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Real Options: A Review of Select Theories and Applications

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Abstract

This paper is an introduction to the concepts and methods used in the field of real options as they related to investments. The analog between financial and real options is explained. The discrete version of a model is introduced, then solutions to the canonical model in continuous time using dynamic programming and contingent claims analysis are discussed. Finally, the paper covers extensions of the canonical model to various other option structures.

I. Introduction

This research paper aims to provide an introduction into developments in the field of real options theory over the last few decades. It aims to serve as a basis of lecture notes for my course at the University of the Philippines. Real options theory draws parallels with the financial options. This is insofar as real options confers to anyone who possesses them, the right to acquire, let go of, purchase or sell real assets, much like the owner of a financial option can at some point, exercise the right to buy or sell underlying financial assets. A real option therefore refers to the discretion to take a course of action. From a firm's point of view, this often refers to the option to buy or sell capital assets, or to undertake investment or incur certain other expenditures such as hiring (or the opposite of all these). Some of the seminal articles on real options effects (McDonald and Siegel, 1986, Pindyck and Dixit, 1994) point out that this discretion has economic value as it imparts flexibility on the part of firms to exploit new information about their potential actions over the passage of time. These seminal papers therefore emphasize that firms can exploit the option to delay new investment because there is a value to waiting due to inherent uncertainty in the outcomes of actions in future states. The value to waiting on investment is analogous to the value of a call option (to buy) an underlying share of asset, which is dependent on the variance of the value of the underlying shares, exercise price, the duration of time in which the option is valid, among other things. Over the years, other authors have drawn analogs between other kinds of firm actions and other types of financial options (such as put options, which is the option to sell an underlying asset). The analogs between real and financial options have led authors to emphasize the role played by uncertainty in business decisions. Meanwhile, principles for valuing and pricing financial options have also been applied to the valuation of real options. This has given rise to modes of investment analysis with very different (invest/do not invest) threshold criteria from traditional methods of analysis such as discounted cash flow and net present value (NPV) analysis.

The proposed research will explore extensions of the original theory with potential applications of real options analysis to the Philippines and/or to regional economies to shed light on policy debates and firms and investment valuation.

II. Real Options and the Economics of Uncertainty

Real options theory is an extension of the theories developed in financial options to real activities such as investments, disinvestment, hiring and firing. Financial options can be valuable in uncertain environments because they confer on their owners the ability to exploit a wider variety of scenarios compared to relatively stable and more deterministic environments. Similarly, real options are valuable because agents that possess them can better exploit volatile environments for gain. This section will review the economics of uncertainty and the role played by uncertainty in influencing the behavior of economic agents. I will also review situations that give rise to real options, provide an overview of the varieties of real options identified in the literature and discuss the analogy between financial and real options valuation.

In valuing real options, focus will be on providing economic intuition behind the derivation of valuation formulas and potential actions of economic agents when confronted with uncertainty.

Models of investment prior to the development of the canonical model thus far ignore two characteristics of investments: Irreversibility of sunk costs and since investment can be delayed, a firm can wait for new information about prices, costs and other market conditions before committing resources.

When investments are irreversible and decisions to invest can be postponed, the typical criterion for investment – invest when the present value of expected cash flows is at least as large as cost is incorrect. In most cases, the opportunity to delay investment is feasible (as firms wait for new information about economic conditions).

The Analogy Between Real and Financial Options

An irreversible opportunity to invest is like a financial call option. A call option gives the owner the right, for some specified period, to pay an exercise price and in return receive an asset (e.g., a share of stock) that has some value. Exercising the call option is irreversible. Although the investors can sell the asset to other investors, they cannot recover the option or the money they paid to exercise it. A firm with an investment opportunity likewise has the option to spend the money (the “exercise price”) now or in the future in return for an asset (an example being a project) of some value. Again, it can sell the asset to another firm, but it cannot reverse the investment. As with a financial call option, this option to invest is valuable because of the potential for a growing net payoff if the value of the asset rises. If the asset falls in value, the firm need not invest and will lose only what it spent to obtain the investment opportunity.

Firms get investment opportunities through patents or ownership of land or natural resources. More generally, they arise from a firm’s managerial resources, technological knowledge, reputation, market position and possibly scale, which enable the firm to undertake investments that individuals or other firms cannot undertake.

The key is that the option to invest is valuable. A substantial amount of the market value of most firms arises from their options to invest and grow, as opposed to the capital they have in place.

When a firm makes an irreversible investment expenditure, it exercises or “kills” its option to invest. It gives up the opportunity of waiting for new information that might affect the desirability or timing of the expenditure, it cannot disinvest should market conditions change adversely. The value of this lost option is an opportunity cost that must be included as part of the investment. Hence, the rule of NPV – invest when the value of a unit of capital is at least as large as the purchase and installation cost - must be modified. The value of the unit MUST exceed the cost of the purchase and installation by an amount equal to or above the value of keeping the investment option alive.

Recent studies have shown that the opportunity cost of investing can be large and the investment rules ignoring it can be erroneous. In addition, the opportunity cost of investing is highly sensitive to the future value of the investment project, so that changing economic conditions affecting the perceived riskiness of future cash flows can have a large impact on investment spending. This can explain why models of neoclassical investment are not empirically successful.

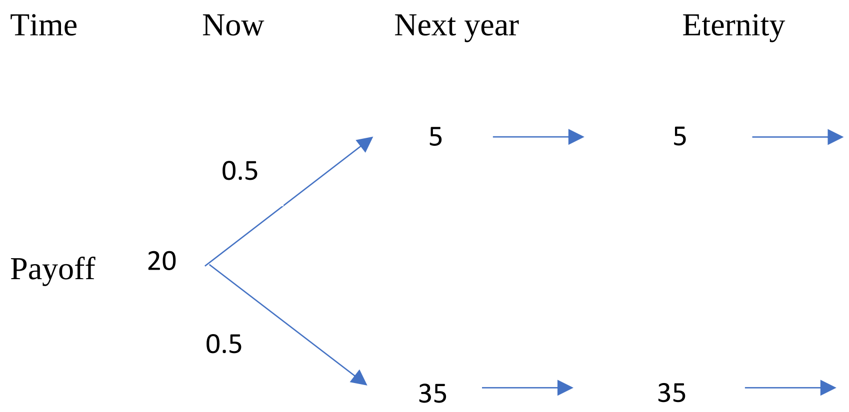
III. Application of Real Options: Binomial Decision Trees

Binomial decision trees (or the binomial lattice model) have been used to value financial options. However, they can also be used to solve real options valuation problems. The method uses a discrete time framework for working out problems. Binomial decision trees can be used to model the flexibility possessed by risk-neutral firms in maximizing uncertain project values at each node over time. At each node or time period, risk-neutral firms are modeled as having the flexibility to decide between two courses of action that yield uncertain project outcomes, the projects being real assets. The risk-neutral probabilities at each node may be derived from market data or from assumptions based on theory. Risk-neutral probabilities can be used to determine the expected value of payoffs in the next period. In the following example, decision tree analysis can be used to determine the timing of an investment with uncertain outcomes.

Consider a firm with the managerial flexibility to decide the timing of an investment worth 205 now. That investment immediately yields 20 but faces an uncertain payoff in the next period. With probability of 50%, the project earns 5 in the next period, and with probability 50%, the project earns 35.

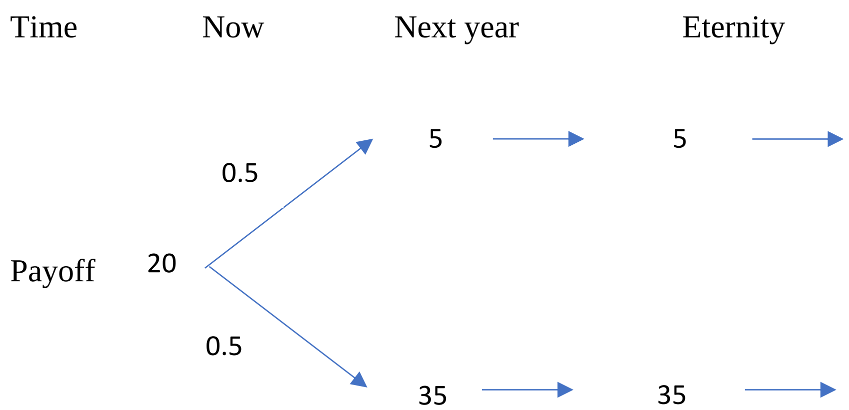
The following diagram depicts the pertinent points of the binomial model. Suppose in the current time period (now), an investor has the option to undertake an irreversible investment in an asset that now costs 205 and now immediately yields 20 with probability 100%. If undertaken next year the investment yields 5 with probability 50% or 35 with probability 50%. What is the value to the investor of the option to invest at each time period? Assume that the discount rate used for the project is 10%.

To value real options in practice, one must first identify whether they exist in a firm’s activities. Options typically exist when the activity involves phases and information arriving in stages that can affect decisions of the firm.



Expected NPV = (20/0.1) – 205 = - 5

Decision: Do not invest now



But if the investor waits until next year to decide

If unfavorable outcome (5), (5/0.1) – 205 = - 155

Decision: Do not invest

If favorable outcome (35), (35/0.1) – 205 = 145

Decision: Invest

Expected payoff when investor waits till next year to invest = (-155 x 0) (145 x 0.7) = 72.5
Final Decision: Do not invest now, invest one year from now.

Since the present value of waiting one year to invest is (72.5/1.1) = 65.91 > PV of investing now (which is - 5) it pays for the investor to wait an additional year to make the irreversible investment. The binomial decision tree problem can be converted into the following table:

Table 1: Outcome of Binomial Tree Model

	Now	Next year	
Outcome		Bad	Good
Probability	100%	50%	50%
Development cost	205	205	205
Payoff	20	5	35
NPV of investment	(20/0.1) - 205 = - 5	(5/.1) – 205 = -155	(35/.1) – 205 = 145
Action		Don't invest	Invest
Future values		0	145
Expected values		145 x 0.7 = 72.5	
= Sum (probability x outcome)		(-155 x 0) (145 x 0.7) = 72.5	
PV of alternatives @ 10%	1 x – 5 = - 5	(72.5/1.1) = 65.91	

Note in the example that the investor earns a way greater return from the investment by delaying it and waiting for greater information about the market to flow before buying the asset in the following year.

Binomial tree analysis has been criticized on the basis of using arbitrary discount rates. Critics wanted a more rigorous discount rate. The following sections discuss the real options model in continuous time, which has since become the canonical model for recent modeling of investment.

IV. Solving the Canonical Real Option Model

The canonical real option model of investment owes itself to the seminal literature written by MacDonald and Siegel (1986) and Dixit and Pindyck (1994).[1] The standard real options model of investment timing predicts that, since waiting allows investors to obtain new information about market conditions, increased uncertainty discourages investment. In other words, when market conditions are uncertain, investors possess a valuable call option that is lost when an irreversible decision is made. Hence, there is value to waiting to invest.

In the earliest canonical models of real option theory, firms are expected to invest less when the level of uncertainty is high or increases. The reason is that higher uncertainty raises the option value of waiting, so that it is optimal to wait to invest rather than to immediately exercise their option and undertake costly and irreversible investment. These results are obtained under the assumption that a firm holds monopoly rights over the investment opportunity and the investment does not affect prices or the structure of markets.

[1] Much of this section is based on the latter.

The seminal papers on real options and investment analyze the problem of the timing of sunk investment cost I, in order to obtain a project with value V. Next, suppose V is a stochastic variable and follows a specific form of Generalized Brownian Motion (GBM), an Ito Process, generally:

$$dV = a(V,t)dt + b(V,t)dz$$

Or more specifically, under risk neutrality

$$dV = \alpha V dt + \sigma V dz \qquad (1)$$

The first term on the RHS is the deterministic component. The second term is stochastic. V follows a GBM process with increment dz (of a Wiener process):

$$dz = \varepsilon(dt)^{\frac{1}{2}} \quad , \qquad \varepsilon \sim N(0,1)$$

ε is a serially uncorrelated and normally distributed random variable. Since ε_t has a mean of zero and unit standard deviation, E(dz) = 0 and var (dz) = E[$(dz)^2$] =dt . Hence, the investment project is akin to an American call option.

Equation (1) implies that the current value of the project is known but its future values are lognormally distributed with a variance that grows linearly over time. Despite the fact that information arrives over time, the future value of the project is always uncertain. McDonald and Siegel (1986) suggest that the value of the investment opportunity above is analogous to a perpetual call option and the decision to invest is akin to the exercise of the call option. Analogous to the solution of financial call options, there are also two ways of solving real options problems. One through dynamic programming, and another via contingent claims techniques.

Suppose there is a sunk, fixed and irreversible investment cost I. To solve the problem of investment using dynamic programming, a rule is needed that maximizes the value of the investment opportunity, F(V):

$$F(V) = \max \mathrm{E} \underbrace{[(V_T - I)\mathrm{e}^{-\rho T}]}_{Payoff\ from\ investing\ at\ time\ t=T} \qquad (2)$$

At some unknown future time t = T, the investment will be made. ρ is the discount rate and maximization is subject to (1) for V. Assume that $\rho > \alpha$ and denote $\delta = \rho - \alpha$. $\rho = \delta + \alpha$ is the total expected return = dividend ratio plus rate of capital gain. The discount rate for this project is ρ. For the problem to make economic sense, it must be the case that α < ρ. If α = ρ, the value of the option to invest will be the same as the underlying asset and it will never be rationally exercised (why?).

At this point, one can ask two questions: (a) How much is this opportunity worth? and (b) at what point is it optimal to launch the project?

Assume that a decision maker has the opportunity to invest in a project whose value V is stochastic over time period $t \in [0, \infty]$. Let z be a Wiener process. Note that the payoff to the investment, V – I is the present discounted value of the payoff stream from the time the option to invest is exercised. It is necessary to assume that $\alpha < \rho$. Otherwise, the value of the project increases indefinitely as t approaches infinity. Note that I is lost permanently, because the value of the investment has an irreversible sunk cost. The value F(V) will be expanded later using Ito's lemma. This yields:

$$dF = \frac{\delta F}{\delta V} dV + \frac{\delta F}{\delta t} dt + \frac{1}{2}\frac{\delta^2 F}{V^2}(dV)^2$$

In the deterministic case where σ = 0 (stochastic component = 0):

$V(t) = V_0 e^{\alpha t}$ where $V_0 = V(0)$, so given the current V, the value of the option to invest at some future time T is

$$F(V) = [(Ve^{-\alpha T} - I)e^{-\rho T}] \quad (3)$$

Suppose $\alpha \leq 0$, then V(t) will remain constant or fall over time. So it is clearly optimal to invest immediately if V > I and not invest otherwise. Hence, F(V) = Max [V – I, 0}, if $\alpha \leq 0$.

In the case of $0 < \alpha < \rho$: F(V) > 0 even if V < I since eventually, V will exceed I, it may be better to wait than to invest now. To see this, maximize F(V) above with respect to T to get:

$$\frac{dF(V)}{dT} = -(\rho - \alpha)Ve^{-(\rho-\alpha)T} + \rho I e^{-\rho T} = 0$$

Which implies that (solving for T)

$$T^* = Max\left\{\frac{1}{\alpha}\log\left[\frac{\rho I}{(\rho-\alpha)V}\right], 0\right\} \quad (4)$$

$$= Max\{ \quad > 0 \quad , = \ 0\}$$

Therefore, if V is not too much larger than I, $T^* > 0$. So the optimal timing of investment is not the current period, time t = 0, but later at time t > 0. The reason for delaying the investment is that in present value terms, the cost of the investment declines by a fraction of $e^{-\rho T}$, whereas the payoff is reduced by the smaller factor of $e^{-(\rho-\alpha)T}$. **Hence, even without uncertainty, there is already value in waiting to invest.**

If one sets $T^* = 0$, one gets the critical value of V (= V^*) at which one should invest immediately: $V \geq V^*$, where

$$V^* = \left(\frac{\rho}{\rho - \alpha}\right) > I$$

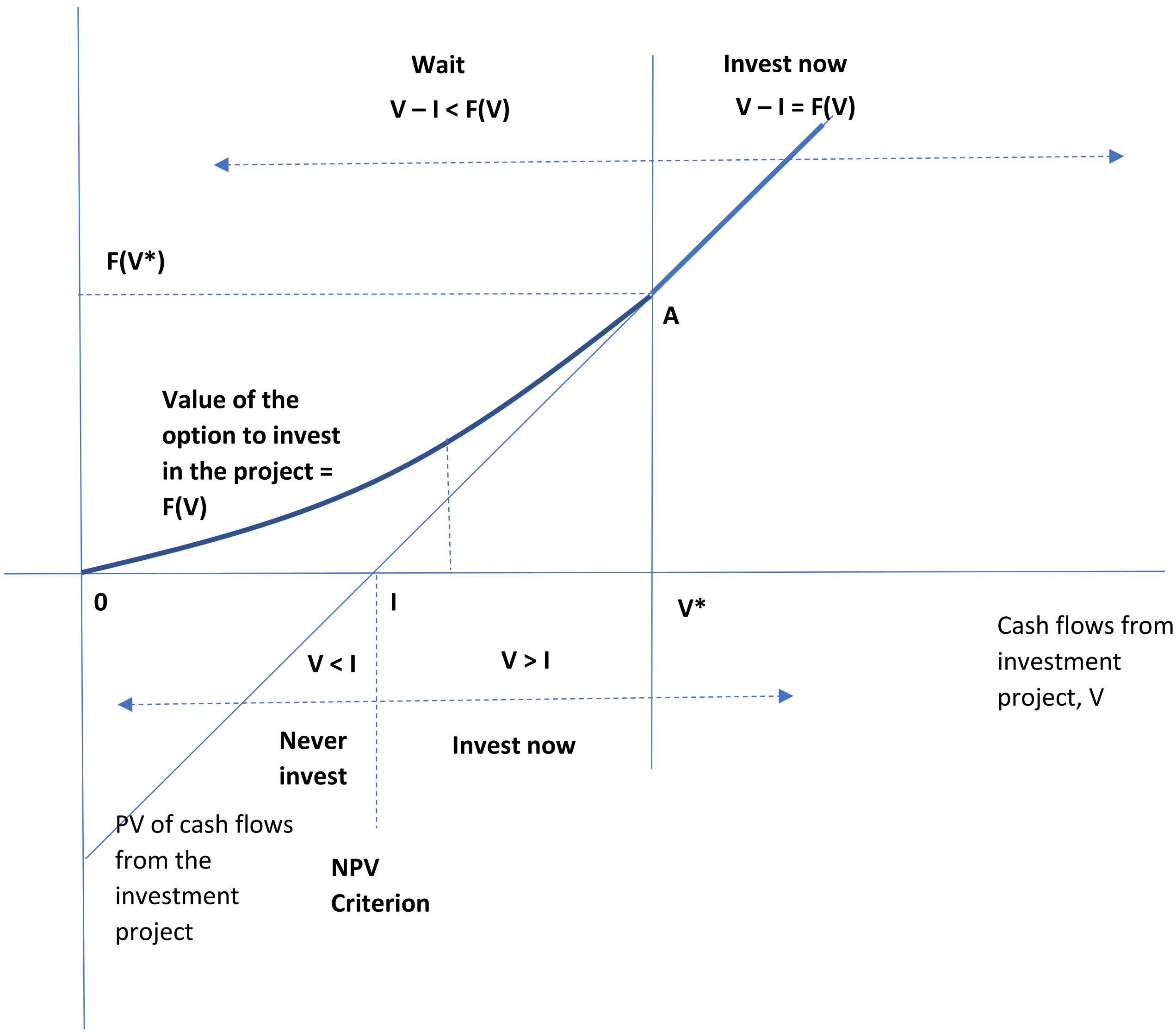
The condition above says that the value of the investment should cross this critical value V* so that the project should be worthwhile pursuing.

Finally, substitute (4) into (3) to get:

$$F(V) = \begin{cases} \left(\frac{\alpha I}{\rho - \alpha}\right)\left(\frac{(\rho - \alpha)V}{\rho I}\right)^{\frac{\rho}{\alpha}}, & V \leq V^* \\ V \; - \; I & , V > V^* \end{cases} \tag{5}$$

A graph of this is depicted in Figure (5.1) in the book by Pindyck and Dixit (1994) for I = 1, $\rho = 0.10$ and $\alpha = 0$. A general diagram is in the next page. The diagram is also applicable when the environment is not deterministic. It will be explained in greater detail in the next section.

Figure 1



Wait
V – I < F(V)
Invest now
V – I = F(V)
F(V*)
A
Value of the option to invest in the project = F(V)
0
I
V*
Cash flows from investment project, V
V < I
V > I
Never invest
Invest now
PV of cash flows from the investment project
NPV Criterion

Stochastic Case

Next, suppose σ > 0? The problem is how to determine the point at which it is optimal to invest I in return for an asset worth V when the environment is uncertain. The investment rule will now take the form of a critical value of V^* such that it is optimal to invest once $V \geq V^*$. We will see that a higher value of σ will result in a higher value of V^* (a greater value to waiting). In general, both growth (α > 0) and uncertainty (σ > 0) can create a value to waiting and hence delay the timing of investment.

Consider an investor making an investment decision under an uncertain environment. The decision to invest under uncertainty can also be modeled in the graph in Figure 1. Think of the vertical axis as measuring net present value of investment projects. The horizontal axis measures the uncertain cash flows from the project, V or investment cost I. The upward sloping line is the present value of the cash flows to the project. The NPV rises along with the cash flows. It intersects the horizontal axis at I, the cost of investment. At this point of intersection, PV = I and NPV = 0, at which point conventional decision-making following the NPV criterion would recommend the investment be made. However, the option value of waiting exceeds V at point I (**V – I < F(V)**) and therefore it is optimal to delay investment at V = I. It is only optimal to invest when F(V) = V*.

But the real options criterion is very different. Under real options, investors decisions are dynamic in nature. Unlike the static decision-making using the NPV criterion (invest now or not at all), investors make investment decisions (to invest or not to invest) in each period. The value of the option to invest, F(V), is compared to an optimal threshold V called V*. F(V) is also growing as V increases (why?). The optimal threshold V* is dependent on several factors and is > I. So, the question is about the optimal timing of investment an optimal timing of investment exists.

The option to invest has value F(V) because:

- The passage of time resolves uncertainty;

- If a year from now, the conditions deteriorate, the investor can decide not to invest in a bad project; and

- By waiting the investor is cutting off some of the left tail of the distribution of outcomes.

The real options approach gives a more restrictive investment strategy since the value of waiting for information about uncertain future trends, which affect the project's cash flow (area AI0 in the graph above) is explicitly taken into account in the project evaluation.

When F(V) < V*, the option to invest is not exercised since the option to invest still does not exceed the real option investment threshold and the investor waits to invest. In the zone below V*, the investor obtains the value of the unexercised option.

When F(V) > V*, the option to invest is exercised and the investment takes place. It is the optimal time to invest. Thereafter, F(V) = V – I and the investor obtains the excess of cash flows over the cost of investment.

There exists a critical value of V^* so that it is optimal to wait if $V < V^*$ (continuation region)

$$F\big(V(t)\big) = e^{-\rho dt}E[F(V(t+dt)]$$

There will also be a stopping region: invest if $V > V^*$

$$\Omega\big(V(t)\big) = V(t) - I = \text{F(V)} = \text{payoff at stopping region}$$

It is optimal to wait if $V < V^*$ (continuation region). The tangency point of F(V) with V – I is at the critical value $V^* = \left(\frac{\rho I}{\rho-\alpha}\right)$. Beyond that and to the right is the stopping region. Note that F(V) increases when α increases as does the critical value V^* (growth α pushes the tangency point to the right). Growth in V creates a value to waiting and increases the value of the investment opportunity (= the option to invest).

Also note from Figure 1 that the more volatile is V, the more likely it is that V* will be breached. So, the option to invest is more valuable the more volatile is V.

There are two ways of solving the canonical model. The first is via dynamic programming. The second is via contingent claims analysis.

Dynamic Programming (DP) Solution

The problem is to invest I and get V, with V a GBM

$$dV = \alpha V dt + \sigma V dz$$

The payout rate is δ and the expected rate of return is ρ so that $\alpha = \rho - \delta$. We want to solve for the value of the investment opportunity (the option to invest), F(V) and the decision rule V*.
Since the investment opportunity F(V), yields no cash flows up to time T when the investment is made, the only return from holding the option to invest is capital appreciation.

The solution via dynamic programming must satisfy the Bellman equation:

$$\rho F dt = EdF \quad (6)$$

F is F(V) and this can be expanded using Ito’s lemma.

Derivation of the DP Solution

The LHS term is the total expected return on the investment opportunity, while the RHS is the rate of capital appreciation. The DP method transforms the problem into an optimal stopping problem.

Recall that in DP, one breaks down a larger problem into a set of smaller problems. A function that is defined over all time periods t is divided into a discrete step Δt and the full solution is formed by solving for the value of the function one step at a time.

Let π(x(t), t) be the rate of profit flow from the investment. Hence, total profit is π(x(t), t) Δt and total discounting over a discrete step is $\frac{1}{1+\rho\Delta t}$.

Hence, the value of the investment at continuous time t is

$$F(V,t) = Max\left\{\pi(V(t),t) + \frac{1}{1+\rho}E[F(V(t+1),t+1)]\right\}$$

This is the Bellman equation in continuous time. On each discrete step Δt, we have

$$F(V,t) = Max\left\{\pi(V(t),t)\Delta t + \frac{1}{1+\rho\Delta t}E[F(V(t+1),t+\Delta t)]\right\}$$

Multiply both sides by $1 + \rho\Delta t$ to get

$$(1+\rho\Delta t)F(V,t) = Max\{\pi(V(t),t)\Delta t(1+\rho\Delta t) + E[F(V(t+1),t+\Delta t)]\}$$

Subtract $F(V,t)$ from both sides to get

$$\rho\Delta t F(V,t) = Max\{\pi(V(t),t)\Delta t(1+\rho\Delta t) + E[F(V(t+1),t+\Delta t) - F(V(t),t)]\}$$

$$= Max\{\pi(V(t),t)(1+\rho\Delta t)\Delta t + E[\Delta F(V(t),t)]\}$$

Then, divided both sides by Δt and let t -> 0, which leads to

$$\rho F(V,t) = Max\left\{\pi(V(t),t) + \frac{1}{dt}E[dF(V(t),t)]\right\}$$

Assume that there are no profits as the project generates cash flow only at the time the investment is undertaken so that $\pi(V(t),t) = 0$ so that the Bellman Equation reduces to

$$\rho F(V,t)dt = E[dF(V(t),t)] \qquad (6.1)$$

LHS = RHS

This is a no-arbitrage condition. The LHS is the discounted normal return that an investor would require from holding an option. The RHS is the expected total return per unit of time from holding the option.

If this condition holds, then the firm is equating the expected return from delaying the investment with the opportunity cost of delay.

Since we assumed that V follows Geometric Brownian Motion (GBM), we can expand the RHS of (6.1) using Ito's lemma, the properties of the Wiener process for z and E(dz) = 0 to obtain the total differential of the continuous time stochastic process, which is the RHS of equation (6):

RHS of (6.1) $= E[dF(V,t)] = E\left[\frac{\partial F(V)}{\partial V}\right] \alpha V dt + \sigma V dz + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}(\alpha V dt + \sigma V dz)^2 + \cdots$

$$= E\left[\frac{\partial F(V)}{\partial V}\alpha V dt + \frac{\partial F(V)}{\partial V}\sigma V dz + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2 dt\right]$$

$$= \frac{\partial F(V)}{\partial V}\alpha V dt + \frac{1}{2}\frac{\partial^2 F(V)}{\partial V^2}\sigma^2 V^2 dt = \rho F(V) dt$$

$$= \alpha V F'(V) + \frac{1}{2}\sigma^2 V^2 F''(V) = \rho F(V) \qquad = \text{LHS of (6.1)}$$

The LHS of (6.1) is expected rate of capital appreciation. The RHS is the total expected return to the investment opportunity. $\alpha = \rho - \delta$. To ensure that am optimum exists, Assume $\delta < \rho$ or $\delta > 0$. With this, rewrite the Bellman equation as:

$$(\rho - \delta) V F'(V) + \frac{1}{2}\sigma^2 V^2 F''(V) - \rho F(V) = 0 \qquad (7)$$

Based on (7), F(V) must satisfy the following boundary conditions:

the

$$F(0) = 0 \qquad (8)$$

$$F(V^*) = V^* - I \qquad \text{value-matching condition} \qquad (9)$$

$$\frac{dF(V^*)}{dV} = 1 \qquad \text{smooth pasting condition} \qquad (10)$$

The first boundary condition states that if V goes to zero, it will stay at zero so that the option to invest will have no value. Meanwhile, V^* is the price at which it is optimal to invest and condition $F(V^*) = V^* - I$ states that upon investing, the firm receives a net payoff of $V^* - I$. The last boundary condition is called the "smooth pasting" condition. If F(V) were not continuous and smooth at the critical option exercise point V^*, an investor could do better by exercising the option at a different point.

As for the second boundary condition, on $I = V^* - F(V^*)$. So when a firm invests, it gets the project value V*, but also gives up the opportunity or option to invest, which is valued at F(V).

$$V^* \quad = \quad F(V^*) \quad + \quad I$$

Critical value = opportunity cost of investment + direct cost of investment

This can be solved using the option pricing approach. This way, one avoids the problem of choosing ρ and α.

We have 2 other optimality conditions for the solution

$$F(V^*) = V^* - I$$

This is the valid payoff at the optimal stopping point. V^* is the critical value of V at which it is optimal to invest

$$\frac{dF(V^*)}{dV} = 1$$

Determines the unique stopping point. It is also called the smooth pasting condition. If F(V) were not continuous or smooth at the exercise point V^*, one can do better by exercising at a different point.

The solution boundary that satisfies the boundary and optimality conditions can be derived as follows:

$\alpha V F'(V) + \frac{1}{2}\sigma^2 V^2 F''(V) - \rho F(V)$= 0 is a second order differential equation. To solve for F(V), solve this equation subject to the boundary conditions. To satisfy F(0) = 0, the solution must have the form

$$F(V) = AV^{\beta_1} \qquad (11)$$

Where A is a constant, and the root β_1 is a constant whose value depends on σ, ρ and δ. To solve for V^*, insert (11) into (9) and (10) and rearrange to get the optimal investment rule

$$V^* = \frac{\beta_1}{\beta_1 - 1} I \qquad (12)$$

So that

$$A = \frac{(V^* - I)}{V^{*\beta_1}} = \frac{(\beta_1 - 1)^{\beta_1}}{[(\beta_1)^{\beta_1})I^{\beta_1 - 1}]}$$

Given the above, $V^* > I,$ so the simple net present value rule is violated. Uncertainty and irreversibility drive a wedge between V^* and I in (12).

Next, note that (7) is a second order differential equation in F(V) of form $ar^2 + br + c = 0$. Equation (7) is linear in F. For such second order differential equations, the general solution can be expressed as a linear combination of any 2 independent solutions.

If we try solution AV^{β}, insert into (7) as F(V) and get:

$$\frac{1}{2}\sigma^2\beta(\beta - I) + (\rho - \delta)\beta - \rho = 0$$

So that the two roots that solve the above are

$$\beta_1\,,\ \beta_2 = \frac{1}{2} - \frac{(\rho - \delta)}{\sigma^2} \pm \sqrt{\left(\frac{(\rho - \delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} \quad > 1$$

So that the general solution to (7) can be written as

$$F(V) = A_1V^{\beta_1} + A_2V^{\beta_2}$$

The boundary condition F(V) = F(0) = 0 implies that A_2=0, leaving $A_1V^{\beta_1}$. Note that the quadratic expression (= Q) is an inverted parabola (See page 143 of Dixit and Pindyck (1994)). One can then use comparative statics to determine the effects of various parameters on β_1 and hence on size of wedge and V^* in (12).

We can see from the roots that as σ (level of uncertainty) increases, β_1 declines and the wedge falls and the larger is the excess return the firm will demand before it is willing to make the irreversible investment. Also, as δ increases, β_1 increases (lower wedge) and as ρ increases, β_1 falls and the wedge increases.

Derivation of the Solution via Contingent Claims Analysis

Contingent Claims Analysis (CCA) Chapter 4 section 2 of Dixit and Pindyck

The problem is to invest I and get V, with V a GBM

$$dV = \alpha Vdt + \sigma Vdz \qquad (12)$$

The payout rate is δ and the expected rate of return is ρ so that α = ρ - δ. We want to solve for the value of the investment opportunity (the option to invest), F(V) and the decision rule V*.

Let x be the price of an asset or dynamic portfolio of financial assets perfectly correlated with V and denote by ρ_{vm} the correlation between V and the market portfolio. x evolves according to dx = μxdt + σxdz. Furthermore, according to the Capital Asset Pricing Model (CAPM), the expected return on V is $\mu = r + \phi\, \rho_{vm}\ \sigma$, where r is the risk-free rate and ϕ is the market price of risk. Assume that α, the expected percentage change of V, is less than its risk adjusted rate of return, μ. The difference between μ and α is denoted by δ and $\delta = \mu - \alpha$. Analogous to the financial call option, if V were the price of a share of common stock, δ would be the dividend rate on the stock. Hence, the total rate of return on holding the stock would be $\mu = \alpha + \delta$, or the dividend rate plus the rate of capital gain. Note that α is the return to holding the option, while δ is that part of the return that can only be realized if one actually exercises the call option (to invest I).

If the dividend rate $\delta = 0$, a call option on the stock would never be exercised prematurely and would always be held up to maturity. The reason is that the entire return of the stock is captured in its price movements and hence, by the call option. Hence, there is no cost to keeping the option alive. But if the dividend rate $\delta > 0$, keeping the option to invest alive and not exercising it carries a positive opportunity cost. This cost is the dividend stream that is foregone by holding the option rather than buying the stock. Since δ is a proportional dividend rate, the higher is the price of the stock, the greater is the flow of dividends. Hence, at some high enough price threshold, the opportunity cost of foregoing growing dividends becomes worthwhile enough to exercise the option.

For the present investment problem, μ is the expected rate of return from owning the completed investment project, the equilibrium established by the capital market, and it includes an appropriate risk premium. Since $\mu = \alpha + \delta$, it follows that if $\delta > 0$, then the expected rate of capital gain on the project, α, is less than μ. Hence, δ is an opportunity cost of delaying implementation of the investment project and instead, keeping the option to invest alive, and the investment would never be worthwhile, no matter how high the NPV of the project is. For that reason, $\delta > 0$ is assumed. If, on the other hand, δ is very large, the value of the option will be very small because the opportunity cost of waiting is large. As δ approaches, infinity, the value of the option goes to zero. In effect, the only choices are to invest now or never, and the standard NPV rule will again apply.

The parameter δ can be interpreted in different ways. For instance, it could reflect competitors' entry and expansion of capacity. It could simply reflect cash flows from the project. If the project is infinitely lived, then equation (12) can represent the evolution of V during the operation of the project and δV is the rate of cash flow the project yields. Since δ is assumed to be constant, this interpretation is consistent with the point that future cash flows are a constant proportion of the project's market value.

The CCA method is similar to valuation of financial options similar to (Black and Scholes). It is applicable when the risk dz_t can be spanned by assets that exist in financial markets – it requires a deep set of assets. The market has to be in a state of equilibrium; no arbitrage. Under this method, one can value F(V) without any assumption about the discount rate or the investor's risk attitude (i.e., without knowing r). The price of the *option* is *relative to* other assets that are traded in the market.

To obtain a solution for F(V) and the optimal investment rule via contingent claims analysis, one needs to construct a riskless portfolio (as in Black-Scholes). This can be done by holding the option to invest worth F(V) (= going long on the option) and going short n = F'(V) units of the project. Or, equivalently, of the asset or portfolio x that is perfectly correlated with V.

The value of this portfolio is $\Phi = F - F'(V)V$.

This portfolio is dynamic. As V changes, F'(V) changes, so the composition of the portfolio will also change.

The short position in this portfolio will require will require a payment of $\delta VF'(V)$ dollars per time period. Otherwise, no rational investor will ever enter into the long side of the transaction.

The return from this portfolio over dt equals the change in the value of the portfolio over dt (capital appreciation): $dF - ndV$ and dividend payout $\delta Vndt$

Hence, the total return is $dF - F'(V)dV - \delta VF'(V)\,dt$

$$= dF - ndV - \delta VF'(V)\,dt$$

The change in the value of the portfolio is $dF - ndV$, meanwhile, the dividend payout is $\delta Vndt$. Hence, the total risk-free rate of return on the investment in the portfolio with the long and short positions is

$dF -$ $F'(V)dV$ $-$ $\delta VF'(V)\,dt$

Capital appreciation cost of shorting the project

dF can be expanded using Ito's lemma:

$$dF = \frac{\delta F}{\delta V} dV + \frac{\delta F}{\delta t} dt + \frac{1}{2}\frac{\delta^2 F}{V^2}(dV)^2$$

In revised notation, Ito's lemma is:

$$dF = F'(V)dV + \frac{1}{2} F''(V)\sigma^2 V^2\,dt$$

Hence, the total riskless return on the portfolio is

$F'(V)dV + \frac{1}{2} F''(V)\sigma^2 V^2 \text{ dt} - F'(V)dV - \delta\text{VF'(V) dt}$

First 3 terms is capital appreciation, the last term is the cost of shorting the project

$= \frac{1}{2} F''(V)\sigma^2 V^2 \text{ dt} - \delta\text{VF'(V) dt}$

This is the risk-free rate of return on the riskless portfolio.

Hence, the no-arbitrage condition is that the risk-free rate of return equals the value of the risk-free portfolio:

$r\Phi dt = \frac{1}{2} F''(V)\sigma^2 V^2 \text{ dt} - \delta\text{VF'(V) dt}$

$r[\text{F(V)} - \text{F}'\text{(V)}]dt = \frac{1}{2} F''(V)\sigma^2 V^2 \text{ dt} - \delta\text{VF'(V) dt}$

One can rearrange this and obtain the characteristic equation (using CCA):

$$= \frac{1}{2} F''(V)\sigma^2 V^2 \text{ dt} + (r - \delta)\text{VF'(V)dt} - \text{rF(V)}dt = 0 \qquad (13)$$

Compare this with the dynamic programming solution

$$= \frac{1}{2} F''(V)\sigma^2 V^2 \text{ dt} + \alpha\text{VF'(V)dt} - \rho\text{F(V)}dt = 0$$

Note that the risk-neutral valuation is characterized by replacing ρ by r. and replacing the expected return α by r – δ.

Hence, under contingent claims solution, there are the same boundary, value-matching and smooth pasting conditions as in the dynamic programming solution.

We have the same solution for F(V)

$$F(V) = AV^{\beta_1} \text{ for V} \leq \text{V*}$$

$$F(V) = \text{V - I} \;\; \text{for V} > \text{V*}$$

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

With the same critical value V* and A as before. Therefore, the contingent claims solution to the investment problem is equivalent to a dynamic programming solution under assumption of risk neutrality.

The solutions give the value of the investment opportunity and the optimal investment rule. That is, the critical value of V^* at which it is optimal to invest (the critical value which maximizes the firm's market value). The solution is obtained by showing that a hedged (risk-free) portfolio can be constructed consisting of the option to invest and a short position in the project. However, F(V) must be the solution to equation (13) even if the option to invest (or the project) does not exist and could not be included in the hedge portfolio. All that is required is spanning – it must be possible to find or construct am asset or dynamic portfolio of assets x, that replicates the stochastic dynamics of V. As Merton (1977) has shown, the value function can be replicated with a portfolio consisting only of asset x and risk-free bonds.

V. Extensions of the Canonical Model

The Value of Growth Options in investing in Offshore Oil Reserves (Paddock, Siegel and Smith, 1988 (PSS))

PSS apply option pricing methods to value the multistage investment options in oil exploration and development embedded in conventional petroleum leases. For any given oil field or prospect that is not too far "in the money," random variation in oil price creates an incentive for the company to delay exploration and development. Moreover, this also applies to highly profitable projects if substantial time remains before the lease expires. PSS valuation incorporates the impact of price risk directly, using contingent claims analysis, and demonstrates that holding all else constant (including the expected value of the cash flow stream), increased price volatility increases the value of marginal investments and leads to delays in the timing of commencement.

PSS break down investment in oil industry into 3 components:

1) Exploration
2) Development
3) Extraction

They focus on the development of a proven reserve because this stage involves the greatest capital expenditure and this stage also has the greatest option value. Hence, this stage involves the valuation of an undeveloped reserve and the decision as to WHEN to develop it. PSS reckon that the real option value of an undeveloped reserve and that of a call option on a stock are analogous to one another. The table below shows the analog between financial options and real options.

Table 2: Analogy Between Call Options and Undeveloped Oil Reserves

Call Option	Undeveloped Reserve
Stock price	Current value of undeveloped reserve (V)
Exercise price of option	Cost of investment to develop the reserve (D)
Stock dividend	Cash flow net of depletion as proportion of V (d)
Volatility of stock price	Volatility of developed reserve value (s)
Time to expiration of option	Relinquishment requirement (t)
Risk free interest rate	Risk free interest rate (r)

Pindyck and Dixit (1994) discuss PSS in Chapter 12. First, they derive the value of a developed reserve. Let B_t be the number of barrels of oil in a developed reserve, V_t be the value per barrel of the developed oil reserve and R_t be the instantaneous return to the owner of the developed reserve. R_t will have 2 components: the flow of profit from production and the capital gain on the oil remaining in the reserve.

Production from a developed reserve is modeled as exponentially declining at negative growth rate ω to reflect depletion.

$$\frac{dB}{B_t} = -\omega dt \qquad (14)$$

Given this, the return is

$$R_t = \omega B_t \Pi_t dt + d(B_t V_t)\,\frac{dB}{B_t} = -\omega dt \qquad (15)$$

$$R_t = \omega B_t \Pi_t dt + B_t dV_t - \omega V_t B_t dt \quad (16)$$

Π_t is the after-tax profit from producing and selling a barrel of oil.

Next, assume that the rate of return on the developed reserve (the return as a fraction of total value of the developed reserve) follows a Brownian motion process:

$$\frac{R_t dt}{B_t V_t} = \mu_v dt + \sigma_v dz \quad (17)$$

μ_v is the risk-adjusted rate of return required by a competitive capital market. Combining equations (14) and (15) yields the equation of motion for V, the unit value of a developed reserve:

$$dV = (\mu_v - \delta_t)V dt + \sigma_v dz \quad (18)$$

δ_t is the payout rate from a unit of producing developed reserve. This payout rate equals

$$\delta_t = \omega(\Pi_t - V_t)/V_t \quad (19)$$

Value of Undeveloped Reserves

Next, calculate the value of an undeveloped reserve and the optimal development rule. Let F (V,t) be the value of a one-barrel unit of undeveloped reserve. Using equation (4) and doing the usual derivations, one finds that F (V,t) must satisfy:

$$\frac{1}{2}\sigma_V^2 V^2 F_{VV} + (r-\delta)VF_V - rF = -F_t \qquad (20)$$

This equation will be solved subject to boundary conditions. Letting D be the per barrel cost of developing the reserve (D is the exercise price of the option), the conditions are:

$$F(0,t) = 0 \qquad (21)$$

$$F(V,t) = \max(V_t - D, 0) \qquad (22)$$

$$F(V^*,t) = V^* - D \qquad (23)$$

$$F_V(V^*,t) = 1 \qquad (24)$$

Condition (22) states that expiration, the option to develop will be exercised if $V_t > D$. The other boundary conditions are standard.

Equation (20) cannot be solved analytically, but it can be solved using finite difference methods. The critical value V*/D is not very sensitive to the time to expiry if that time is greater than 1 or 2 years. Hence, for many such investments in oil reserves, it is a reasonable approximation to ignore the relinquishment requirement altogether and simply treat the option to develop as perpetual. Then the term F_t disappears and the equation can be solved analytically.

Equation for undeveloped reserve. F (V,t) must satisfy (6)

$$\frac{1}{2}\sigma_V^2 V^2 F_{VV} + (r-\delta)VF_V - rF = -F_t \qquad (25)$$

F (V,t) be the value of a one-barrel unit of undeveloped reserve
V = value of developed reserve = Can equal BP, P = price of oil per barrel
Let B_t be the number of barrels of oil in a developed reserve, V_t be the value per barrel of the developed oil reserve
δ = dividend yield for V
σ_V^2 = volatility for V
D = development cost

To satisfy F(0, t) = 0, the solution must have the form

$$F(V) = AV^{*\beta_1} \quad (26)$$

Where A is a constant, and the root β_1 is a constant whose value depends on σ, ρ and δ
To solve for V^*, insert (26) into (24) and (25) and rearrange to get the optimal investment rule

$$V^* = \frac{\beta_1}{\beta_1 - 1} D \quad (27)$$

So that

$$A = \frac{(V^* - D)}{V^{*\beta_1}} = \frac{(\beta_1 - 1)^{\beta_1}}{[(\beta_1)^{\beta_1}) I^{\beta_1 - 1}]} \quad (28)$$

Given the above, $V^* > D,$ so the simple net present value rule is violated. Uncertainty and irreversibility drive a wedge between V^* and D.

So, the root β_1 is

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \quad (29)$$

The Appendix to this paper demonstrates that the methods discussed above are also applicable to a situation where the basic uncertainty is over the demand for the project's output. Hence, output price is exogenous, the value of the project is determined as well as the value of the option to invest in terms of the stochastic process for the output price.

VI. Other Real Options Theories – Exceptions to the Canonical Model

Over time, researchers have shown that real options exist whenever investment decisions attain a multi-stage character, involve sequential investments or multiple option like features. The multi-stage project is akin to a compound option – success achieved at each stage gives one the option to do additional stages. This can also be true of research and development activities. Investments such as these have very different characteristics relative to the canonical model, and hence, can produce different outcomes.

Growth and Compound Options in a Model of Real Estate by Bar-Ilan and Strange (1996)

The standard real options model of investment timing predicts that, since waiting allows investors to obtain new information about market conditions, increased uncertainty discourages investment. In other words, when market conditions are uncertain, investors possess a valuable call option that is lost when an irreversible decision is made.

Bar-Ilan and Strange (1996) applied investment lags on irreversible investments and they found that a lag can reduce the effects of uncertainty in an investment, since the investor has more time to act on an unexpected fall in the price or changes in the investment. Bar-Ilan and Strange (1996) show that, when it takes time to build and funds must be committed up front, flexibility at the completion date also gives investors a put option (an option to abandon).

The option to abandon a project which is the right to sell cash flows for the remainder of the project's life in exchange for salvage value, are analogous to American put options. When the present value of the remaining cash flows falls below the liquidation value of an investment, the asset may be sold. Hence, to abandon a project is to exercise a put option. These put options are particularly important for large capital-intensive projects such as infrastructure investments. They are also important for investments involving new products where market acceptance is uncertain.

Bar-Ilan and Strange (1996) assume that there is an investment lag with a time-to-build and an abandonment option available for each project. Since an option to abandon exists, losses (bad news) are bounded from below in bad states, limited to the initial development costs. Meanwhile time-to-build forces the firm to invest earlier, in order to be able to capture opportunities in the near future (good news, prizes). A situation such as this were there are compound options can cause the rational firm to invest sooner in a high uncertainty environment. Bar-Ilan and Strange show that for some parameter values, the overall effect of an investment lag is to lower the trigger price at which investment is started to below the price that would trigger investment in a world of certainty.

When long investment lags exist, uncertainty can actually encourage investment, in the sense of reducing the trigger price at which it is optimal to start construction. The price of the underlying good might rise over the intervening period, which would raise profits, while the downside risk from a price fall is limited by the option of abandoning the project. Waiting to start investment still has a value, since the firm learns more about the evolution of prices. It is costly, however, since if prices rise strongly, a firm that has not started to build will not be able to exploit this immediately, and will have foregone potential profits. By investing/building immediately, the investor can waive its decision to wait, lose only the initial costs in the case of bad news, and keep its competitive advantage in the market in the case of good news.

Wheaton (1987) finds that “the lag between issuing a construction permit and the completion of an office building is between 18 and 24 months”. These kinds of long lags tend to mitigate the negative effects of uncertainty on investment, and under some circumstances, even to stimulate investment. They act as negative real option phenomena since the investor can interrupt its decision and lose only the initial costs in the case of bad news, and keep its competitive advantage in the market in the case of good news. This also explains why there is often excess capacity in the real estate sector.

Research and Development

Growth options of the type in Bar-Ilan and Strange (1996) can be applicable to other projects with long development or gestation times, such as research and development activities.

Note that if firms have long delays in completing projects — perhaps because of time-to-build or time-to-develop — then uncertainty can have a positive effect on investment. As time-to-develop increases, uncertainty can have an even more positive effect on investment. As an illustration, consider a pharmaceutical company developing a new drug that notices a mean-preserving increase in demand uncertainty has occurred.

The costs of bad outcomes of the development process (e.g., the drug or vaccine turns out to be ineffective or unsafe during trials) combined with the option to abandon, means the firm has a limited lower bound because the it can simply cancel the project. It loses only its sunk research and development costs. However, good draws (the product turns out to be even more useful and profitable than expected) are not similarly constrained. Therefore, a rise in mean-preserving risk will mean higher expected profits when the product is marketed.

Real Options in Internet Development

The combination of the call (growth) and put (abandonment) options that characterize real estate and R and D investments can also facilitate investment in internet startups (websites, apps, others). Since developing websites and apps takes time, building any of these qualifies as investing in a "call-option" on the future success of technology and the internet. Growth options can be invoked to explain the dot-com boom of the late 1990s. At that time, firms were unsure about the internet but that uncertainty encouraged investment. The worst outcome for firms would be losing their development costs, while the best outcome looked ever more profitable as the range of products sold over the internet expanded and networks grew and more and more consumers connected to them, creating to large potentially exploitable economies of scale and of scope.

Synthesis

The above models suggest that real options sometimes involve interactions between the gestation and duration of investment activities and preemption of rivals. Even a slight possibility of rival preemption changes the conventional attractiveness of waiting. Longer duration increases the possibility of rival preemption and decreases the project value over time. As this negative effect is much stronger than the positive effect, the effect of duration on investment becomes negative in the presence of rival preemption.

Two contributions of the growth options literature: (a) They present an analytic solution to the investment problem with lags; (b) they show it is possible that an increase in uncertainty hastens the decision to invest. The price that triggers investment under uncertainty may be lower than the trigger price under certainty. Thus, investment lags offset uncertainty and tend to reduce inertia, contrary to conventional wisdom.

These results suggest that for investment characterized by long lags, policymakers have less reason to be concerned with uncertainty. The more immediate investment with long lags is also consistent with chronic excess supply observed in certain industries (e.g., real estate). Their results contrast with papers that show that an increase in uncertainty delays investment.

Projects with different investment lags respond to uncertainty differently. With a short lag, an increase in uncertainty delays investment so the volatility of the economic environment is a hindrance to investment. With a longer lag, an increase in uncertainty may increase investment. In any case, the deterrent effect of uncertainty is smaller than it would have been with a short lag. Hence, policies designed to reduce volatility during periods of uncertainty have smaller effects on projects with long investment lags than on projects with short investment lags.

Papers on strategic growth options tend to fall under two categories: (a) those that suggest that firms will make a preemptive move to invest immediately under threat of competition; (b) those that anticipate future growth through investment in acquisition of assets that can enhance growth opportunities available to the firm, enabling them to gain a competitive advantage relative to potential rivals.

VII. Conclusion

If waiting is an option, the conventional canonical result of McDonald and Siegel can arise. In the canonical model, a firm can delay investment to avoid learning of low prices (of the good to be produced by the investment) after it has made an irreversible decision to enter/invest. Since the likelihood of observing a low price rises with uncertainty, so does the benefit of waiting. The opportunity cost of waiting is income that could have been earned from the project, which depends on the price of the good during the delay. Since the firm has the capacity to enter the market immediately, the opportunity cost of a short delay in entering a market is independent of uncertainty. In other words, the negative impact of uncertainty on investment may not matter if time to market is very short. If, however, time to market is very long, a delay could lead prices to rise as well. Hence, to avoid being out of the market when prices rise, firms with long build times invest immediately.

In Bar-Ilan and Strange (1996), with longer lags, a firm that delays invest cannot enter the market immediately. Thus, the opportunity cost of waiting does not depend on the (price of the good) during the delay/today.

Instead, it depends on the price of the good in the future. Longer lags increase the likelihood of extreme prices in the interim period. With long investment lags, the price of the good might rise during the period of delay, which increases profits. Meanwhile downside risk from a fall in prices is limited by the option of abandoning the project. While waiting to start the investment still has a value, in that the firm learns more about the evolution of prices. However, waiting is costly, because the firm will be unable to immediately exploit any increase in price (or any other good news), and will have missed out on some potential profits.

But because the firm can abandon the investment project (an abandon option exists), even if not costlessly, the firm’s profits in bad states are bounded from below. Because the firm can exploit the upside of the investment with protection from the downside by having an option to abandon the project, the opportunity cost of waiting rises with uncertainty.

This means that the effect of uncertainty on the timing of an irreversible investment is ambiguous. An increase in uncertainty may in fact hasten investment (which is the opposite of what happens in the canonical model) by providing the firm with a growth option that it can exercise. This is consistent with the "good news principle". With long investment lags, the firm hurries to invest sooner rather than later in order to avoid learning of high prices while it is still out of the market.

In the Bar-Ilan and Strange model, the option to abandon allows the firm to vary output by exiting when output price is low. This makes profit a convex function of the stochastic price and increases expected profit in a more uncertain environment. Hence, a firm will invest at a lower price when time to build (investment lags) forces the firm to decide in advance whether to be active or not a few periods ahead. The effect competes with the option value of waiting. With longer lags, the threshold value of investment is lower.

Paddock, Siegel and Smith show that decision over investment in the development of oil reserves is akin to possessing a call option. When there is considerable uncertainty in the price of an underlying commodity (for instance, oil reserves), investing in extraction of the commodity may not be currently viable but can still be worthy because of the potential to create value if commodity prices increase.

The development of an offshore petroleum lease consists of sequential investments in exploration, development and extraction (production) of oil. During the development stage, managers can get a better sense about revenues or costs regarding the investment, and based on this information, they can review and recalibrate decisions on future stages of investment. Note that the value of investment in this case can explicitly incorporate various sources of managerial flexibility that allow the firm to commit itself sequentially to investment decisions in future stages.

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Note: The Canonical Model of Real Options in the Face of Uncertainty is that of McDonald and Siegel (1986) further expounded on by Dixit and Pindyck (1994). Explained in Pindyck Real Options Lecture Notes downloadable on the web. .

Appendix

The Appendix to this paper demonstrates that the methods discussed above are also applicable to a situation where the basic uncertainty is over the demand for the project's output. Hence, output price is exogenous, the value of the project is determined as well as the value of the option to invest in terms of the stochastic process for the output price.

Alternative Model: Treat the price of the project's output, rather than the value of the project, as a geometric random walk (and possibly one or more factor inputs as well). It would also allow the project to be shut down (permanently or temporarily) if prices fell below the variable cost. In this case, option pricing methods can be used to find the value of the project and the optimal investment rule.

Suppose output price, P, follows a stochastic process,

$$dP = \alpha P dt + \sigma P dz \qquad \text{(A.1)}$$

Assume that $\alpha < \mu$, where μ is the expected rate of return on P, adjusted for market risk, or an asset perfectly correlated with P, and let $\alpha = \mu - \delta$ as before. If the project's output is a storable commodity (e.g., oil or copper), δ will represent the net marginal convenience yield from storage, or the flow of benefits (less storage costs) that the marginal stored unit provides. Assume for simplicity that δ is constant. (For most commodities, the marginal convenience yield fluctuates as the total amount of storage fluctuates.) Also assume that: (a) the marginal and average cost of production is equal to a constant, c; (b) the project can be shut down at no cost if P falls below c and can later be restarted if P rises above c; and (c) the project produces one unit of output per period and is infinitely lived, and the (sunk) cost of investing in the project is I.

In this case, there are now two problems to solve. The first is to find the value of the project, V(P). To solve this problem, remember that the project is itself a set of options. Specifically, once the project has been built, the firm has, for each future time t, an option to produce a unit of output, that is, an option to pay c and receive P. Hence, the project is equivalent to a large number of operating options, and it can be valued accordingly.

The second problem is to find the value of the firm's option to invest in the project, given the project's value, and the optimal exercise (investment) rule. The solution involves finding a critical P* at which the firm invests only of $P \geq P^*$. The two steps of this problem can be solved sequentially using the same methods used in the previous section.

It is assumed that existing assets span the uncertainty over P, the project (as well as the option to invest) can be valued using CCA.

As before, construct a risk-free portfolio, one in which the project is held long and V_p units of the output are held short. This portfolio has a value $V(P) - V_p P$ and yields an instantaneous cash flow of $j(P - c)dt - \delta V_p P dt$, where j = 1 if $P \geq c$ so that the firm is producing and j = 0 otherwise. Recall that $\delta V_p P dt$ is the payment required to maintain the short position. The total return on the portfolio is $dV - V_p dP + j(P - c)dt - \delta V_p P dt$. Since this return is risk-free, set it equal to r (V -

V_p P)dt. Expanding dP using Ito's Lemma, substituting equation (A.1) for dP and rearranging yield the following differential equation for V:

$$\left(\frac{1}{2}\right)\sigma^2 P^2 V_{pp} + (r-\delta)PV_p - rV + j(P-c) = 0 \qquad \text{(A.2)}$$

This equation must be solved subject to the following boundary conditions

$$V(0) = 0 \qquad \text{(A.3)}$$

$$V(c) = V(c^*) \qquad \text{(A.4)}$$

$$V_p(c) = V_p(c^*) \qquad \text{(A.5)}$$

$$\lim_{P\to\infty} {}^{P}/_{\delta} - {}^{c}/_{r} \qquad \text{(A.6)}$$

Condition (A.3) is an implication of (A.1): if P is ever zero, it will remain zero and the project has no value. Condition (A.4) says that as P becomes very large, the probability that over any finite period it will fall below cost and production will cease becomes very small. Hence, the value of the project approaches the differences between the two perpetuities: a flow of revenue, P, that is discounted at the risk adjusted rate μ but is expected to grow at rate α and a flow of cost c, which is constant and hence is discounted at rate r. Finally, conditions (A.5) and (A.6) say that a project's value is a continuous and smooth function of P.

The solution to equation (A.2) has two parts, one of P < c and another for $P \geq c$. Check that by substitution, that the following satisfies equation (A.2), as well as the boundary conditions (A.4) and (A.5):

$$V(p) = \begin{cases} A_1 P^{\beta_1} & ; \quad P < c \\ A_2 P^{\beta_1} + {}^{P}/_{\delta} - {}^{c}/_{r} & ; \quad P \geq c \end{cases} \qquad \text{(A.7)}$$

where

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad \text{(A.8)}$$

and

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad \text{(A.9)}$$

The constants A_1 and A_2 can be found by applying the boundary conditions (A.4) and (A.5):

$$A_1 = \frac{r - \beta_2 (r-\delta) c^{(1-\beta_1)}}{r\delta(\beta_1 - \beta_2)} \qquad \text{(A.10)}$$

$$A_2 = \frac{r-\beta_1(r-\delta)c^{(1-\beta_2)}}{r\delta(\beta_1-\beta_2)} \qquad \text{(A.11)}$$

The solution for V(P) (equation (A.7)) can be interpreted as follows. When P < c, the project is not producing. Then $A_1 P^{\beta_1}$ is the value of the project's options to produce when P increases. When $P \geq c$, the project is producing. If, irrespective of changes in P, the firm had no choice but to continue producing throughout the future, the present value of the future flow of profits would be given by ${}^{P}/_{\delta} - {}^{c}/_{r}$. However, should P fall, the firm can stop producing and avoid losses. The value of its option to stop producing is $A_2 P^{\beta_2}$.