{\sim\alpha} \qquad\qquad q.e.d.
\end{aligned}
$$

Remark. We note here that Lemma 1 (ii) is true even if $C^p E^\alpha \cap E^{\sim\alpha} = \emptyset$ since that means that $\varphi\left(\Theta \backslash E^\alpha \left| \bigcap_{k\geq 1} \left(B_*^p\right)^k E^\alpha \right.\right) = 0$ which still admits elements for $\left\{\theta \mid \ \varphi\left(\Theta \backslash E^\alpha \left| \bigcap_{k\geq 1} \left(B_*^p\right)^k E^\alpha \right.\right) \leq 1-p\right\}$. Thus, $C^{\overline{1-p}} E^{\sim\alpha}$ is always non-empty since $C^p E^\alpha$ contains at least one element, θ.

Lemma 2. (Belief Deduction)

i) If E^α is common p-belief, then for all i, E^α is q-believed, *i.e.* $C^p E^\alpha \subseteq B_i^q E^\alpha$, where $q = 1 - \frac{\gamma^k(1-p)}{n^k}$, $\gamma \in [1, n]$, and $k \geq 0$.

ii) If $E^{\sim\alpha}$ is common $(\overline{1-p})$-belief, then for all i, $E^{\sim\alpha}$ is $\overline{r}-$believed, *i.e.* $C^{\overline{1-p}} E^{\sim\alpha} \subseteq B_i^{\overline{r}} E^{\sim\alpha}$, where $r = 1 - \frac{\gamma^k p}{n^k}$, $\gamma \in [1, n]$, and $k \geq 0$.

When E^α is common p-belief, it is true that everyone p-believes E^α (Proposition 1). However, this does not give a finer information about the extent of i's belief now that each one *knows* that there is a common p-belief on E^α. Lemma 2 therefore provides a more precise information on each i's belief on E^α.

Proof:

i) If E^α is common p-belief then there exists an evident p-belief event F such that $\theta \in F$ and $F \subseteq B_i^p E^\alpha$, for all i. Monderer and Samet (1989) showed that $C^p E^\alpha$ is the largest of such event F, i.e. $\theta \in C^p E^\alpha \subseteq B_i^p(C^p E^\alpha) \subseteq B_i^p E^\alpha$. Since $(B_*^p)^k E^\alpha$ is decreasing in k, we have from P4 and P5:

$$\begin{aligned} B_i^p(C^p E^\alpha) &\Rightarrow \varphi\left(C^p E^\alpha \mid B_i^p(C^p E^\alpha)\right) \geq p \\ &\Rightarrow \varphi\left(\bigcap_{i=1}^{n} B_i^p\left(\bigcap_{k\geq 1} (B_*^p)^{k-1} E^\alpha\right) \middle| B_i^p(C^p E^\alpha)\right) \geq p. \end{aligned}$$

By countable subadditivity, we have the following inequality:

$$\begin{aligned} &\varphi\left(\bigcup_{i=1}^{n}\left(\Theta \backslash B_i^p\left(\bigcap_{k\geq 1} (B_*^p)^{k-1} E^\alpha\right)\right) \middle| B_i^p(C^p E^\alpha)\right) \\ \leq\ &\sum_{j=1}^{n} \varphi\left(\Theta \backslash B_i^p\left(\bigcap_{k\geq 1} (B_*^p)^{k-1} E^\alpha\right) \middle| B_i^p(C^p E^\alpha)\right) = \gamma(1-p), \end{aligned}$$

where $\gamma \in [1, n]$ is a measure of set overlapping, i.e. $\gamma = 1 \Rightarrow$ disjoint sets.

This therefore leads to the following expressions:

$$\begin{aligned} &\Rightarrow \varphi\left(B_i^p\left(\bigcap_{k\geq 1} (B_*^p)^{k-1} E^\alpha\right) \mid B_i^p(C^p E^\alpha)\right) \geq 1 - \frac{(1-p)\gamma}{n} \\ &\Rightarrow B_i^{1-\frac{(1-p)\gamma}{n}}\left(\bigcap_{k\geq 1} (B_*^p)^{k-1} E^\alpha\right) \end{aligned}$$

Following the above derivation, we have $B_i^{1-\frac{(1-p)\gamma}{n}}\left(\bigcap_{k\geq 1} (B_*^p)^{k-1} E^\alpha\right) \Rightarrow$ $B_i^{1-\frac{(1-p)\gamma^2}{n^2}}\left(\bigcap_{k\geq 1} (B_*^p)^{k-2} E^\alpha\right)$. By induction, we derive $B_i^q E^\alpha$, for all i where $q = 1 - \frac{\gamma^k(1-p)}{n^k}$ and that $C^p E^\alpha \subseteq B_i^p(C^p E^\alpha) \subseteq B_i^q E^\alpha$.

ii) In a similar fashion, $B_i^{\overline{1-p}}(C^{\overline{1-p}} E^{\sim\alpha})$ implies that $\varphi\left(C^{\overline{1-p}} E^{\sim\alpha} \middle| B_i^{\overline{1-p}}(C^{\overline{1-p}} E^{\sim\alpha})\right) \leq$

$1-p$. Then, we obtain the following.

$$\Rightarrow \ \varphi\left(\bigcap_{i=1}^{n} B_i^{\overline{1-p}}\left(\bigcap_{k\geq 1}\left(B_*^{\overline{1-p}}\right)^{k-1} E^{\sim\alpha}\right)\middle| B_i^{\overline{1-p}}(C^{\overline{1-p}}E^{\sim\alpha})\right) \leq 1-p$$

$$\Rightarrow \ \varphi\left(\bigcup_{i=1}^{n}\left(\Theta\backslash B_i^{\overline{1-p}}\left(\bigcap_{k\geq 1}\left(B_*^{\overline{1-p}}\right)^{k-1} E^{\sim\alpha}\right)\right)\middle| B_i^{\overline{1-p}}(C^{\overline{1-p}}E^{\sim\alpha})\right) \geq p\gamma$$

$$\Rightarrow \ \varphi\left(B_i^{\overline{1-p}}\left(\bigcap_{k\geq 1}\left(B_*^{\overline{1-p}}\right)^{k-1} E^{\sim\alpha}\right)\middle| B_i^{\overline{1-p}}(C^{\overline{1-p}}E^{\sim\alpha})\right) \leq 1-\frac{p\gamma}{n}$$

$$\Rightarrow \ B_i^{\overline{1-\frac{p\gamma}{n}}}\left(\bigcap_{k\geq 1}\left(B_*^{\overline{1-p}}\right)^{k-1} E^{\sim\alpha}\right)$$

By induction, this leads to $B_i^{\overline{r}}E^{\sim\alpha}$, where $r = 1-\frac{\gamma^k p}{n^k}$, $\gamma \in [1,n]$, and $k \geq 0$ and that $C^{\overline{1-p}}E^{\sim\alpha} \subseteq B_i^{\overline{1-p}}(C^{\overline{1-p}}E^{\sim\alpha}) \subseteq B_i^{\overline{r}}E^{\sim\alpha}$. *q.e.d.*

By using Lemma 1 and 2, we are now prepared to present a theorem on selection.

Theorem 1. Every i p-believes E^α and chooses α if and only if E^α is common p-belief with $p > \frac{1}{2}$. If $p = \frac{1}{2}$, then players are indifferent in choosing between α and $\sim\alpha$.

Proof:

$(\rightarrow)$ Suppose E^α is common p-belief with $p > 1/2$. By duality, $C^pE^\alpha = C^{\overline{1-p}}E^{\sim\alpha}$ and that event E^α is known to have higher common belief than $E^{\sim\alpha}$. By deduction, $B_i^qE^\alpha$ and $B_i^{\overline{r}}E^{\sim\alpha}$ are obtained and since $p > 1/2$, this implies from Lemma 2 that $q > r$. Every i therefore puts higher probability on event E^α than on $E^{\sim\alpha}$. Thus, every i chooses α.

$(\leftarrow)$ We prove this by contraposition such that (either $\sim C^pE^\alpha$ or $p \leq 1/2$) $\Rightarrow$ (either $B_i^pE^\alpha$ for some i or $\sim\alpha$ is chosen).

Case 1. Suppose $\sim C^pE^\alpha$. Then there exists a $\theta' \in \bigcap_{k\geq 1}\left(\bigcap_{i=1}^{s} B_i^p\right)^k E^\alpha \subseteq B_i^pE^\alpha$ for all $i = 1$ to s, where $s < n$. Thus, there are some $i \in I$ who do not p-believe E^α.

Case 2. Suppose C^pE^α and $p \leq 1/2$. By duality, $C^pE^\alpha = C^{\overline{1-p}}E^{\sim\alpha}$ and that event E^α is known to have lower common belief than $E^{\sim\alpha}$. By deduction, $B_i^qE^\alpha$ and $B_i^{\overline{r}}E^{\sim\alpha}$ are obtained and since $p \leq 1/2$, this implies from Lemma 2 that $q \leq r$. Thus, every i

places on event $E^{\sim\alpha}$ greater than or equal probability on E^{α}. Thus, every i chooses $\sim \alpha$ if $p < 1/2$ and is indifferent between α and $\sim \alpha$ if $p = 1/2$. *q.e.d*

In the following example, we show a case where players are indifferent between α and $\sim \alpha$ when both E^{α} and $E^{\sim\alpha}$ are common $\frac{1}{2}$-belief.

Example 1. Suppose $\Theta = \{a, b, c, d, e, f\}$ and $i \in I = \{1, 2, 3\}$. Let all the states in Θ have equal prior of $\frac{1}{6}$ and assume that $\Psi_1 = \{\{c, d\}, \{a, f\}, \{b, e\}\}$, $\Psi_2 = \{\{b, c, d, e\}, \{a, f\}\}$, and $\Psi_3 = \{\{a, f, c, d\}, \{b, e\}\}$. Now, consider the events $E^{\alpha} = \{a, b, c\}$ and $E^{\sim\alpha} = \{d, e, f\}$ which are both common 0.5-belief. Then, at $\theta = c$, the posteriors of all players on E^{α} and $E^{\sim\alpha}$ are as follows:

$$B_1^{0.5}E^{\alpha} = \{\theta \mid \varphi(\{a, b, c\} \,|\{c, d\}\,)\} = \{c, d\} = \{\theta \mid \varphi(\{d, e, f\} \,|\{c, d\}\,)\} = B_1^{0.5}E^{\sim\alpha}$$
$$B_2^{0.5}E^{\alpha} = \{b, c, d, e\} = B_2^{0.5}E^{\sim\alpha}$$
$$B_3^{0.5}E^{\alpha} = \{a, f, c, d\} = B_3^{0.5}E^{\sim\alpha}$$

From Lemma 1, we confirm that $C^{0.5}E^{\alpha} = C^{0.5}E^{\sim\alpha}$, i.e.

$$B_*^{0.5}E^{\alpha} = B_*^{0.5}B_*^{0.5}E^{\alpha} = ... = \{c, d\} \text{ and}$$
$$B_*^{0.5}E^{\sim\alpha} = B_*^{0.5}B_*^{0.5}E^{\sim\alpha} = ... = \{c, d\}$$

And since both events are equally commonly believed by all players, then each of them is indifferent in choosing between α or $\sim \alpha$.

It is worth noting that the theorem does not say whether the event that contains the true state will always be chosen. It is therefore possible that an event that has higher common belief will lead players to select an action favoring that event even if the true state is nonexistent. We show this pathological case in our next example.

Example 2. Let $\Theta = \{a, b, c, d, e, f\}$ and $i \in I = \{1, 2, 3\}$. Let all the states in Θ have equal prior of $1/6$ and assume that $\Psi_1 = \{\{c, d, e\}, \{a, b, f\}\}$, $\Psi_2 = \{\{c, d, e\}, \{a, b\}, \{f\}\}$, and $\Psi_3 = \{\{c, d.e\}, \{a\}, \{b, f\}\}$. Now, consider that event $E^{\alpha} = \{a, b, c\}$ is common 1/3-belief while $E^{\sim\alpha} = \{d, e, f\}$ is common 2/3-belief. Then, at $\theta = c$, the posteriors of all players on E^{α} and $E^{\sim\alpha}$ are as follows:

$$B_1^{1/3}E^{\alpha} = B_2^{1/3}E^{\alpha} = B_3^{1/3}E^{\alpha} = B_1^{2/3}E^{\sim\alpha} = B_2^{2/3}E^{\sim\alpha} = B_3^{2/3}E^{\sim\alpha} = \{c, d, e\}$$
$$B_*^{1/3}E^{\alpha} = B_*^{1/3}B_*^{1/3}E^{\alpha} = \; ... = \{c, d, e\}$$

$$B_*^{2/3}E^{\sim\alpha} = B_*^{2/3}B_*^{2/3}E^{\sim\alpha} = \ ... = \{c, d, e\}$$

Thus, it is possible to have higher probability on $E^{\sim\alpha}$ even if the true state c is not in $E^{\sim\alpha}$.

5. A Weaker Form of Common Belief.

In this section, we propose a weaker form of common p-belief called iterated quasi-common p-belief. It is a generalization to many players of a two-person iterated p-belief that maintains a symmetric binary relation, *i.e.* in this case of n players, between i and the rest. Morris (1999) showed, in the case of two individuals, that iterated p-belief is weaker than common p-belief. That is, if an event is common p-belief for every $p < 1$, then that event is iterated p-belief, but not vice versa. In this section, we claim that the converse is also true provided that the number of players becomes larger. Thus, the iterated quasi-common p-belief converges with the standard notion of common p-belief as will be presented in Theorem 2.

The relevance of studying iterated process of beliefs, as argued by Morris (1999), is that it is the most appropriate relaxation of CKA to use for the analysis of best response dynamics. For example, when an incomplete information game is repeatedly played, this procedure allows an individual to learn over time how her strategy based on her beliefs "performed" against that of the others. Although we will not be extending this dynamic setup to many players, we will nonetheless show how this generalized iterated process can shed light on the structure of beliefs in terms of the number of players involved.

An event E is iterated quasi-common p-belief for agent i if i p-believes E, i p-believes that others p-believe E, i p-believes that others p-believe that i p-believes E, and so on. We formally define this as follows.

Definition 4. An event E is iterated quasi-common p-belief at θ if it is so for all agents, *i.e.* $\theta \in Q^pE \equiv \bigcap_{i\in I} Q_i^pE$, where:

$$Q_i^pE \equiv B_i^pE \cap B_i^pB_{*\backslash i}^pE \cap B_i^pB_{*\backslash i}^pB_i^pE \cap ... \equiv \bigcap_{k\geq 0} B_i^p\left(B_{*\backslash i}^pB_i^p\right)^k E \cap \bigcap_{k\geq 0}(B_i^pB_{*\backslash i}^p)^{k+1}E$$

and $B^p_{*\backslash i}E \equiv \bigcap B^p_j E$, for all $j \in I\backslash i = \{1, 2, ..., i-1, i+1, ..., n\}$ and for a given $i \in I$.

The iterated quasi-common p-belief also has a simpler (fixed point) characterization in the spirit of Morris' (1999) two-person iterated p-belief.

Definition 5. A set ψ is joint p-evident if for all $G \subseteq \psi$ and for all i, $B^p_i G \subseteq \psi$.

Proposition 3. Event E is iterated quasi-common p-belief at θ if and only if there exists a joint p-evident set ψ such that (i) $B^p_i E \subseteq \psi$, for all i and (ii) $\theta \in G$, for all $G \subseteq \psi$.

Proof:

($\rightarrow$) Assume that ψ is a joint p-evident set where (i) and (ii) hold. By (i), $B^p_i E \subseteq \psi$ for all i which implies that $B^p_{*\backslash i}E \subseteq \psi$. Since ψ is joint p-evident, it follows that for all i, $B^p_i(B^p_{*\backslash i}E) \subseteq \psi$. Note also that the definition of joint p-evident set ψ also implies that $B^p_{*\backslash i}G \subseteq \psi$, for all $G \subseteq \psi$ and for all i. Thus, we have $B^p_{*\backslash i}(B^p_i E) \subseteq \psi$. By induction, we obtain $B^p_i(B^p_{*\backslash i}B^p_i)^k E \subseteq \psi$ and $(B^p_i B^p_{*\backslash i})^{k+1}E \subseteq \psi$, for all i and for all $k \geq 0$. Since the intersection of these iterative sets in k is a subset of ψ, θ is contained in the intersection by (ii). The intersection, though, is equivalent to $Q^p E$, thus E is iterated quasi-common p-belief at θ.

($\leftarrow$) Assume that E is iterated quasi-common p-belief at θ. Let ψ be a collection of subset G where $G \in \left\{B^p_i(B^p_{*\backslash i}B^p_i)^k E,\ (B^p_i B^p_{*\backslash i})^{k+1}E,\ \text{for all } i \in I \text{ and } k \geq 0\right\}$. Since by definition of $Q^p E$, $\theta \in \bigcap\limits_{i\in I}\left(\bigcap\limits_{k\geq 0} B^p_i\left(B^p_{*\backslash i}B^p_i\right)^k E \cap \bigcap\limits_{k\geq 0}(B^p_i B^p_{*\backslash i})^{k+1}E\right)$, it therefore shows that $\theta \in G$ for any $G \subseteq \psi$ and therefore (ii) holds. Notice also that by construction of ψ, (i) holds at $k = 0$. Thus, ψ is joint p-evident. *q.e.d.*

Now we present that the two belief processes can be made equivalent at a sufficiently large number of agents.

Theorem 2. For a large number of players, every event E is common p-belief at θ if and only if it is iterated quasi common p-belief at θ.

To prove this theorem, we will need the following lemmas:

Lemma 3. For any $i \in I$, $B^p_{*\backslash i}E \subseteq B^p_i E$ for a large number of n.

Remark: This means that the set of payoff states where a group of people other than i (as many as possible) collectively p-believes E, then these states are also p-believed by i. The intuition behind this lemma is that the combined information of more people provides a finer belief on E at θ, as compared to the belief on E of a single arbitrary player not part of the group. This, however, assumes that the information set that contains θ of those players other than i should not be identical to one another. Otherwise, the increase in the number of players cannot refine the information that pins down the actual θ. We state this assumption formally as follows:

Assumption: For any $i, g \in I$ and $\theta \in \Theta, \Psi_i(\theta) \neq \Psi_g(\theta)$, where $i \neq g$.

Proof:

Note that the true statement $B_*^p E \equiv \bigcap\limits_{i \in I} B_i^p E \subseteq B_i^p E$ can be expressed as $\varphi \left(B_i^p E \,|B_*^p E \right) = 1$. That is, given that all players collectively p-believe E, then the probability that anyone of them believes E by at least p is 1. Similarly, we will prove $B_{*\backslash i}^p E \subseteq B_i^p E$ by showing that $\varphi \left(B_i^p E \middle| B_{*\backslash i}^p E \right) = 1$ as the number of players increases. Hence, we derive:

$$\varphi \left(B_i^p E \middle| B_{*\backslash i}^p E \right) = \frac{\varphi \left(B_i^p E \cap B_{*\backslash i}^p E \right)}{\varphi(B_{*\backslash i}^p E)} = \frac{\varphi \left(\bigcap\limits_{i \in I} B_i^p E \right)}{\varphi \left(\bigcap\limits_{i \in I \backslash i} B_j^p E \right)}$$

Let $\varphi \left(\bigcap\limits_{i \in I} B_i^p E \right) \geq r$, where $r \in (0, 1)$. Then, it is also true that $\varphi \left(\bigcap\limits_{i \in I \backslash i} B_j^p E \right) \geq r$ since $B_*^p E \subseteq B_{*\backslash i}^p E$. Observe that $\varphi \left(\bigcap\limits_{i \in I} B_i^p E \right) \geq r \Leftrightarrow \varphi \left(\bigcup\limits_{i=I}^{n} (\Theta \backslash B_i^p E) \right) \leq 1 - r$. Then, $\varphi \left(\Theta \backslash B_i^p E \right) \leq (1 - r)\frac{\gamma}{n} \Rightarrow \varphi \left(B_i^p E \right) \geq 1 - \frac{(1-r)\gamma}{n}$, for all $i \in I$, where $\gamma \in [1, n]$ such that if $\gamma = 1$ then $\bigcup\limits_{i=1}^{n} (\Theta \backslash B_i^p E)$ is a union of disjoint sets. Similarly, since $\bigcup\limits_{j=1}^{n-1} \left(\Theta \backslash B_j^p E \right)$ is a union of $n - 1$ players, we obtain from the same arguments as above the value of the denominator as $\varphi \left(B_j^p E \right) \geq 1 - \frac{(1-r)\gamma}{n-1}$, for all $j \in I \backslash i$. Thus,

$$\lim_{n \to \infty} \varphi \left(B_i^p E \middle| B_{*\backslash i}^p E \right) \equiv \frac{1 - \frac{(1-r)\gamma}{n}}{1 - \frac{(1-r)\gamma}{n-1}} = 1$$

q.e.d.

Lemma 4. (Decreasing Sequence)

Let $x_k E \equiv \bigcap\limits_{i \in I} B_i^p (B_{*\backslash i}^p B_i^p)^k E$ and $y_k E \equiv \bigcap\limits_{i \in I} (B_i^p B_{*\backslash i}^p)^k E$.

(i) For any $k \geq 0$, $x_k E \supseteq y_{k+1} E$

(ii) For any $k \geq 0$, $x_k E \supseteq x_{k+1} E$ and $y_{k+1} E \supseteq y_{k+2} E$

Proof:

(i) By Lemma 3, $B_{*\backslash i}^p E \subseteq B_i^p E$, for all i. Notice that $B_i^p E$ and $B_{*\backslash i}^p E$ are events (*i.e.* measurable with respect to ζ_i and $\zeta_{*\backslash i}$ respectively, for any $E \in \Sigma$). Using P3 and P2, we get $B_i^p B_{*\backslash i}^p E \subseteq B_i^p B_i^p E \equiv B_i^p E \Rightarrow y_1 E \subseteq x_0 E$. By applying again P3, $B_{*\backslash i}^p B_i^p B_{*\backslash i}^p E \subseteq B_{*\backslash i}^p B_i^p E \Rightarrow (B_i^p B_{*\backslash i}^p)^2 E \subseteq B_i^p B_{*\backslash i}^p B_i^p E \Rightarrow y_2 E \subseteq x_1 E$. By doing the same process repeatedly, we obtain $y_{k+1} E \subseteq x_k E$ for any $k \geq 0$.

(ii) Since Lemma 3 is true for any event (e.g. $B_i^p E$) measurable in Σ, it shows that $B_{*\backslash i}^p (B_i^p E) \subseteq B_i^p (B_i^p E) \equiv B_i^p E$. Using P3 and P2, we have $B_i^p B_{*\backslash i}^p B_i^p E \subseteq B_i^p E \Rightarrow x_1 E \subseteq x_0 E$. By applying P3 repeatedly, we obtain $B_{*\backslash i}^p B_i^p B_{*\backslash i}^p B_i^p E \subseteq B_{*\backslash i}^p B_i^p E \Rightarrow B_i^p (B_{*\backslash i}^p B_i^p)^2 E \subseteq B_i^p (B_{*\backslash i}^p B_i^p) E \Rightarrow x_2 E \subseteq x_1 E$. Thus, by following the process continuously, it is clear that $x_{k+1} E \subseteq x_k E$ for all $k \geq 0$.

Now notice that we can write:

$x_{k+1} E \equiv \bigcap\limits_{i \in I} B_i^p (B_{*\backslash i}^p B_i^p)^{k+1} E \equiv \bigcap\limits_{i \in I} (B_i^p B_{*\backslash i}^p)^{k+1} B_i^p E \equiv y_{k+1} F$ and

$x_{k+2} E \equiv \bigcap\limits_{i \in I} B_i^p (B_{*\backslash i}^p B_i^p)^{k+2} E \equiv \bigcap\limits_{i \in I} (B_i^p B_{*\backslash i}^p)^{k+2} B_i^p E \equiv y_{k+2} F$, for all $k \geq 0$ and where F is a measurable event in Σ. Thus, it is also clear that $y_{k+2} E \subseteq y_{k+1} E$, for all $k \geq 0$. *q.e.d.*

Proof of Theorem 2:

($\rightarrow$) Suppose E is iterated quasi-common p-belief at θ. By Proposition 3, there exists a joint p-evident set ψ.

Let ψ be a collection of subset G where $G \in \{B_i^p (B_{*\backslash i}^p B_i^p)^k E, (B_i^p B_{*\backslash i}^p)^{k+1} E,$ for all $i \in I$ and $k \geq 0\}$ such that $\theta \in G$, for all $G \subseteq \psi$. Since $\bigcap G \subseteq \psi$, it follows that

$\theta \in \bigcap G \equiv Q^pE \equiv \bigcap\limits_{k\geq 0} x_kE \cap \bigcap\limits_{k\geq 0} y_{k+1}E$. By P2, we obtain $\bigcap\limits_{k\geq 0} x_kE \equiv \bigcap\limits_{k\geq 0} B_i^p x_kE$ and $\bigcap\limits_{k\geq 0} y_{k+1}E \equiv \bigcap\limits_{k\geq 0} B_i^p y_{k+1}E$ which leads to the following equality:

$$Q^pE \equiv \bigcap_{k\geq 0} B_i^p x_kE \cap \bigcap_{k\geq 0} B_i^p y_{k+1}E.$$

Using Lemma 4(ii) and P4, $Q^pE \equiv B_i^p \bigcap\limits_{k\geq 0} x_kE \cap B_i^p \bigcap\limits_{k\geq 0} y_{k+1}E$. Then, by Lemma 4(i) and P4,

$$Q^pE \equiv B_i^p \left(\bigcap_{k\geq 0} x_kE \cap \bigcap_{k\geq 0} y_{k+1}E \right).$$

Thus, $Q^pE \equiv B_i^p\left(Q^pE\right)$ and shows that Q^pE is p-evident event. Since it can be shown also that for $k = 0$, $Q^pE \subseteq x_0E \cap y_1E \equiv B_i^pE \cap B_i^p B_{*\setminus i}^p E \subseteq B_i^pE$, we then conclude by Proposition 1 that event E is common p-belief at θ.

($\leftarrow$) Suppose E is common p-belief at θ. Then, by Proposition 1 (see remark), $\theta \in C^pE \equiv \bigcap\limits_{k\geq 0} B_*^p(B_*^p)^kE$. Since $\bigcap\limits_{k\geq 0} B_*^p(B_*^p)^kE \subseteq \bigcap\limits_{k\geq 0} B_i^p(B_i^p)^kE$, then $\theta \in B_i^p(B_i^p)^kE$, for all $k \geq 0$ and for all $i \in I$. Let ψ be a collection of subset G where $G \in \{B_i^p(B_i^p)^kE, \text{for all } i \in I, \text{for some } k \geq 0\}$. Then it shows that for any $G \subseteq \psi$, $\theta \in G$. Moreover, by construction of ψ, $B_i^pE \subseteq \psi$, for all i. Hence, from Proposition 3, it follows that E is iterated quasi-common p-belief at θ. *q.e.d.*

The short proof on the statement that all common p-belief events are iterated quasi-common p-belief comes from the fact that Q^pE is generally a weaker variant of common p-belief *i.e.* $C^pE \subseteq Q^pE$. The converse, however, is not straightforward and it requires to show that Q^pE is a decreasing sequence among its elements x_kE and $y_{k+1}E$ for all $k \geq 0$. And it is shown that this can be achieved when the number of players is sufficiently large.

6. A Case of Agreeing to Disagree.

The immediate question that comes from the result of Theorem 2 is that if convergence is achieved only at a sufficiently large number of players, how would the two belief processes differ at every (small) number of players. More specifically, this question attempts to determine at which probability will an iterated quasi-common

p-belief event for a given number of players be commonly believed[6]. The following proposition answers this question concretely.

Proposition 4. For a given number of players n and for any $p < 1$, if event E is iterated quasi-common p-believed, then it is commonly believed with probability $1 - \frac{n(1-p)}{n-1}$ *i.e.* $Q^p E \subseteq C^{1-\frac{n(1-p)}{n-1}} E$.

Remark. It is easily seen from the claim of Proposition 4 that as n approaches infinity, we get $Q^p E \subseteq C^p E$, which supports the idea behind the first part of the proof of Theorem 2.

Proof:

Since we have by definition $Q^p E \equiv \bigcap_{i\in I} \left(\bigcap_{k\geq 0} B_i^p \left(B_{*\backslash i}^p B_i^p \right)^k E \cap \bigcap_{k\geq 0} (B_i^p B_{*\backslash i}^p)^{k+1} E \right)$, then for all i, $\theta \in \bigcap_{k\geq 0} B_i^p \left(B_{*\backslash i}^p B_i^p \right)^k E$ and $\theta \in \bigcap_{k\geq 0} (B_i^p B_{*\backslash i}^p)^{k+1} E$. Let $\psi = \{B_i^p E,\ B_i^p B_{*\backslash i}^p E,$ for all $i \in I\}$ i.e ψ is the smallest collection of joint p-evident events derived from $Q^p E$. Thus, for every $Q_i^p E$ there exists a set $\left\{ B_i^p E \cap B_i^p B_{*\backslash i}^p E \right\}$. From P5, $B_i^p B_{*\backslash i}^p E \Rightarrow \varphi(B_{*\backslash i}^p E \left| B_i^p (B_{*\backslash i}^p E)\right.) \geq p$. Then we have, $\varphi\left(\bigcap_{j\in I\backslash i} B_j^p E \left| B_i^p (B_{*\backslash i}^p E) \right.\right) \geq p$ which implies the following:

$$\begin{aligned}
\varphi\left(\bigcup_{j\in I\backslash i} (\Theta\backslash B_j^p E) \left| B_i^p (B_{*\backslash i}^p E) \right.\right) &\leq \sum_{j=1}^{n-1} \varphi\left(\Theta\backslash B_j^p E \left| B_i^p (B_{*\backslash i}^p E) \right.\right) = \gamma(1-p) \\
\varphi\left(B_j^p E \left| B_i^p (B_{*\backslash i}^p E) \right.\right) &\geq 1 - \frac{\gamma(1-p)}{n-1},
\end{aligned}$$

where $\gamma \in [1, n]$ is the measure of set-overlapping i.e. $\gamma = 1$ means that all the subsets are disjoint. Thus, $B_i^p B_{*\backslash i}^p E \Rightarrow B_i^q E$, where $q = 1 - \frac{\gamma(1-p)}{n-1}$. However since q is always at least as big as p for all $p < 1$ and for all $n \geq 2$ (otherwise $p > 1$, a contradiction), then by P6, $B_i^q E \subseteq B_i^p E$ which implies that $Q_i^p E \Rightarrow B_i^q E$.

[6] This type of comparison was used and examined by Morris (1999) between (two-player) iterated p-belief and common p-belief. His result differs from this study since it focuses on the number of states in Θ and makes use of the fixed point characterization of iterated p-belief which only gives a "loose" bound for a divergence to occur between the two belief processes. In Proposition 4, we examine the role of the number of players in comparing the two processes by directly employing their definitions in order to give a tighter bound for divergence.

Now, note that $B_i^q E$ is also a measurable event and that $B_i^q E = B_i^q(B_i^q E)$. So we have $\varphi\left(B_i^q E \,|\Psi_i(\theta)\right) \geq q$ which implies $\varphi\left(\Theta \backslash B_i^q E \,|\Psi_i(\theta)\right) \leq 1-q$. Then,

$$\begin{aligned} \varphi\left(\bigcup_{i=1}^{n}\left(\Theta \backslash B_i^q E\right)|\Psi_i(\theta)\right) &\leq \sum_{i=1}^{n} \varphi\left(\Theta \backslash B_i^q E \,|\Psi_i(\theta)\right) = \frac{n}{\gamma}(1-q) \\ \varphi\left(\Theta \backslash \bigcup_{i=1}^{n}\left(\Theta \backslash B_i^q E\right)|\Psi_i(\theta)\right) &\equiv \varphi\left(\bigcap_{i=1}^{n} B_i^q E \,|\Psi_i(\theta)\right) \geq 1-\frac{n}{\gamma}(1-q) \end{aligned}$$

Then, by definition of $(1-\frac{n}{\gamma}(1-q))-$belief, we have $\theta \in B_i^{1-\frac{n}{\gamma}(1-q)}\left(\bigcap_{i=1}^{n} B_i^q E\right)$. Thus,

$$Q^p E \equiv \bigcap_{i=1}^{n} B_i^q E \subseteq B_i^{1-\frac{n}{\gamma}(1-q)}\left(\bigcap_{i=1}^{n} B_i^q E\right) \equiv B_i^{1-\frac{n}{n-1}(1-p)}\left(Q^p E\right)$$

which shows that $Q^p E$ is an evident $(1-\frac{n}{n-1}(1-p))$-belief event where $\theta \in Q^p E$. Moreover, since it is always true that $p \geq 1-\frac{n}{n-1}(1-p)$ (otherwise, $p>1$), then by P6, $Q^p E \subseteq B_i^p E \subseteq B_i^{1-\frac{n}{n-1}(1-p)} E$. Hence, we conclude by Proposition 1 that E is a common $(1-\frac{n}{n-1}(1-p))$-belief at θ. *q.e.d.*

Aumann (1976) showed that it is impossible to agree to disagree when the posteriors of a certain event is common knowledge. In the case of common p-belief, Monderer and Samet (1989) showed that posteriors can no longer coincide and could differ by at most $2(1-p)$. This bound, however, was later improved by Neeman (1996a) to $1-p$.

Theorem 3. If the posteriors of the event E are iterated quasi-common p-belief at some $\theta \in \Theta$, then the maximum difference of any two posteriors is $\frac{n(1-p)}{n-1}$.

Proof:

From Proposition 4, E is common $(1-\frac{n(1-p)}{n-1})$-belief. Then, by following the result of Neeman (1996a), the maximum distance between any two posteriors is $\frac{n(1-p)}{n-1}$. *q.e.d.*

Remark: One can see here that as the number of players increases, the difference of any two posteriors approaches $1-p$ which is the maximum bound set by Neeman (1996a) for common p-belief. This again confirms the assertion that an increasing number of players allows iterated quasi-common p-belief to approximate common p-belief. Furthermore, note that at $p=1$, the posteriors of a certain event are common

1-belief and therefore coincide with one another (Aumann, 1976 and Brandenburger & Dekel, 1987) regardless of the number of players.

We show in a simple example where the maximum posterior difference reaches the limit predicted by Theorem 3.

Example 3. Let $\Theta = \{a, b, c, d\}$ and $I = \{1, 2\}$. Set $\varphi(\{a\}) = \varphi(\{b\}) = 1 - p$ and $\varphi(\{c\}) = \varphi(\{d\}) = \frac{(2p-1)}{2}$, for $p < 1$. Suppose also that $\Psi_1 = \{\{a, b\}, \{b, c\}, \{d\}\}$ and $\Psi_2 = \{\{a, c\}, \{b, d\}\}$. Now, consider the event $E = \{a, c\}$. At $\theta = c$, the posteriors of the two players are:

$$B_1^p E = B_{*\backslash 2}^p E = \{\theta \mid \varphi(E \mid \{b, c\}) = 2p - 1\} = \{b, c\} \text{ and}$$
$$B_2^p E = B_{*\backslash 1}^p E = \{\theta \mid \varphi(E \mid \{a, c\}) = 1\} = \{a, c\}$$

Furthermore, these posteriors can be shown to be iterated quasi-common p-belief at θ. Notice first that $B_1^p B_{*\backslash 1}^p E = B_1^p B_{*\backslash 1}^p B_1^p E = ... = \{c\}$ and $B_2^p B_{*\backslash 2}^p E = B_2^p B_{*\backslash 2}^p B_2^p E = ... = \{c\}$. Then let ψ be a collection of subset G where $G \in \{B_i^p \left(B_{*\backslash i}^p B_i^p\right)^k E,$ $(B_i^p B_{*\backslash i}^p)^{k+1} E,$ for $i = \{1, 2\}$ and $k \geq 0\}$ which shows that $\psi = \{\{a, c\}, \{b, c\}, \{c\}\}$. Observe that $B_i^p E \subseteq \psi$ for $i = \{1, 2\}$; and for all $G \subseteq \psi, c \in G$. Finally, $B_i^p G \subseteq \psi$ for all $G \subseteq \psi$.

Note now that the difference of the posteriors is $2(1 - p)$, which is the bound set by Theorem 3 for two players. However, the posterior of player 1 must be at least p (*i.e.* $2p - 1 \geq p$) in order to obtain $\{b, c\}$. Thus, the result is trivial since the limit $p = 1$ is already reached and so the bound for the difference cannot anymore be extended.

7. Conclusion.

This study which takes a second look at the notion of common belief has achieved a twofold objective. The first is it provides necessary and sufficient conditions for selecting an equilibrium in an environment where payoff states are commonly believed. The main ingredient in our setup is the duality of beliefs that allows individuals to obtain a meaningful comparison between a particular event and its (complementary) alternative. Given this and through a deductive approach, we showed that all agents will end up choosing an action favorable to the event that they all p-believe if and only if such event is commonly believed to have higher probability of occurrence (*i.e.* $p > 1/2$) than its alternative.

The second is the introduction of a weaker form of common belief dubbed as iterated quasi-common p-belief. Its weakness lies on the iterated binary process that makes believing of an event unstable at small number of players. Nonetheless, when the population becomes larger, it is shown that this converges to the standard notion of common p-belief. It therefore gives common belief a general characterization in terms of the number of players and which also allows for the differences in the players' posterior beliefs to be depicted in a more general fashion.

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