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A Note on the Parametric Linear Complementarity Problem

by

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Abstract. Consider the parametric linear complementarity problem $w = Mz + q + \lambda p$, $w \geq 0$, $z \geq 0$, $w^T z = 0$, where $p \neq 0$, $0 \neq q \geq 0$, and $\lambda \geq 0$. We show that a necessary condition for every complementary map $z(\lambda)$ to be isotone for every nonzero $q \geq 0$ and every p is that M be either a P-matrix or a P_1*-matrix ($M \in P_1$* iff $M \in P_1 \backslash Q$ and $\det(M) = 0$). Cottle's necessary and sufficient conditions for strong and uniform isotonicity for P-matrices are restated for P_1*-matrices.

Key Words. Parametric linear complementarity problem, isotone solutions, matrices, complementary cones.

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1. Introduction

For a given matrix $M \in \mathbb{R}^{n\times n}$ and vector $q \in \mathbb{R}^n$, the linear complementarity problem LCP(q,M) is that of finding $w, z \in \mathbb{R}^n$ such that

$$w = Mz + q,\ \ w \geq 0,\ \ z \geq 0,\ \ w^Tz = 0. \tag{1}$$

The set of solutions $(w;z)$ of the LCP(q,M) is denoted by $S(q,M)$ and the set of all $q \in \mathbb{R}^n$ for which the LCP(q,M) has a solution is denoted by $K(M)$.

A parametric linear complementarity problem (PLCP) is a family of linear complementarity problems $\{\text{LCP}(q+\lambda p,M) \mid \lambda \in \Lambda\}$ where $p \neq 0$ and Λ is a closed interval in $\mathbb{R}$. The PLCP arose in the study of elastoplastic structures (Ref. 1) and has also been applied to the computation of economic equilibria (Refs. 2 and 3) and portfolio selection (Ref. 4).

In this paper, we focus on the PLCP in which $q \geq 0$ and $\Lambda = [0,\lambda^*]$ or $\Lambda = \mathbb{R}_+$. We consider the problem of determining conditions under which the z-component of the solution $(w(\lambda);z(\lambda))$ of the LCP$(q+\lambda p,M)$ is monotone nondecreasing in Λ, i.e., each coordinate of $z(\lambda)$ is monotone nondecreasing in Λ. The monotonicity of $z(\lambda)$ is well-defined when M is a P-matrix (i.e., all principal minors of M are positive) since, in this case, the LCP(q,M) has a unique solution for every $q \in \mathbb{R}^n$ (Ref. 5). (We note that the only class of matrices M for which the LCP(q,M) has a unique solution for every $q \in K(M)$ is the class of P-matrices (Ref. 6)).

When M is not a P-matrix, the LCP may not have a solution and when it has, the solution may not be unique. Thus, $z(\lambda)$ becomes a point-to-set mapping. In this case, Kaneko (Ref. 7) proposed a more general definition of monotonicity by introducing the concept of a complementary map. A complementary map is a function z: $\Lambda \mapsto \mathbb{R}^n$, where $z(\lambda)$ is the z-component of an element $(w(\lambda);z(\lambda)) \in S(q+\lambda p,M)$. A complementary map $z(\lambda)$ is said to be isotone iff, for each $j = 1,2, \ldots, n$, $z_j(\lambda)$ is monotone nondecreasing with respect to λ. The PLCP$(q+\lambda p,M)$ is said to have isotone solutions iff every complementary map $z(\lambda)$ is isotone.

Under the assumption that M is a P-matrix, Cottle (Ref. 8) proved that the PLCP$(q+\lambda p,M)$ has isotone solutions for every $q \geq 0$ and every p iff M is a

Minkowski matrix (i.e., a P-matrix whose off-diagonal entries are nonpositive). Under the assumption that M is a Z-matrix (i.e., the off-diagonal entries of M are nonpositive), Kaneko (Ref. 7) proved that the PLCP($q+\lambda p,M$) has isotone solutions for every $q \geq 0$ and every p iff M is a Minkowski matrix. Thus, under different assumptions on M, Cottle and Kaneko arrived at the same necessary and sufficient condition for isotonicity for every $q \geq 0$ and every p. This is not surprising since, without making any assumption on M, isotonicity for every $q \geq 0$ and every p requires M to be a P-matrix. This is shown below by proving that when $q = 0$, the PLCP($0+\lambda p,M$) has isotone solutions for every p iff M is a P-matrix. Thus the PLCP reduces to Cottle's case.

Theorem 1.1. The PLCP($0+\lambda p,M$) has isotone solutions for every p iff M is a P-matrix.

Proof. ($\Rightarrow$) Let $0 \neq p \geq 0$. Since $\lambda \geq 0$, $(\lambda p;0)$ is a solution of the LCP($0+\lambda p,M$). Moreover, it is unique; otherwise, if $(w;z)$ is another solution, then $0 \neq z \geq 0$ and for $\lambda' > \lambda$, $(\lambda' p;0)$ is a solution of the LCP($0+\lambda' p,M$), contrary to isotonicity. By choosing $\lambda = 0$ and $\lambda = 1$, we see that for every $p \geq 0$, the LCP(p,M) has a unique solution. Thus, M is strictly semimonotone and hence, a Q-matrix (Refs. 6 and 9).

We next show that for any $p \ngeq 0$, the LCP(p,M) has a unique solution. Suppose that $(w^1;z^1)$ and $(w^2;z^2)$ are distinct solutions of the LCP(p,M). Then $z^1 \neq z^2$, say $z^1_j > z^2_j$ for some index j. Let δ be a small positive number such that $z^1_j > (1+\delta)z^2_j$. Then $((1+\delta)w^2;(1+\delta)z^2)$ is a solution of the LCP($(1+\delta)p,M$), contradicting isotonicity.

($\Leftarrow$) If $(w;z)$ is the solution of the LCP(p,M) and $\lambda_1 < \lambda_2$, then $(\lambda_1 w;\lambda_1 z)$ and $(\lambda_2 w;\lambda_2 z)$ are the unique solutions of the LCP($\lambda_1 p,M$) and LCP($\lambda_2 p,M$), respectively. Clearly, $\lambda_1 z \leq \lambda_2 z$. Hence, the PLCP($0+\lambda p,M$) has isotone solutions. □

Theorem 1.1 suggests that, by excluding $q = 0$, it is possible to have isotonicity for every nonzero $q \geq 0$ and every p where M is not a P-matrix. To illustrate, consider the matrix

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The isotonicity for every nonzero $q \geq 0$ and every p can be seen from the complementary cones shown in Figure 1. (The complementary cones are indicated by lines with two oppositely directed arrowheads). For example, let $q = [2\ \ 1]^T$ and $p = [-1\ -1]^T$. Here, $\Lambda = [0,3/2]$.

Figure 1

The z-components of the solutions of the LCP($q+\lambda p,M$) are given below:

(a) $0 \leq \lambda \leq 1$: $z_1(\lambda) = 0$, $z_2(\lambda) = 0$
(b) $1 \leq \lambda < 3/2$: $z_1(\lambda) = 0$, $z_2(\lambda) = \lambda - 1$
(c) $\lambda = 3/2$: $z_1(\lambda) = [0,\infty)$, $z_2(\lambda) = [1/2,\infty)$

The graphs of $z_1(\lambda)$ and $z_2(\lambda)$ (Figure 2) clearly show the isotonicity of every complementary map.

Figure 2

We note that M is not a P-matrix; it is a P_1-matrix (i.e., a P_0-matrix with exactly one zero principal minor). In fact, M is a P_1*-matrix ($M \in P_1$* iff $M \in P_1 \backslash Q$ and $\det(M) = 0$).

This paper investigates the PLCP($q+\lambda p,M$) where $0 \neq q \geq 0$ and shows that a necessary condition for the PLCP($q+\lambda p,M$) to have isotone solutions for every nonzero $q \geq 0$ and every p is that M be either a P-matrix or a P_1*-matrix. Cottle's (Ref. 8) results on strong and uniform isotonicity for P-matrices are restated, with slight modifications, for P_1*-matrices.

2. Further Definitions and Notations

The jth column of a matrix A is denoted by $A_{\cdot j}$, the ith row by $A_{i\cdot}$, and the ijth entry by A_{ij}. If $M \in \mathbb{R}^{n\times n}$ and $J \subseteq \{1,2, \ldots, n\}$, then the principal submatrix of M obtained by deleting the rows and columns of M corresponding to indices not in J is denoted by M_{JJ} and the corresponding subvectors of w, z, and q in the

LCP(q,M) are denoted by w_J, z_J, and q_J, respectively.

The cone generated by the columns of a matrix A is denoted by pos[A]. In the LCP(q,M), the pair $\{I_{\cdot j}, -M_{\cdot j}\}$ (j=1,2, ..., n), where I is the identity matrix, is called a complementary pair. If $A \in \mathbf{R}^{n \times n}$ and if for each $j=1,2,\ldots,n$, $A_{\cdot j} \in \{I_{\cdot j}, -M_{\cdot j}\}$, then pos[$A$] is called a complementary cone. The LCP(q,M) has a solution iff q belongs to some complementary cone. Thus, $K(M)$ is the union of all the complementary cones. A complementary cone pos[A] is said to be nondegenerate iff its interior, int(pos[A]), is nonempty (equivalently, the columns of A are linearly independent); otherwise, it is said to be degenerate. If pos[A] is an m-dimensional complementary cone, then its relative interior, ri(pos[A]), is its interior in $\mathbf{R}^m$. The complementary cones form a partition of $\mathbf{R}^n$ iff their union is $\mathbf{R}^n$ and they are nondegenerate with pairwise disjoint interiors.

There is a 1-1 correspondence between the principal submatrices and the complementary cones in which a principal submatrix M_{JJ} is associated with the complementary cone pos[A] where $A_{\cdot j} = -M_{\cdot j}$ for $j \in$ J and $A_{\cdot j} = I_{\cdot j}$ for $j \in \{1,2, \ldots, n\} \setminus$J. In this correspondence, a principal submatrix is nonsingular iff the associated complementary cone is nondegenerate. (In the case of $M_{\phi\phi}$, which is associated with pos[I], the convention is that det($M_{\phi\phi}$) = 1.)

For each q in a complementary cone pos[A], there is, by definition, an $x \geq 0$ such that $Ax = q$. The solution of the LCP(q,M) obtained by setting each variable in (w;z) associated with $A_{\cdot j}$ equal to x_j and the rest equal to zero is said to be induced by pos[A].

In the rest of the paper, we assume that $M \in \mathbf{R}^{n \times n}$ and q, $p \in \mathbf{R}^n$.

3. Necessary Conditions for Isotonicity

Definition 3.1. M is an E^*-matrix (or $M \in E^*$) iff the LCP(q,M) has a unique solution for every q such that $0 \neq q \geq 0$.

Remark 3.1. The class E of strictly semimonotone matrices has the property that $M \in E$ iff the LCP(q,M) has a unique solution for every $q \geq 0$ (Ref. 9). Thus, $E \subseteq E^*$.

Definition 3.2. M is an E'-matrix (or $M \in E'$) iff $M \in E^* \setminus E$.

Lemma 3.1. If $q \geq 0$, $0 \neq p \geq 0$, and the PLCP$(q+\lambda p,M)$ has isotone solutions, then the LCP$(q+\lambda p,M)$ has a unique solution for each $\lambda \geq 0$.

Proof. Suppose $0 \leq \lambda_1 < \lambda_2$. Since $q+\lambda_1 p \geq 0$, it follows that $(q+\lambda_1 p;0)$ is a solution of the LCP$(q+\lambda_1 p,M)$. If $(w;z)$ is another solution, then $0 \neq z \geq 0$. This and the fact that $(q+\lambda_2 p;0)$ is a solution of the LCP$(q+\lambda_2 p,M)$ contradict isotonicity. □

Remark 3.2. If the PLCP$(q+\lambda p,M)$ has isotone solutions for every nonzero $q \geq 0$ and every p, then, by choosing $0 \neq p \geq 0$ and $\lambda = 0$, it follows from Lemma 3.1 that the LCP(q,M) has a unique solution for every nonzero $q \geq 0$, i.e., $M \in E^*$.

Lemma 3.2. If $q \geq 0$ and the PLCP$(q+\lambda p,M)$ has isotone solutions for every p, then the nondegenerate complementary cones have pairwise disjoint interiors.

Proof. Let pos$[A]$ and pos$[B]$ be distinct nondegenerate complementary cones and suppose that $q^1 \in$ int(pos$[A]$) $\cap$ int(pos$[B]$). Let $p = q^1 - q$. Since pos$[A]$ and pos$[B]$ are distinct, there is an index k such that $\{A_{.k}, B_{.k}\}$ is a complementary pair, say $A_{.k} = -M_{.k}$ and $B_{.k} = I_{.k}$. Let $(w^A(1);z^A(1))$ and $(w^B(1);z^B(1))$ denote the solutions of the LCP$(q+1p,M)$ induced by pos$[A]$ and pos$[B]$, respectively. Since $q+1p$ is interior to both pos$[A]$ and pos$[B]$, we have

$$z_k^A(1) > 0 \tag{2}$$

$$z_k^B(1) = 0 \quad (\text{since } w_k^B(1) > 0). \tag{3}$$

Let δ be small positive number such that $q+(1+\delta)p$ is interior to both pos$[A]$ and pos$[B]$ and let $(w^A(1+\delta);z^A(1+\delta))$ and $(w^B(1+\delta);z^B(1+\delta))$ be the solutions of the LCP$(q+(1+\delta)p,M)$ induced by pos$[A]$ and pos$[B]$, respectively. Then

$$z_k^A(1+\delta) > 0, \tag{4}$$

$$z_k^B(1+\delta) = 0 \quad (\text{since } w_k^B(1+\delta) > 0). \tag{5}$$

Conditions (2) and (5) contradict isotonicity. □

Lemma 3.3. See Ref. 6. Let $A \in \mathbb{R}^{n \times p}$ and $q \in \mathbb{R}^n$ be given. If $q \in$ ri(pos[A]), then there exists a $u \in \mathbb{R}^p$, with $u > 0$, such that $q = Au$.

Remark 3.3. Lemma 3.3 implies that if pos[A] is a complementary cone in the LCP(q,M) and $q \in$ ri(pos[A]), then the LCP(q,M) has a solution whose coordinates associated with the columns of A are positive. In particular, if $q \in$ ri(pos[$-M$]), then the LCP(q,M) has a solution (w;z) in which $z > 0$.

Lemma 3.4. If $M \in Q$ and $q^0 \geq 0$, then a necessary condition for the PLCP($q^0+\lambda p$,M) to have isotone solutions for every p is that M be nonsingular.

Proof. If M is singular, then pos[$-M$] is degenerate. Pos[$-M$] $\neq \{0\}$ since $M \in Q$; hence, there is a $q^* \neq 0$ such that $q^* \in$ ri(pos[$-M$]). By Remark 3.3, the LCP(q^*,M) has a solution (w^*;z^*) where $z^* > 0$. Let $p^* = q^* - q^0$ and consider the PLCP($q^0+\lambda p^*$,M). Note that $(w^*;z^*) \in S(q^0+1p^*,M)$. Choose $\lambda_1 > 1$. If $q^0+\lambda_1 p^* \in$ pos[$-M$], then the LCP($q^0+\lambda_1 p^*$,M) has a solution ($w(\lambda_1)$;$z(\lambda_1)$) induced by pos[$-M$] in which $z(\lambda_1)$ has a zero coordinate since pos[$-M$] is degenerate, contradicting isotonicity. If $q^0+\lambda_1 p^* \notin$ pos[$-M$], then it belongs to some complementary cone pos[A], where $A \neq -M$. Hence, pos[A] must have a generator from I. Consequently, the LCP($q^0+\lambda_1 p^*$,M) has a solution induced by pos[A] in which the z-component has a zero coordinate, contradicting isotonicity. □

Definition 3.3. Given a PLCP($q+\lambda p$,M), a proper principal submatrix M_{JJ} of order m, and corresponding subvectors q_J and p_J. The PLCP($q_J+\lambda p_J$,M_{JJ}) is called a proper principal subproblem of order m.

Theorem 3.1. Let $M \in E^* \cap \mathbb{R}^{n \times n}$ ($n > 1$) and $q \geq 0$ be given. If the PLCP($q+\lambda p$,M) has isotone solutions for every p, then every proper principal subproblem PLCP($q_J+\lambda p_J$,M_{JJ}) has isotone solutions for every $p_J \in \mathbb{R}^m$, where m is the order of M_{JJ}.

Proof. Whether $M \in E$ or $M \in E'$, its proper principal submatrices are E-matrices (Refs. 9 and 10). Hence, we need only prove the theorem for the proper principal subproblem of order $n-1$. Without loss of generality, let the principal submatrix of order $n-1$ be the one obtained by deleting the nth row and the nth column of M and denote it by M^* and the corresponding subvector of q

by q^*.

Suppose that there is a $p^* \in \mathbb{R}^{n-1}$ such that the PLCP$(q^*+\lambda p^*,M^*)$ has a complementary map that is not isotone. Then there exist λ_1, λ_2 such that $0 \leq \lambda_1 < \lambda_2$ and $(w^*(\lambda_1);z^*(\lambda_1)) \in S(q^*+\lambda_1 p^*,M^*)$, $(w^*(\lambda_2);z^*(\lambda_2)) \in S(q^*+\lambda_2 p^*,M^*)$ with $z_k^*(\lambda_1) > z_k^*(\lambda_2)$ for some index k. This implies that $z_k^*(\lambda_1) > 0$ which, in turn, implies that $\lambda_1 > 0$; for, if $\lambda_1 = 0$, then the LCP(q^*+0p^*,M^*) = LCP(q^*,M^*) would have two distinct solutions $(w^*(\lambda_1);z^*(\lambda_1))$ and $(q^*;0)$ which is impossible since $M^* \in E$. Choose p_n^1 and p_n^2 such that

$$p_n^1 > (1/\lambda_1)\left[-\sum_{j=1}^{n-1} M_{nj} z_j^*(\lambda_1) - q_n\right],$$

$$p_n^2 > (1/\lambda_2)\left[-\sum_{j=1}^{n-1} M_{nj} z_j^*(\lambda_2) - q_n\right],$$

and let $p_n = \max\{p_n^1, p_n^2\}$. Then

$$\lambda_1 p_n > -\sum_{j=1}^{n-1} M_{nj} z_j^*(\lambda_1) - q_n ,$$

$$\lambda_2 p_n > -\sum_{j=1}^{n-1} M_{nj} z_j^*(\lambda_2) - q_n .$$

Define

$$w_n(\lambda_1) = \sum_{j=1}^{n-1} M_{nj} z_j^*(\lambda_1) + q_n + \lambda_1 p_n$$

$$w_n(\lambda_2) = \sum_{j=1}^{n-1} M_{nj} z_j^*(\lambda_2) + q_n + \lambda_2 p_n.$$

Then $w_n(\lambda_1) > 0$, $w_n(\lambda_2) > 0$.

Let $p = [p^*,p_n]^T$. It is easy to verify that $(w^*(\lambda_1),w_n(\lambda_1);z^*(\lambda_1),0) \in S(q+\lambda_1 p,M)$ and $(w^*(\lambda_2),w_n(\lambda_2);z^*(\lambda_2),0) \in S(q+\lambda_2 p,M)$. Since $z_k^*(\lambda_1) > z_k^*(\lambda_2)$, the PLCP$(q+\lambda p,M)$ has a complementary map that is not isotone, contrary to the hypothesis. □

Recall from Remark 3.2 that if the PLCP$(q+\lambda p,M)$ has isotone solutions for every nonzero $q \geq 0$ and every p, then $M \in E^*$, i.e., either (a) $M \in E$ or (b) $M \in E'$. We now show that in case (a), $M \in P$ and in case (b), $M \in P_1^*$.

(We dispense with the case $n = 1$. In this case, if $M \in E$, then $M = [M_{11}]$, where $M_{11} > 0$; hence, M is a P-matrix. If $M \in E'$, then $M = [0]$, a P_1*-matrix.)

Theorem 3.2. If $M \in E \cap \mathbb{R}^{n \times n}$ $(n > 1)$ and $0 \neq q \geq 0$, then a necessary condition for the PLCP$(q+\lambda p,M)$ to have isotone solutions for every p is that M be a P-matrix.

Proof. By hypothesis and Theorem 3.1, every principal subproblem PLCP$(q_J+\lambda p_J,M_{JJ})$ has isotone solutions for every p_J. Since the principal submatrices of M are Q-matrices, they are nonsingular by Lemma 3.4; hence, the complementary cones are nondegenerate. By Lemma 3.2, the complementary cones have pairwise disjoint interiors; hence, they form a partition of $\mathbb{R}^n$. It follows that M is a P-matrix (Ref. 5). □

Theorem 3.3. If $M \in E' \cap \mathbb{R}^{n \times n}$ $(n > 1)$ and $0 \neq q \geq 0$, then a necessary condition for the PLCP$(q+\lambda p,M)$ to have isotone solutions for every p is that M be a P_1*-matrix.

Proof. Let M_{JJ} be a proper principal submatrix of order $n-1$. Then $M_{JJ} \in E$ (Ref. 10). By Theorem 3.1, the proper principal subproblem PLCP$(q_J+\lambda p_J,M_{JJ})$ has isotone solutions for every $p_J \in \mathbb{R}^{n-1}$. By Theorem 3.2, M_{JJ} is a P-matrix. Hence, all the proper principal submatrices of M are P-matrices. Moreover, M is singular and $M \notin Q$ since $M \in E'$ (Ref. 11). Hence, $M \in P_1$*. □

4. Strong Isotonicity

Consider the PLCP$(q+\lambda p,M)$ where $M \in P_1$*. In this case, (a) $K(M)$ is a closed halfspace whose boundary is the hyperplane pos$[-M]$, (b) the normal to pos$[-M]$ can be chosen to be a positive vector, and (c) the LCP(q,M) has a unique solution for every $q \in$ int$(K(M))$ (Ref. 6). Thus, Λ is of the form $[0,\lambda^*]$ or $[0,\infty)$. These properties enable us to use a slightly modified version of Cottle's (Ref. 8) monotonicity-checking algorithm and to extend his results on strong and uniform isotonicity to PLCPs involving P_1*-matrices. The modification is made in Step 4 where a stopping rule is introduced. Step 4 now reads:

"Step 4. Change of basis. Case 1: $M_{rr}^{(k-1)} > 0$. Pivot on $M_{rr}^{(k-1)}$. Return to Step 1 with the transformed tableau.
Case 2: $M_{rr}^{(k-1)} = 0$. Stop; the complementary map is isotone."

($M_{rr}^{(k-1)}$ is a diagonal entry of the principal pivotal transform of M at the $(k-1)$th iteration. For convenience, the modified Cottle algorithm is given in the Appendix.) This modification does not affect the finiteness of the algorithm since the only change is the addition of a stopping rule. Moreover, we show that when the stopping rule in Step 4 occurs, then $\lambda^{(k)} = \lambda^*$ and the PLCP($q+\lambda p,M$) has isotone solutions.

Lemma 4.1. Let $M \in P_1^* \cap \mathbf{R}^{n\times n}$ ($n > 1$) be given. Let M' be the principal pivotal transform of M resulting from a principal pivot on a proper principal submatrix M_{JJ} and let $K = \{1,2,\ldots,n\} \setminus J$. Then
(a) M'_{KK} is singular;
(b) M' has positive diagonal entries iff the order of M'_{KK} is greater than 1.

Proof. (a) M'_{KK} is the Schur complement of M_{JJ} in M. Hence, M'_{KK} is singular since M is singular (Ref. 6).

(b) ($\Rightarrow$) If the diagonal entries of M' are positive, then there is no principal submatrix of order 1 that is singular. Hence, M'_{KK} is of order greater than 1.

($\Leftarrow$) Since P_1-matrices are invariant under principal pivots (Ref. 6), it follows that $M' \in P_1$ with M'_{KK} as its only singular principal submatrix. If the order of M'_{KK} is greater than 1, then all principal submatrices of order 1 (i.e., the diagonal entries) must be positive. □

Lemma 4.2. Let $M \in P_1^* \cap \mathbf{R}^{n\times n}$ ($n > 1$) and $q \in \text{pos}[-M]$. If $(w^B;z^B)$ is a basic solution of the LCP(q,M) and $(w;z)$ is a solution distinct from $(w^B;z^B)$, then $z > z^B$.

Proof. Since $q \in \text{pos}[-M]$, we must have $w = 0$; otherwise, q can be written as

$$q = \sum_{j\in J} z_j(-M_{\cdot j}) + \sum_{j\notin J} w_j I_{\cdot j}$$

where $J \subseteq \{1,2,\ldots,n\}$ and at least one w_j is positive. Let the normal to the hyperplane pos$[-M]$ be v, chosen such that $v > 0$. Then we get the following contradiction

$$0 = v^T q = \sum_{j \in I} z_j v^T(-M_{.j}) + \sum_{j \notin I} w_j v^T I_{.j} = 0 + \sum_{j \notin I} w_j v_j > 0.$$

Similarly, $w^B = 0$. Hence, we have

$$Mz^B = -q, \quad Mz = -q$$

and so, $M(z - z^B) = 0$, i.e., $z - z^B$ belongs to the null space of M. Since $(w^B;z^B)$ and $(w;z)$ are distinct, $z - z^B \neq 0$. Now, $M \in E'$ since $M \in P_1{}^*$ (Ref. 10); hence, the null space of M is generated by a positive vector (Ref. 11). Hence, either $z - z^B > 0$ or $z - z^B < 0$. Since the columns of $-M$ are linearly dependent and $(w^B;z^B)$ is basic, z^B has a zero coordinate; hence, if $z - z^B < 0$, then z has a negative coordinate, contrary to the nonnegativity of z. It follows that $z - z^B > 0$. □

Theorem 4.1. Given the PLCP$(q+\lambda p,M)$ where $0 \neq q \geq 0$ and $M \in P_1{}^* \cap \mathbb{R}^{n \times n}$ $(n > 1)$. If, in the modified Cottle algorithm, the stopping rule in Step 4 occurs, then the PLCP$(q+\lambda p,M)$ has isotone solutions.

Proof. By hypothesis, M has itself as its only singular principal submatrix; hence, every principal pivotal transform of M has only one singular principal submatrix. By Lemma 4.1(b), the stopping rule in Step 4 can be obtained by a block principal pivot on a principal submatrix of order $n-1$ which is equivalent to a sequence of $(n-1)$ simple principal pivots. Assume that the algorithm has generated a sequence of simple principal pivots in which z_j is exchanged for w_j $(j=1, 2, \ldots, n-1)$. Let M' be the resulting principal pivotal transform of M. Then $M'_{nn} = M^{(n-1)}_{nn} = 0$. The variables $z_1, z_2, \ldots, z_{n-1}$ have become basic and the stopping rules in Steps 1 and 3 have not occurred. These imply that

$$p_j^{(n-1)} \geq 0, \quad j=1,2,\ldots,n-1$$

$$p_n^{(n-1)} < 0.$$

The values of the basic variables are given by

$$z_j = Q_{j1}^{(n-1)} + \lambda^{(n-1)} p_j^{(n-1)}, \quad j=1,2,\ldots,n-1;$$

$$w_n = Q_{n1}^{(n-1)} + \lambda^{(n-1)} p_n^{(n-1)}.$$

The next critical value of λ is

$$\lambda^{(n)} = -Q_{n1}^{(n-1)} / p_n^{(n-1)}.$$

As λ is increased from $\lambda^{(n-1)}$ to $\lambda^{(n)}$, the variables z_j $(j=1,2, \ldots, n-1)$ do not decrease, z_n remains at zero (since z_n remains nonbasic), and w_n goes to zero:

$$w_n = Q_{n1}^{(n-1)} - (Q_{n1}^{(n-1)} / p_n^{(n-1)}) p_n^{(n-1)} = 0.$$

This results in a basic solution $(w^B;z^B)$ of the LCP$(q+\lambda^{(n)}p,M)$ in which $w^B = 0$, implying that $q+\lambda^{(n)}p \in \text{pos}[-M]$. Moreover, no decrease in any z_j has occurred and, by Lemma 4.2, any solution $(w;z)$ of the LCP$(q+\lambda^{(n)}p,M)$ distinct from $(w^B;z^B)$ satisfies $z > z^B$, proving isotonicity. Since $q \in \text{int}(K(M))$ and pos$[-M]$ is the boundary of $K(M)$, it follows that $\lambda^{(n)} = \lambda^*$. □

Definition 4.1. The PLCP$(q+\lambda p,M)$ has strongly isotone solutions iff it has isotone solutions for every p.

Theorem 4.2. Let $M \in P_1^* \cap \mathbb{R}^{n \times n}$ $(n > 1)$ and let $0 \neq q \geq 0$. The PLCP$(q+\lambda p,M)$ has strongly isotone solutions iff $(M_{JJ})^{-1} q_J \geq 0$ for every proper principal submatrix M_{JJ} and corresponding subvector q_J.

Proof. (⇒) In every proper principal subproblem PLCP$(q_J+\lambda p_J,M_{JJ})$, M_{JJ} is a P-matrix. By Theorem 3.1, the PLCP$(q_J+\lambda p_J,M_{JJ})$ has isotone solutions for every p_J. By Theorem 1 of Ref. 8, $(M_{JJ})^{-1} q_J \geq 0$ for all proper principal submatrices M_{JJ} and corresponding subvector q_J.

(⇐) Note that the LCP(q,M) has a unique solution for each $q \in \text{int}(K(M))$ (Ref. 6). Moreover, by Lemma 4.1(b), a sequence of simple principal pivots on M, indexed by k, has a positive pivot element at each iteration $k < n$. Thus, Cottle's algorithm for P-matrices can be applied on the iterations $k < n$. The proof that the algorithm develops an isotone complementary map up to the kth iteration is exactly the same as Cottle's proof of Theorem 1 (Ref. 8) and is not repeated here. When $k = n$, the stopping rule in Step 4 has occurred and, by Theorem 4.1, the complementary map is isotone. □

5. Almost Uniform Isotonicity

Definition 5.1. The PLCP($q+\lambda p,M$) has almost uniformly isotone solutions iff it has strongly isotone solutions for every nonzero $q \geq 0$ and every p.

Definition 5.2. M is an almost K-matrix iff M is a singular Z-matrix whose proper principal submatrices are K-matrices. ($M \in K$ iff $M \in P \cap Z$. K-matrices are also called Minkowski matrices).

Theorem 5.1. Let $M \in P_1^* \cap \mathbf{R}^{n\times n}$ ($n > 1$). The PLCP($q+\lambda p,M$) has almost uniformly isotone solutions iff M is an almost K-matrix.

Proof. ($\Rightarrow$) We need only show that M is a Z-matrix since M is singular and its proper principal submatrices are P-matrices. Suppose that $M_{ij} > 0$ for some $i \neq j$. Since $M_{ii} > 0$, it follows that the vectors $-M_{\cdot i}$ and $-M_{\cdot j}$ lie strictly on the same side of the hyperplane spanned by the vectors $I_{\cdot 1}, \ldots, I_{\cdot i-1}, I_{\cdot i+1}, \ldots, I_{\cdot n}$. Consider the nondegenerate complementary cone pos[A] = pos[$I_{\cdot 1},\ldots,I_{\cdot i-1},-M_{\cdot i},I_{\cdot i+1},\ldots,I_{\cdot n}$] and its face F = pos[$I_{\cdot 1}, \ldots, I_{\cdot i-1}, I_{\cdot i+1}, \ldots, I_{\cdot n}$] opposite the edge generated by $-M_{\cdot i}$. Let

$$q^0 = \sum_{k\neq i} I_{\cdot k}$$

and set $p^0 = -M_{\cdot j} - q^0$.

Consider the PLCP($q^0+\lambda p^0,M$). Since $-M_{\cdot i}$ and $-M_{\cdot j}$ lie strictly on the same side of the hyperplane of F and $q^0 \in$ ri(F), it follows that for sufficiently small λ_1 ($0 < \lambda_1 < 1$), $q^0+\lambda_1 p^0 \in$ int(pos[A]). Hence, pos[A] induces a solution $(w(\lambda_1);z(\lambda_1))$ of the LCP($q^0+\lambda_1 p^0,M$) with $z_i(\lambda_1) > 0$. Noting that $-M_{\cdot j} = q^0 + 1p^0$, we easily verify that the LCP(q^0+1p^0,M) has a solution $(0;z(1))$ where $z_k(1) = 1$ ($k = j$) and $z_k(1) = 0$ ($k \neq j$). Since $i \neq j$, $z_i(1) = 0$, contrary to isotonicity. Thus $M_{ij} \leq 0$ for all $i \neq j$ and M is a Z-matrix.

($\Leftarrow$) Every proper principal submatrix M_{JJ} of M is a K-matrix; hence, $(M_{JJ})^{-1} \geq 0$ (Ref. 12). If $0 \neq q \geq 0$, then $(M_{JJ})^{-1}q_J \geq 0$ for all proper principal submatrices M_{JJ} and corresponding subvector q_J. By Theorem 4.2, the PLCP($q+\lambda p,M$) has isotone solutions for every nonzero $q \geq 0$ and every p. □

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Appendix

Modified Cottle Monotonicity-Checking Algorithm

(The algorithm checks the isotonicity of $z(\lambda)$ when M is a P_1*-matrix)

Step 0. *Initialization*

(i) Construct the matrix $Q \in \mathbb{R}^{n \times n}$ satisfying the following conditions:

(a) $Q_{\cdot 1} = q$;

(b) If q_k is the first positive coordinate of q, then $Q_{k\cdot} = [q_k \; 0 \; \dots \; 0]$;

(c) If the first column and the kth row of Q are deleted, the remaining submatrix is the identity matrix of order $n-1$.

(ii) Set up the following tableau:

	1	0	...	0	λ	z_1	z_2	...	z_n
w_1	Q_{11}	Q_{12}	...	Q_{1n}	p_1	M_{11}	M_{12}	...	M_{1n}
w_2	Q_{21}	Q_{22}	...	Q_{2n}	p_2	M_{21}	M_{22}	...	M_{2n}
...	...	...	...	...	...	...	...	...	...
w_n	Q_{n1}	Q_{n2}	...	Q_{nn}	p_n	M_{n1}	M_{n2}	...	M_{nn}

Start with $\lambda^0 = 0$ as "critical value" and set $z_j^0 = 0$ ($j = 1,2, \dots, n$).

Step 1. *Monotonicity test.* If the λ-column is nonnegative, stop; $z(\lambda)$ is isotone.

Step 2. *Determination of the next critical value.* Determine the "critical index" r by the relation

$$-(1/p_r^{(k-1)})Q_{r\cdot}^{(k-1)} = \text{lexicomin}\{-1/p_j^{(k-1)}Q_{j\cdot}^{(k-1)} \mid p_j^{(k-1)} < 0\}.$$

Set $\lambda^{(k)} = -Q_{r1}^{(k-1)}/p_r^{(k-1)}$.

Step 3. *Decrease check.* If the new critical value is greater than its predecessor and the λ-column contains a negative entry in the row of a basic z-variable, stop; $z(\lambda)$ is not isotone.

Step 4. *Change of basis.*

Case 1. $M_{rr}^{(k-1)} > 0$. Pivot on $M_{rr}^{(k-1)}$ and go to Step 1 with the new tableau.

Case 2. $M_{rr}^{(k-1)} = 0$. Stop; $z(\lambda)$ is isotone.

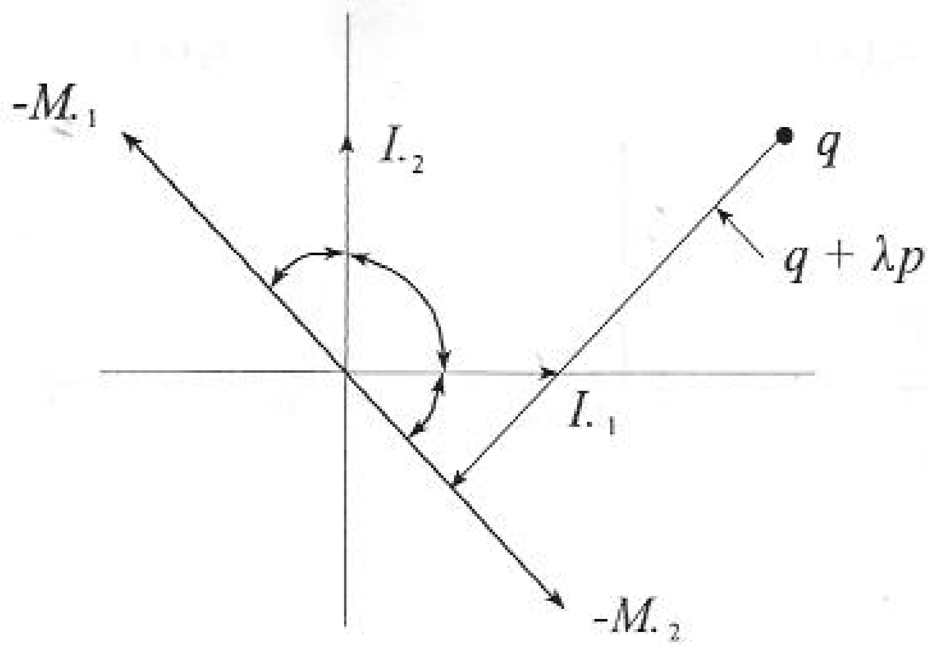

Figure 1

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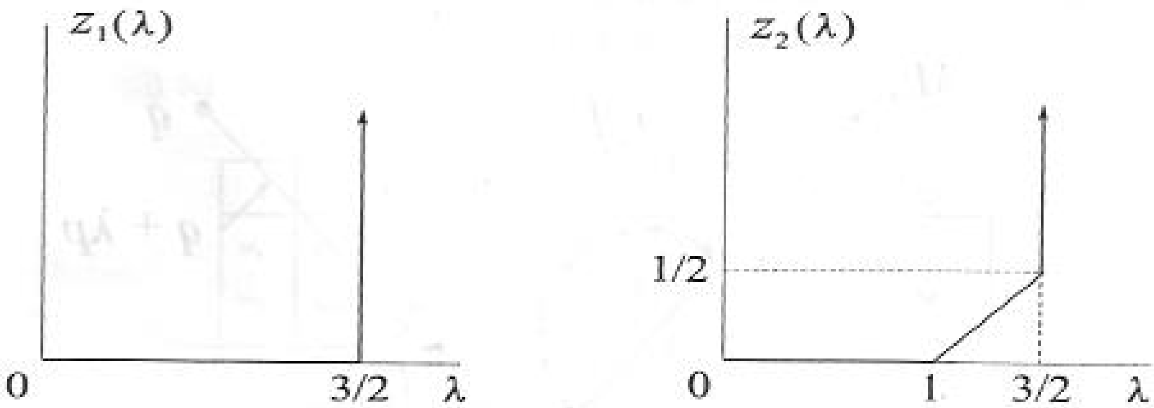

Figure 2

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