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An Intertemporal Model of Optimal Commercial Bank Reserves

by

*Johnny Noe E. Ravalo**

*Senior Lecturer, School of Economics, University of the Philippines

ABSTRACT

The standard bank reserve model is extended by explicitly introducing the dimension of time. This feature has not been addressed in the literature which seems to be a very surprising omission given that the concensus view of this financial institution is that of a (differentiated) microeconomic firm maximizing over terminal wealth. The results indicate that the solution across periods is stationary. All the information needed by the bank to make its choice is embodied in the various interest rates as well as in the penalty structure that institutionally defines the cost of illiquidity. Neither the contemporaneous level of portfolio variables nor its comparative levels in previous periods become relevant to the optimal rule.

It is further shown that the stationary solution is in fact Markov stationary, that is, the intraperiod optima will be exactly the same as that in the static case. This literally suggests that the optimization in a multi-period horizon can be taken as a series of repeated one-period models, leading to the same set of optima. This result was shown to be robust, particularly when the allocation problem is modeled explicitly as a Markov process.

The model is extended [illegible] ***what is labelled as an "income-commitment effect" and an "income-option effect".***

I. Formulating an Intertemporal Model of Reserve Holdings

The standard models on optimal bank reserve take the allocation decision as a static problem. Although the solution is derived in a straightforward manner, there is little evidence that this line of research has provided insights which banks actually use at the level of day-to-day operations. In the context of representing observed behavior, this form of theorizing subsequently falls short of its task despite the volume of research material that it has spawned.

One problem with the standard models is that the static optimization *per se* is not given a critical role in defining the bank's portfolio. The timing is such that the reserve decision must be made at the beginning of the period (i.e. *ex ante*), only to find out at period's end (i.e. *ex post*) whether the chosen level turns out to be adequate. Under a one-period model, the consequence of this reserve decision simply does not impact operations enough. It is not clear, for example, what the effect of the current decision is upon non-reserve funds or whether it has influence on future reserve decisions.

This suggests that the standard models can be extended (1) by accounting for an intertemporal framework and (2) by explicitly modelling the impact of the reserve decision on non-reserve funds. This provides the needed flexibility because the choice of period t reserves will now influence the contemporaneous operations of the bank through its impact on non-reserve funds. To the extent that current operations determine end-of-the-period available resources, then there is a potential link is created between future operations and the reserve decision in the current period.

1.1 The Base One-Period Model

The bank needs to divide at the beginning of the period its total resources between reserves and fixed-term loans (i.e. non-reserve funds). Stochastic withdrawals occur at period's end, to be financed out of the pool of reserves. Apart from the normal interest income and expense, an illiquidity cost is incurred if reserves are either insufficient to cover the withdrawals or fall short of the minimum set by the monetary authorities.

In any period j, the bank expects its rate of return to be:

$$\pi_j = r_A\alpha_j + r_R\gamma_j - r_D D_{j-1} - \int_{\gamma_j}^{\bar{\delta}} \left\{\Omega + r_I(\delta_j - \gamma_j)\right\} f(\delta)\, d\delta$$

$$= r_A\alpha_j + r_R\gamma_j - r_D D_{j-1} - \Omega\omega_j(\gamma_j) - \int_{\gamma_j}^{\bar{\delta}} r_I(\delta_j - \gamma_j)\, f(\delta)\, d\delta \qquad (1)$$

where: $\omega_j(\gamma_j) = \int_{\gamma_j}^{\bar{\delta}} f(\delta)\, d\delta$ (2)

= probability that stochastic withdrawals would cause illiquidity

D_j = deposits at the end of period j
W_j = equity at the end of period j
α_j = loans disbursed at the beginning of period j
γ_j = reserves at the beginning of period j
δ_j = stochastic withdrawals in period j
$f(\delta)$ = density function for δ
r_A = loan rate
r_R = return on reserves
r_D = deposit rate
r_I = interbank call loan rate

Maximization of (1) is subject to the allocation constraint:

$$\alpha_j + \gamma_j = D_{j-1} + W_{j-1} = 1 \qquad \forall\, j < T \tag{3}$$

which implicitly defines α_j as a residual of the decision on γ_j and subject to the condition that reserves cannot be less than required minimum:

$$\gamma_j \geq kD_{j-1} \tag{4}$$

Note that the notation--and subsequently the specification of the objective function--are slightly different from those in the basic (static) model. Specifically, the variable γ_j now reflects the total holding of reserves rather than the increment over the required minimum as previously modeled. This correspondingly affects the way the probability $\omega_j(\gamma_j)$ is defined and subsequently necessitates that the minimum reserve requirement be set as an explicit inequality constraint. Note further that the probability density of withdrawals as a proportion of total resources, $f(\delta)$, is assumed stable over time.

Formulated in this form, the model can be then solved by the Kuhn-Tucker theorem. Defining the lagrangian as:

$$Z(\gamma_j) = r_A\left\{D_{j-1} + W_{j-1} - \gamma_j\right\} + r_R\gamma_j - r_D D_{j-1} - \Omega\omega_j(\gamma_j) - \int_{\gamma_j}^{\bar{\delta}} r_I(\delta_j - \gamma_j)\, f(\delta)\, d\delta + \epsilon\left\{\gamma_j - kD_{j-1}\right\} \tag{5}$$

the relevant Kuhn-Tucker conditions are:

$$\frac{dZ}{d\gamma_j} = -r_A + r_R - \Omega\frac{d\,\omega_j}{d\,\gamma_j} - r_1 \frac{d\int_{\gamma_j}^{\bar{\delta}} (\delta_j - \gamma_j)\ f(\delta)\ d\delta}{d\,\gamma_j} + \varepsilon \qquad \leq \quad 0$$

$$= -r_A + r_R + \Omega f(\delta_j = \gamma_j) + r_I\omega_j + \varepsilon \qquad \leq \quad 0$$

$$\frac{dZ}{d\varepsilon} = \gamma_j - kD_{j-1} \qquad \geq \quad 0$$

$$\gamma_j \geq 0 \quad \text{and} \quad \gamma_j \frac{dZ}{d\gamma_j} = 0$$

$$\varepsilon \geq 0 \quad \text{and} \quad \varepsilon \frac{dZ}{d\varepsilon} = 0$$

Subsequently, we have the following four cases to consider.

case 1.1: if $\gamma_j > 0$ and $\varepsilon > 0$

$$\varepsilon = r_A - r_R - \left\{ r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j) \right\} \tag{6.1}$$

$$\gamma_j = kD_{j-1} \tag{6.2}$$

case 1.2: if $\gamma_j > 0$ and $\varepsilon = 0$

$$\varepsilon = r_A - r_R - \left\{ r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j) \right\}$$

$$r_A - r_R = \left\{ r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j) \right\} \qquad (7.1$$

$$\gamma_j = kD_{j-1} \tag{7.2}$$

Case 1.3: if $\gamma_j = 0$ and $\varepsilon > 0$

$$\varepsilon < r_A - r_R - \left\{ r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j) \right\} \tag{8.1}$$

$$\gamma_j = kD_{j-1} \tag{8.2}$$

case 1.4: if $\gamma_j = 0$ and $\varepsilon = 0$

$$\varepsilon < r_A - r_R - \left\{ r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j) \right\} \tag{9.1}$$

$$\gamma_j > kD_{j-1} \tag{9.2}$$

Cases 1.3 and 1.4 can be disregarded because equations (**8.2**) and (**9.2**) are both inconsistent with the presumption that $\gamma_j = 0$. Case 1.1 is feasible but it is unlikely that the lagrange multiplier defined in (**6.1**) will be positive as presumed. Since $\left\{r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j)\right\} > 0$, the right-hand side of (**6.1**) can only be positive if r_I is significantly lower than r_A and if the fixed penalty Ω is practically nonexistent. Under this specific set of condition, however, banks will be much less concerned with the possible consequences of illiquidity since the obvious benefits of a high loan rate relative to the costs of a low penalty structure (i.e. low r_I and Ω) effectively eliminates any meaningful trade-off.

Case 1.2 therefore is the solution to the nonlinear programming problem. Note that this solution finds that banks will not be bound passively to the mandatory minimum but will instead set the optima higher than kD_{j-1}. The exact value of such optima γ_j^* is implied by (**7.1**) which can readily be interpreted as the condition at the margin of equating the cost due to and the benefit due from the holding of reserves. On one hand, the cost at the margin of holding an extra unit of reserve, $r_A - r_R$, is the foregone return from loans net of any direct monetary return on reserve holdings. On the other hand, banks exchange this cost for the primary benefit of decreasing the expected cost of illiquidity:

$$\frac{-\partial \int_{\gamma_j}^{\bar{\delta}} \left\{\Omega + r_I(\delta_j - \gamma_j)\right\} f(\delta)\, d\delta}{\partial \gamma_j} = \left\{r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j)\right\} > 0 \tag{10}$$

This is the result for a one-period model. It can be shown quite easily that (10) is convex downwards in γ_j while $r_A - r_R$ is clearly a constant function. A stable equilibria is then found where the marginal benefit function $\left\{r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j)\right\}$ intersects the marginal cost function $r_A - r_R$ from above.

1.2 Extending Into A Second Period

The model easily accounts for a second period by maximizing the discounted stream of profits over the two periods. Defining β as the discount factor, the problem is now of the form:

$$
\begin{aligned}
\max_{\gamma_j, \gamma_{j+1}} \quad \Pi &= \pi_j + \beta\pi_{j+1} \\
&= \left[r_A\alpha_j + r_R\gamma_j - r_D D_{j-1} - \int_{\gamma_j}^{\bar{\delta}} \left\{\Omega + r_I(\delta_j - \gamma_j)\right\} f(\delta)\, d\delta \right] + \beta\pi_{j+1} \\
&= r_A(\alpha_j + \beta\alpha_{j+1}) \;+\; r_R(\gamma_j + \beta\gamma_{j+1}) \;-\; r_D(D_{j-1} + \beta D_j) \\
&\quad - \Omega\left[\omega_j(\gamma_j) + \beta\omega_{j+1}(\gamma_{j+1})\right] \\
&\quad - r_I\left[\int_{\gamma_j}^{\bar{\delta}} (\delta_j - \gamma_j) f(\delta) d\delta + \beta\int_{\gamma_{j+1}}^{\bar{\delta}} (\delta_{j+1} - \gamma_{j+1}) f(\delta) d\delta\right]
\end{aligned} \tag{11}
$$

subject to the following control constraints:

$$\gamma_j \geq kD_{j-1} \tag{12.1}$$
$$\gamma_{j+1} \geq kD_j \tag{12.2}$$
$$\alpha_j + \gamma_j = D_{j-1} + W_{j-1} \tag{12.3}$$
$$\alpha_{j+1} + \gamma_{j+1} = D_j + W_j \tag{12.4}$$

Defining the lagrangian as:

$$
\begin{aligned}
Z(\gamma_j,\gamma_{j+1}) = {} & r_A\left\{(D_{j-1} + W_{j-1} - \gamma_j) + \beta(D_j + W_j - \gamma_{j+1})\right\} + r_R(\gamma_j + \beta\gamma_{j+1}) \\
& - r_D(D_{j-1} + \beta D_j) - \Omega\left[\omega_j(\gamma_j) + \beta\omega_{j+1}(\gamma_{j+1})\right] \\
& - r_I\left[\int_{\gamma_j}^{\bar{\delta}} (\delta_j - \gamma_j) f(\delta) d\delta + \beta\int_{\gamma_{j+1}}^{\bar{\delta}} (\delta_{j+1} - \gamma_{j+1}) f(\delta) d\delta\right] \\
& + \varepsilon_j\left\{\gamma_j - kD_{j-1}\right\} + \varepsilon_{j+1}\left\{\gamma_{j+1} - kD_j\right\} \qquad (13)
\end{aligned}
$$

the appropriate Kuhn-Tucker conditions are therefore:

$$\frac{d\,Z}{d\,\gamma_j} = -r_A + r_R - \Omega\frac{d\,\omega_j}{d\,\gamma_j} - r_I\,\frac{d\int_{\gamma_j}^{\bar{\delta}} (\delta_j - \gamma_j)\ f(\delta)\ d\delta}{d\,\gamma_j} + \varepsilon_j \qquad \leq 0$$

$$\frac{d\,Z}{d\,\gamma_{j+1}} = \beta\left[-r_A + r_R - \Omega\frac{d\,\omega_{j+1}}{d\,\gamma_{j+1}} - r_I\,\frac{d\int_{\gamma_{j+1}}^{\bar{\delta}} (\delta_{j+1} - \gamma_{j+1})\ f(\delta)\ d\delta}{d\,\gamma_{j+1}}\right] + \varepsilon_{j+1} \qquad \leq 0$$

$$\frac{d\,Z}{d\,\varepsilon_j} = \gamma_j - kD_{j-1} \qquad \leq 0$$

$$\frac{d\,Z}{d\,\varepsilon_{j+1}} = \gamma_{j+1} - kD_j \qquad \leq 0$$

$$\gamma_j \geq 0 \quad \text{and} \quad \gamma_j\,\frac{d\,Z}{d\,\gamma_j} = 0$$

$$\gamma_{j+1} \geq 0 \quad \text{and} \quad \gamma_{j+1}\,\frac{d\,Z}{d\,\gamma_{j+1}} = 0$$

$$\varepsilon_j \geq 0 \quad \text{and} \quad \varepsilon_j\,\frac{d\,Z}{d\,\varepsilon_j} = 0$$

$$\varepsilon_{j+1} \geq 0 \quad \text{and} \quad \varepsilon_{j+1}\,\frac{d\,Z}{d\,\varepsilon_{j+1}} = 0$$

With 4 variables γ_j, γ_{j+1}, ε_j and ε_{j+1}, the solution to the above problem can be any of 16 possible cases. The analysis of the one-period model suggests that cases where the choice variables vanish by supposition (i.e. $\gamma_j=0$, $\gamma_{j+1}=0$ or both $\gamma_j=0$ and $\gamma_{j+1}=0$) can be safely disregarded because they turn out to be inconsistent with the respective constraint, regardless of whether the latter is binding or not. Of the sixteen cases, only the following four cases explicitly allow for positive values for the two choice variables.

<u>case 2.1: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\varepsilon_j > 0$ and $\varepsilon_{j+1} > 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{ r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j) \right\} \tag{14.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{ r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1}) \right\} \right] \tag{14.2}$$

$$\gamma_j = kD_{j-1} \tag{14.3}$$

$$\gamma_{j+1} = kD_j \tag{14.4}$$

<u>case 2.2: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\varepsilon_j > 0$ and $\varepsilon_{j+1} = 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{ r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j) \right\} \tag{15.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{ r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1}) \right\} \right]$$

$$r_A - r_R - \left\{ r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1}) \right\} \tag{15.2}$$

$$\gamma_j = kD_{j-1} \tag{15.3}$$

$$\gamma_{j+1} > kD_j \tag{15.4}$$

<u>case 2.3: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\varepsilon_j = 0$ and $\varepsilon_{j+1} > 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{ r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j) \right\}$$

$$r_A - r_R = \left\{ r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j) \right\} \tag{16.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}\right] \tag{16.2}$$

$$\gamma_j > kD_{j-1} \tag{16.3}$$

$$\gamma_{j+1} = kD_j \tag{16.4}$$

<u>case 2.4: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\varepsilon_j = 0$ and $\varepsilon_{j+1} = 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\}$$

$$r_A - r_R = \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\} \tag{17.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\} \tag{17.2}$$

$$\gamma_j > kD_{j-1} \tag{17.3}$$

$$\gamma_{j+1} > kD_j \tag{17.4}$$

Cases 2.1-2.3 suggest solutions where at least one of the constraints is binding. Where institutional penalties Ω are high, these cases can be disregarded primarily because **(14.1)**, **(14.2)**, **(15.1)** and **(16.2)** will all likely be negative, inconsistent with the assumption of the binding constraint, whether it be $\varepsilon_j > 0$, $\varepsilon_{j+1} > 0$, or both. However, even if these four critical equations turn out to be consistent with $\varepsilon_j > 0$ and/or $\varepsilon_{j+1} > 0$, it easily can be demonstrated that the resulting analytical implications are equally unacceptable. Particularly, it must follow from the presumption of a binding constraint that:

$$\varepsilon_j > 0 \quad \longrightarrow \quad r_A - r_R > \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\}$$

$$\varepsilon_{j+1} > 0 \quad \longrightarrow \quad r_A - r_R > \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}$$

These would mean however that the suggested respective optimal values will be at a levels after the convex downward marginal function has intersected the flat benefit function, specifically where the marginal cost of holding the reserve is greater than its marginal benefit.

There is no reason why the bank should choose to maintain a level of reserves where at the margin the benefits fall short of the costs. This discrepancy would otherwise be eliminated by reducing reserves further. However, this will not be possible, at least not without cost, since reserves are already kept at the beginning-of-the-period mandatory minimum. Hence, a contradiction is found in the supposition that binding constraints could be compatible with the reserve optima.

In contrast, case 2.4 suggests a solution where both constraints are nonbinding. Banks will choose a level of reserve in every period higher than the required minimum up to the point where the cost and benefit of reserve holdings offset on the margin. Since the proposition of binding constraints has just been shown to be internally flawed, the only consistent interpretation must be the case where the langrange multipliers evaluated at the mandatory minimum [(14.1), (14.2), (15.1) and (16.2)] must indeed be negative in sign, implying that the marginal benefit actually exceeds the marginal cost:

$$\varepsilon_j\Big|_{\gamma_j=kD_{j-1}} < 0 \longrightarrow r_A - r_R < \left\{r_1\omega_j(\gamma_j=kD_{j-1}) - \Omega\omega'_j(\gamma_j=kD_{j-1})\right\}$$
$$\varepsilon_{j+1}\Big|_{\gamma_j=kD_j} < 0 \longrightarrow r_A - r_R < \left\{r_1\omega_{j+1}(\gamma_{j+1}=kD_j) - \Omega\omega'_{j+1}(\gamma_{j+1}=kD_j)\right\}$$

The negative sign of the above lagrange multipliers follows from the tenuous position of holding reserves only at the mandatory minimum. Even if banks had enough funds in the reserve pool to physically service every withdrawal, it will still find itself technically illiquid since the mandatory holding will decline by only a fraction $0 < k < 1$ of the full amount of the withdrawal. In the best possible scenario therefore, banks will have to incur <u>with certainty</u> at least the cost of finding their actual reserve holdings being lower than the mandatory minimum. Increasing reserves beyond this minimum clearly reduces the possibility of incurring any illiquidity cost which is equivalent to stating that the net benefit at the margin of a unit of reserve is still positive, fully eliminated only at higher reserve levels.

1.3 Generalizing Over n Periods

It should be obvious that the recursive nature of the optimization easily extends to several periods. However, the need to specify all the possible combinations in a Kuhn-Tucker problem suggests that the "mechanics" of the optimization increases substantially as the time horizon expands. This is evident from the set of cases to consider, starting with only 4 in a one-period model but expanding substantially to 16 in a two-period model. To generalize the model over several periods, it should suffice therefore to show that the results continue a recursive pattern in a three-period model.

The three-period maximization will take the form:

$$\max_{\gamma_j, \gamma_{j+1}, \gamma_{j+2}} \quad \Pi = \pi_j + \beta\pi_{j+1} + \beta^2\pi_{j+2}$$

$$= \left[r_A\alpha_j + r_R\gamma_j - r_D D_{j-1} - \int_{\gamma_j}^{\bar{\delta}} \left\{ \Omega + r_I(\delta_j - \gamma_j) \right\} f(\delta)\, d\delta \right] + \beta\pi_{j+1} + \beta^2\pi_{j+2}$$

$$
\begin{aligned}
&= r_A(\alpha_j + \beta\alpha_{j+1} + \beta^2\alpha_{j+2}) + r_R(\gamma_j + \beta\gamma_{j+1} + \beta^2\gamma_{j+2}) \\
&\quad - r_D(D_{j-1} + \beta D_j + \beta^2 D_{j+1}) \\
&\quad - \Omega\left[\omega_j(\gamma_j) + \beta\omega_{j+1}(\gamma_{j+1}) + \beta^2\omega_{j+2}(\gamma_{j+2})\right] \\
&\quad - r_I\left[\int_{\gamma_j}^{\bar{\delta}} (\delta_j - \gamma_j) f(\delta) d\delta + \beta\int_{\gamma_{j+1}}^{\bar{\delta}} (\delta_{j+1} - \gamma_{j+1}) f(\delta) d\delta \right. \\
&\quad \left. + \beta^2\int_{\gamma_{j+1}}^{\bar{\delta}} (\delta_{j+1} - \gamma_{j+1}) f(\delta) d\delta \right] \qquad (18)
\end{aligned}
$$

subject to the corresponding three-period set of control constraints:

$$\gamma_j \geq kD_{j-1} \qquad (19.1)$$
$$\gamma_{j+1} \geq kD_j \qquad (19.2)$$
$$\gamma_{j+2} \geq kD_{j+1} \qquad (19.3)$$
$$\alpha_j + \gamma_j = D_{j-1} + W_{j-1} \qquad (19.4)$$
$$\alpha_{j+1} + \gamma_{j+1} = D_j + W_j \qquad (19.5)$$
$$\alpha_{j+2} + \gamma_{j+2} = D_{j+1} + W_{j+1} \qquad (19.6)$$

implying the lagrangian:

$$
\begin{aligned}
Z = r_A&\left\{(D_{j-1} + W_{j-1} - \gamma_j) + \beta(D_j + W_j - \gamma_{j+1}) + \beta^2(D_{j+1} + W_{j+1} - \gamma_{j+2})\right\} \\
&+ r_R\left(\gamma_j + \beta\gamma_{j+1} + \beta^2\gamma_{j+2}\right) - r_D\left(D_{j-1} + \beta D_j + \beta^2 D_{j+1}\right) \\
&- \Omega\left[\omega_j(\gamma_j) + \beta\omega_{j+1}(\gamma_{j+1}) + \beta^2\omega_{j+2}(\gamma_{j+2})\right] \\
&- r_I\left[\int_{\gamma_j}^{\bar{\delta}} (\delta_j - \gamma_j) f(\delta) d\delta + \beta\int_{\gamma_{j+1}}^{\bar{\delta}} (\delta_{j+1} - \gamma_{j+1}) f(\delta) d\delta \right.
\end{aligned}
$$

$$+ \beta^2 \int_{\gamma_{j+2}}^{\bar{\delta}} (\delta_{j+2} - \gamma_{j+2}) f(\delta) d\delta \Bigg]$$

$$+ \varepsilon_j \left\{ \gamma_j - kD_{j-1} \right\} + \varepsilon_{j+1} \left\{ \gamma_{j+1} - kD_j \right\} + \varepsilon_{j+2} \left\{ \gamma_{j+2} - kD_{j+1} \right\} \qquad \mathbf{(20)}$$

The corresponding set of Kuhn-Tucker conditions are defined as:

$$\frac{d\,Z}{d\,\gamma_j} = -r_A + r_R - \Omega \frac{d\,\omega_j}{d\,\gamma_j} - r_I \frac{d \int_{\gamma_j}^{\bar{\delta}} (\delta_j - \gamma_j)\, f(\delta)\, d\delta}{d\,\gamma_j} + \varepsilon_j \leq 0$$

$$\frac{d\,Z}{d\,\gamma_{j+1}} = \beta \left[-r_A + r_R - \Omega \frac{d\,\omega_{j+1}}{d\,\gamma_{j+1}} - r_I \frac{d \int_{\gamma_{j+1}}^{\bar{\delta}} (\delta_{j+1} - \gamma_{j+1})\, f(\delta)\, d\delta}{d\,\gamma_{j+1}} \right] + \varepsilon_{j+1} \leq 0$$

$$\frac{d\,Z}{d\,\gamma_{j+2}} = \beta^2 \left[-r_A + r_R - \Omega \frac{d\,\omega_{j+2}}{d\,\gamma_{j+2}} - r_I \frac{d \int_{\gamma_{j+2}}^{\bar{\delta}} (\delta_{j+2} - \gamma_{j+2})\, f(\delta)\, d\delta}{d\,\gamma_{j+2}} \right] + \varepsilon_{j+2} \leq 0$$

$$\frac{d\,Z}{d\,\varepsilon_j} = \gamma_j - kD_{j-1} \geq 0$$

$$\frac{d\,Z}{d\,\varepsilon_{j+1}} = \gamma_{j+1} - kD_j \geq 0$$

$$\frac{d\,Z}{d\,\varepsilon_{j+2}} = \gamma_{j+2} - kD_{j+2} \geq 0$$

with the following complementary slackness conditions:

$$\gamma_j \geq 0 \quad \text{and} \quad \gamma_j \frac{dZ}{d\gamma_j} = 0$$

$$\gamma_{j+1} \geq 0 \quad \text{and} \quad \gamma_{j+1} \frac{dZ}{d\gamma_{j+1}} = 0$$

$$\gamma_{j+2} \geq 0 \quad \text{and} \quad \gamma_{j+2} \frac{dZ}{d\gamma_{j+2}} = 0$$

$$\varepsilon_j \geq 0 \quad \text{and} \quad \varepsilon_j \frac{dZ}{d\varepsilon_j} = 0$$

$$\varepsilon_{j+1} \geq 0 \quad \text{and} \quad \varepsilon_{j+1} \frac{dZ}{d\varepsilon_{j+1}} = 0$$

$$\varepsilon_{j+2} \geq 0 \quad \text{and} \quad \varepsilon_{j+2} \frac{dZ}{d\varepsilon_{j+2}} = 0$$

With three choice variables and three lagrange multipliers, the above Kuhn-Tucker problem requires an enormous total of 64 (i.e. 2^6) possible cases to consider. Fortunately, the three choice variables all have to be positive to satisfy the minimum reserve constraint, eliminating the majority of the possible permutations. This effectively leaves only the three multipliers to be free to take non-negative values and hence leaving only the following 8 (i.e. 2^3) cases.

<u>case 3.1: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j > 0$, $\varepsilon_{j+1} > 0$ and $\varepsilon_{j+2} > 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{ r_I \omega_j(\gamma_j) - \Omega \omega_j'(\gamma_j) \right\} \tag{21.1}$$

$$\varepsilon_{j+1} = \beta \left[r_A - r_R - \left\{ r_I \omega_{j+1}(\gamma_{j+1}) - \Omega \omega_{j+1}'(\gamma_{j+1}) \right\} \right] \tag{21.2}$$

$$\varepsilon_{j+2} = \beta^2 \left[r_A - r_R - \left\{ r_I \omega_{j+2}(\gamma_{j+2}) - \Omega \omega_{j+2}'(\gamma_{j+2}) \right\} \right] \tag{21.3}$$

$$\gamma_j = kD_{j-1} \tag{21.4}$$

$$\gamma_{j+1} = kD_j \tag{21.5}$$

$$\gamma_{j+2} = kD_{j+1} \tag{21.6}$$

<u>case 3.2: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j > 0$, $\varepsilon_{j+1} > 0$ and $\varepsilon_{j+2} = 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j)\right\} \tag{22.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega_{j+1}'(\gamma_{j+1})\right\}\right] \tag{22.2}$$

$$\varepsilon_{j+2} = \beta^2\left[r_A - r_R - \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega_{j+2}'(\gamma_{j+2})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega_{j+2}'(\gamma_{j+2})\right\} \tag{22.3}$$

$$\gamma_j = kD_{j-1} \tag{22.4}$$

$$\gamma_{j+1} = kD_j \tag{22.5}$$

$$\gamma_{j+2} > kD_{j+1} \tag{22.6}$$

<u>case 3.3: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j > 0$, $\varepsilon_{j+1} = 0$ and $\varepsilon_{j+2} > 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j)\right\} \tag{23.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega_{j+1}'(\gamma_{j+1})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega_{j+1}'(\gamma_{j+1})\right\} \tag{23.2}$$

$$\varepsilon_{j+2} = \beta^2\left[r_A - r_R - \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega_{j+2}'(\gamma_{j+2})\right\}\right] \tag{23.3}$$

$$\gamma_j = kD_{j-1} \tag{23.4}$$

$$\gamma_{j+1} > kD_j \tag{23.5}$$

$$\gamma_{j+2} = kD_{j+1} \tag{23.6}$$

<u>case 3.4: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j = 0$, $\varepsilon_{j+1} > 0$ and $\varepsilon_{j+2} > 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j)\right\}$$

$$r_A - r_R = \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\} \tag{24.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}\right] \tag{24.2}$$

$$\varepsilon_{j+2} = \beta^2\left[r_A - r_R - \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\}\right] \tag{24.3}$$

$$\gamma_j > kD_{j-1} \tag{24.4}$$

$$\gamma_{j+1} = kD_j \tag{24.5}$$

$$\gamma_{j+2} = kD_{j+1} \tag{24.6}$$

case 3.5: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j = 0$, $\varepsilon_{j+1} = 0$ and $\varepsilon_{j+2} > 0$

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\}$$

$$r_A - r_R = \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\} \tag{25.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\} \tag{25.2}$$

$$\varepsilon_{j+2} = \beta^2\left[r_A - r_R - \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\}\right] \tag{25.3}$$

$$\gamma_j > kD_{j-1} \tag{25.4}$$

$$\gamma_{j+1} > kD_j \tag{25.5}$$

$$\gamma_{j+2} = kD_{j+1} \tag{25.6}$$

case 3.6: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j = 0$, $\varepsilon_{j+1} > 0$ and $\varepsilon_{j+2} = 0$

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\}$$

$$r_A - r_R = \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\} \tag{26.1}$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}\right] \tag{26.2}$$

$$\varepsilon_{j+2} = \beta^2\left[r_A - r_R - \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\} \quad (26.3)$$

$$\gamma_j > kD_{j-1} \quad (26.4)$$

$$\gamma_{j+1} = kD_j \quad (26.5)$$

$$\gamma_{j+2} > kD_{j+1} \quad (26.6)$$

<u>case 3.7: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j > 0$, $\varepsilon_{j+1} = 0$ and $\varepsilon_{j+2} = 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\} \quad (27.1)$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\} \quad (27.2)$$

$$\varepsilon_{j+2} = \beta^2\left[r_A - r_R - \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\} \quad (27.3)$$

$$\gamma_j = kD_{j-1} \quad (27.4)$$

$$\gamma_{j+1} > kD_j \quad (27.5)$$

$$\gamma_{j+2} > kD_{j+1} \quad (27.6)$$

<u>case 3.8: if $\gamma_j > 0$, $\gamma_{j+1} > 0$, $\gamma_{j+2} > 0$, $\varepsilon_j = 0$, $\varepsilon_{j+1} = 0$ and $\varepsilon_{j+2} = 0$</u>

$$\varepsilon_j = r_A - r_R - \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\}$$

$$r_A - r_R = \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\} \quad (28.1)$$

$$\varepsilon_{j+1} = \beta\left[r_A - r_R - \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\} \quad (28.2)$$

$$\varepsilon_{j+2} = \beta^2\left[r_A - r_R - \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\}\right]$$

$$r_A - r_R = \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\} \tag{28.3}$$

$$\gamma_j > kD_{j-1} \tag{28.4}$$

$$\gamma_{j+1} > kD_j \tag{28.5}$$

$$\gamma_{j+2} > kD_{j+1} \tag{28.6}$$

Just as in the one and two-period models, cases 3.1-3.7 can be eliminated because of internal inconsistencies with either $\varepsilon_j > 0$, or $\varepsilon_{j+1} > 0$, or ε_{j+2}. This leaves case 3.8 as the only consistent set of equations that would define the optima. Note, specifically that such optima is characterized by the system:

$$r_A - r_R = \left\{r_I\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j)\right\} \quad \text{implicitly defining } \gamma_j$$

$$r_A - r_R = \left\{r_I\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1})\right\} \quad \text{implicitly defining } \gamma_{j+1}$$

$$r_A - r_R = \left\{r_I\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2})\right\} \quad \text{implicitly defining } \gamma_{j+2}$$

The first two of these equations exactly solved the optima in the two-period model and the first equation was found to be the solution in the base one-period model. It is obvious therefore that the model easily extends into several periods where the optimal solution in any particular period remains totally unchanged regardless of the total number of periods involved.

The solution is stationary in nature. The same optimal rule is applied to every period and such rule is determined exclusively by the trading signals reflected by interest rates and the probability of illiquidity. It is not essential that the rates are kept constant overtime and this particular assumption is used purely for expediency. Indeed, time subscripts may be added to all the interest rates in the model without consequence.

This seems to suggest that the particular structure of the bank's portfolio is not relevant upon the optimization. This result relies heavily, however, upon the reliability of the respective interest rates to convey all the incentives that define the trade-off between holding the marginal unit of resource as either reserve or loan.

1.4 An Explicit Specification of a Markov Process

It is not obvious from the specification of the model thus far how the choice of reserves in one period directly affects choices in succeeding periods. Clearly, the result of a stationary solution must be taken in the context that there is an evident link across periods that sufficiently relates the choice variables. In short, the model may be susceptible to false dynamics if the only real intertemporal dimension it maintains is the discount process.

Consider again the allocation constraint in (12.4). Since the following equation of motions hold:

$$W_j - W_{j-1} = \pi_j \tag{29}$$

$$D_j - D_{j-1} = r_D D_{j-1} - \delta_j \tag{30}$$

it must follow that:

$$\alpha_{j+1} + \gamma_{j+1} = D_j + W_j = (1+r_D)D_{j-1} - \delta_j + W_{j-1} + \pi_j$$

$$= \left(\alpha_j + \gamma_j\right) + r_A\alpha_j + r_R\gamma_j - \delta_j - \int_{\gamma_j}^{\bar{\delta}} \left\{\Omega + r_I(\delta_j - \gamma_j)\right\} f(\delta)\, d\delta \qquad (31)$$

Written in this form, it becomes clear that the choice variables are in fact dynamically linked since the available resources in the succeeding period is fully dependent upon the resources available at the beginning of the current period, the choice variable in the current period and any illiquidity cost incurred from maintaining that choice *vis-a-vis* the random variable δ_j. Note that (31) qualifies as a Markov process because the current state fully summarizes all the relevant information needed in deciding the state in the succeeding period and that the "one-step transition probabilities" would be independent of the specific periods in consideration since $f(\delta)$ is time-invariant.

Collecting similar terms and using (12.3), we then find that:

$$\alpha_{j+1} + \gamma_{j+1} = (1 + r_A)(W_{j-1} + D_{j-1}) - (r_A - r_R)\gamma_j - \delta_j$$
$$- \Omega\omega_j(\gamma_j) - \int_{\gamma_j}^{\bar{\delta}} r_I(\delta_j - \gamma_j)\, f(\delta)\, d\delta \qquad (32)$$

implying a two-period maximization of the form:

$$\max_{\gamma_j, \gamma_{j+1}} \quad \Pi = \pi_j + \beta\pi_{j+1}$$
$$= r_A\left(W_{j-1} + D_{j-1} - \gamma_j\right) + r_R\gamma_j - r_D D_{j-1} - \Omega\omega_j(\gamma_j)$$
$$- \int_{\gamma_j}^{\bar{\delta}} r_I(\delta_j - \gamma_j)\, f(\delta)\, d\delta + \beta r_A\left(W_{j-1} + D_{j-1} - \gamma_j\right)\left(1 + r_A\right)$$

$$+\ \beta r_A(1+r_A)\gamma_j - \beta r_A \Omega\omega_j(\gamma_j) - \beta r_A \int_{\gamma_j}^{\bar{\delta}} r_l(\delta_j-\gamma_j)\ f(\delta)\ d\delta$$

$$-\ \beta r_A\delta_j - \beta r_A\gamma_{j+1} + \beta r_R\gamma_{j+1} - \beta r_D D_j$$

$$-\ \beta\Omega\omega_{j+1}(\gamma_{j+1}) - \beta\int_{\gamma_{j+1}}^{\bar{\delta}} r_l(\delta_{j+1}-\gamma_{j+1})\ f(\delta)\ d\delta \qquad (33)$$

Solving through the Kuhn-Tucker conditions, the solution will then be:

$$r_A - r_R = \left\{r_l\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j)\right\} \qquad (34.1)$$

$$r_A - r_R = \left\{r_l\omega_{j+1}(\gamma_{j+1}) - \Omega\omega_{j+1}'(\gamma_{j+1})\right\} \qquad (34.2)$$

$$\gamma_j > kD_{j-1} \qquad (34.3)$$

$$\gamma_{j+1} > kD_j \qquad (34.4)$$

which are the exact same optimal values found in (17).

Given the results in section 1.3, it should be clear that the solution with the Markov specification extends directly over several periods. Thus, not only has the solution been shown to be stationary, it is now evident that it will, in fact, be the same regardless of whether the allocation constraints are re-written to prominently reflect its Markovian properties. Using the terminology in Kreps (1990), we now characterize the optima to be Markov stationary. It is worth re-emphasizing that the "Markov" component arises out of the allocation constraint which is binding. Since this is the case, the "stationary" feature is obtained since the bank essentially faces the same problem in every period, contingent only upon the contemporaneous resources it has on hand, the latter subject to the Markov process discussed above.

2. Long-Term Loans: Fixed Maturities Beyond One Period

The model has thus far treated loans as an aggregate. In practice, it is clear that "loan instruments" is an array of highly differentiated products that vary according to (1) the different categories of clients involved (i.e. across the high-risk-low-risk spectrum) as well as (2) the individual characteristics of each loan contract within the same general category of clients. This distinction between different specific loan instruments has not been critical to the reserve decision thus far because the obvious concern of the latter is simply a rule that calculates the optimal reserve pool, irrespective of how the balance is allocated.

The bank has also had the luxury of explicitly collecting the full principal of its loan disbursements at the end of every period. In reality, banks have much less freedom to work with because (1) there is always the uncertainty of defaults and (2) it is much more reasonable to observe loan contracts extending over several planning periods. This has the obvious effect of restricting the amount that can be allocated in every period since a portion of its total resources is either non-collectible or effectively pre-committed.

We now extend the model by relaxing the assumption that loans are homogeneous instruments that are redeemable with certainty at the end of one period. In particular, we consider the case where there are two types of loan instruments that are differentiated by maturity and rate of return. The first type of instrument is a short-term loan contract, α^S, that maintains a fixed one-period maturity and commands a loan rate r_A^S. The other is a longer-term instrument, α_A^L, with a two-period fixed maturity that is offered at a rate $\overline{r}_A^L$.

It is assumed that the full amount of the long-term loan is provided lump-sum by the bank in the period that the loan is transacted. Long-term loans are not, however, dominated by short-term loans because its rate is defined to be greater than the short-term loan rate by the usual term-structure argument. In short, disbursing a short-term loan over two successive periods will remain inferior to allocating the same amount through a two-period long-term loan since $\bar{r}^L_A > r^S_A$. In exchange for this premium, $\bar{r}^L_A - r^S_A$, the amount involved in the long-term loan is pre-committed over its term--since the bank provides the full funding upfront--it is therefore unavailable for optimal allocation in the succeeding period.

The array of assets available to the bank now involve reserves, one-period loans and two-period loans which will have to satisfy the resource constraint:

$$\alpha^S_j + \alpha^L_j = 1 - \gamma_j \tag{35}$$

Any increment in the reserve pool will have to mean an equivalent decrement in the loan portfolio. If the composition of the loan portfolio between short and long-term loans is uncorrelated with the corresponding decrement then, the distinction between these two forms of loan instruments is benign and inconsequential to the reserve decision. The invariance of the composition of the loan portfolio to the level of reserves suggests that the corresponding decrement may be fully absorbed by any weighted average:

$$b\alpha^S_j + (1-b)\alpha^L_j \quad \forall\ 0 \leq b \leq 1 \tag{36}$$

However, since:

$$\alpha_j = b\alpha^S_j + (1-b)\alpha^L_j \quad \text{where } \frac{\partial b}{\partial \gamma_j} = 0 \tag{37}$$

then satisfying the fixed resource constraint:

$$\frac{d\,\alpha_j}{d\,\gamma_j} = \frac{\partial\,\alpha_j}{\partial\,\gamma_j} = -1 \longrightarrow b\frac{\partial\,\alpha_j^S}{\partial\,\gamma_j} + (1-b)\frac{\partial\,\alpha_j^L}{\partial\,\gamma_j} = -1 \tag{38}$$

creates a problem because neither $\frac{\partial\,\alpha_j^S}{\partial\,\gamma_j}$ nor $\frac{\partial\,\alpha_j^L}{\partial\,\gamma_j}$ is formally meaningful.

Consider now a simple alternative. Define $0 \leq \lambda_j(\gamma_j) \leq 1$ to be a variable in period j that is inversely proportional to γ_j, i.e. $\frac{\partial\,\lambda_j(\gamma_j)}{\partial\,\gamma_j} < 0$. Let $\lambda_j(\gamma_j)$ indicate the manner in which the <u>composition</u> of the loan portfolio is affected by the reserve decision such that:

$$\frac{\partial\,\alpha_j^L}{\partial\,\gamma_j} = \frac{\partial\,\lambda_j(\gamma_j)}{\partial\,\gamma_j}\alpha_j \longrightarrow \frac{\partial\,\alpha_j^S}{\partial\,\gamma_j} = -1 - \frac{\partial\,\lambda_j(\gamma_j)}{\partial\,\gamma_j}\alpha_j \tag{39}$$

The intuition is that the bank would prefer to have relatively more of one-period rather than two-period loans in its loan portfolio to coincide with the initiative of increased reserves. This is neither a particularly restrictive assumption nor a strangely artificial construct. In fact, this simply capitalizes on the distinction already made between these two loan instruments: a trade-off between the higher profitability of long-term loans (i.e. the premium $\tilde{r}_A^L - r_A^S$) against the greater liquidity of short-term loans (i.e. the difference in their fixed maturities). This is exactly the same trade-off that is generating the allocation problem in the first place: a choice between profitable loans and liquid reserves.

Given the above, the shortest meaningful horizon for the model would be two periods. Taking the case of three periods, the maximization will be of the form:

$$\begin{aligned}\max_{\gamma_j, \gamma_{j+1}} \quad \Pi = {} & r_A^L \alpha_j^L + r_A^S \alpha_j^S + r_R \gamma_j - r_D D_{j-1} - \int_{\gamma_j}^{\bar{\delta}} \left\{ \Omega + r_I(\delta_j - \gamma_j) \right\} f(\delta) \, d\delta \\ & + \beta r_A^L \alpha_j^L + \beta r_A^L \alpha_{j+1}^L + \beta r_A^S \alpha_{j+1}^S + \beta r_R \gamma_{j+1} - \beta r_D D_j \\ & - \beta \int_{\gamma_{j+1}}^{\bar{\delta}} \left\{ \Omega + r_I(\delta_{j+1} - \gamma_{j+1}) \right\} f(\delta) \, d\delta \\ & + \beta^2 r_A^L \alpha_{j+1}^L + \beta^2 r_A^L \alpha_{j+2}^L + \beta^2 r_A^S \alpha_{j+2}^S + \beta^2 r_R \gamma_{j+2} - \beta^2 r_D D_{j+1} \\ & - \beta^2 \int_{\gamma_{j+1}}^{\bar{\delta}} \left\{ \Omega + r_I(\delta_{j+1} - \gamma_{j+1}) \right\} f(\delta) \, d\delta \end{aligned} \tag{40}$$

subject to the usual constraints on the mandatory minimum reserve level and the following allocation constraints:

$$\alpha_j + \gamma_j = D_{j-1} + W_{j-1} - \alpha_{j-1}^L \quad = D_{j-1} + W_{j-1} - \lambda_{j-1}\alpha_{j-1} \tag{41.1}$$

$$\alpha_{j+1} + \gamma_{j+1} = D_j + W_j - \alpha_j^L \quad = D_j + W_j - \lambda_j \alpha_j \tag{41.2}$$

$$\alpha_{j+2} + \gamma_{j+2} = D_{j+1} + W_{j+1} - \alpha_{j+1}^L \quad = D_{j+1} + W_{j+1} - \lambda_{j+1}\alpha_{j+1} \tag{41.3}$$

Notice that in (40) two-period loans explicitly generate nominal interest revenues of the amount $r_A^L \alpha_j^L$ in period j and j+1 and $r_A^L \alpha_{j+1}^L$ in periods j+1 and j+2. The rate used, r_A^L, is therefore the simple annualized rate that corresponds to the stated rate $\bar{r}_A^L$ over the two-period term such that $r_A^L - r_A^S$ is effectively ambiguous in sign. The allocation constraints subsequently impose that the disbursement of these long term loans are honored by the bank. These constraints then allow total loans to be implicitly defined as:

$$\alpha_j(\gamma_j,\gamma_{j-1}) = D_{j-1} + W_{j-1} - \lambda_{j-1}(\gamma_{j-1})\alpha_{j-1}(\gamma_{j-1}) - \gamma_j \qquad (42.1)$$

$$\alpha_{j+1}(\gamma_{j+1},\gamma_j) = D_j + W_j - \lambda_j(\gamma_j)\alpha_j(\gamma_j) - \gamma_{j+1} \qquad (42.2)$$

$$\alpha_{j+2}(\gamma_{j+2},\gamma_{j+1}) = D_{j+1} + W_{j+1} - \lambda_{j+1}(\gamma_{j+1})\alpha_{j+1}(\gamma_{j+1}) - \gamma_{j+2} \qquad (42.3)$$

Applying the Kuhn-Tucker theorem, the solution can be then found to be:

$$r_f\omega_j(\gamma_j) - \Omega\omega'_j(\gamma_j) = r_A^S - \left(r_A^L - r_A^S\right)\left(\lambda'_j\alpha_j - \lambda_j\right) - r_R$$
$$- \beta\left(r_A^L - r_A^S\right)\left(1 + \lambda_{j+1}\right)\left(\lambda'_j\alpha_j - \lambda_j\right) \qquad (43.1)$$

$$r_f\omega_{j+1}(\gamma_{j+1}) - \Omega\omega'_{j+1}(\gamma_{j+1}) = r_A^S - \left(r_A^L - r_A^S\right)\left(\lambda'_{j+1}\alpha_{j+1} - \lambda_{j+1}\right) - r_R$$
$$- \beta^2\left(r_A^L - r_A^S\right)\left(1 + \lambda_{j+2}\right)\left(\lambda'_{j+1}\alpha_{j+1} - \lambda_{j+1}\right) \qquad (43.2)$$

$$r_f\omega_{j+2}(\gamma_{j+2}) - \Omega\omega'_{j+2}(\gamma_{j+2}) = r_A^S - \left(r_A^L - r_A^S\right)\left(\lambda'_{j+2}\alpha_{j+2} - \lambda_{j+1}\right) - r_R \qquad (43.3)$$

$$\gamma_j > kD_{j-1} \qquad (43.4)$$

$$\gamma_{j+1} > kD_j \qquad (43.5)$$

$$\gamma_{j+2} > kD_{j+1} \qquad (43.6)$$

To properly evaluate the modified optimality condition, consider the first two terms in the right-hand side of (43.1). This is easily interpreted as fore-gone earnings from current-period loans.

$$r_A^S - \left(r_A^L - r_A^S\right)\left(\lambda'_j\alpha_j - \lambda_j\right) \equiv \left\{r_A^L\frac{-\partial\alpha_j^L}{\partial\gamma_j} + r_A^S\frac{-\partial\alpha_j^S}{\partial\gamma_j}\right\} > 0 \qquad (44)$$

consisting of a base return r_A^S on one-period loans plus the weighted incremental return of long-term loans over short-term loans. The latter is a consequence of the fact that having finite resources forces reserves and loans to be direct substitutes, $\frac{\partial \alpha_j(\gamma_j)}{\partial \gamma_j} = -1$, such that:

$$\frac{-\partial \alpha_j(\gamma_j)}{\partial \gamma_j} = \frac{-\partial \alpha_j^L(\gamma_j)}{\partial \gamma_j} + \frac{-\partial \alpha_j^S(\gamma_j)}{\partial \gamma_j} = 1 \tag{45}$$

Stated differently, every unit invested into the portfolio of current-period loans cannot do worse than the return on one-period loans. In addition, a proportion of the interest premium, $r_A^L - r_A^S > 0$, accrues because the availability of an extra unit investible in the loan portfolio is a unit taken away from reserves, implying an increase in $\lambda_j(\gamma_j)$ and a concommitant shift towards long-term loans and away from short-term loans. The full premium itself is not gained for as long one-period loans take a proportion of the loan portfolio. Conversely, only r_A^S would accrue if no two-period loans were carried in the portfolio.

Note that foregone earnings from current-period loans is not invariant to the actual level of reserves held. For convex and linear forms of $\lambda_j(\gamma_j)$,

$$\frac{\partial \left[\frac{-\partial \alpha_j^L(\gamma_j)}{\partial \gamma_j}\right]}{\partial \gamma_j} = \frac{-\partial \left\{\lambda_j' \alpha_j - \lambda_j\right\}}{\partial \gamma_j} = 2\lambda_j'(\gamma_j) - \lambda_j''(\gamma_j)\alpha_j \tag{46}$$

will be unambiguously negative. Depending on the assumed sign of $\frac{\partial^3 \lambda_j(\gamma_j)}{\partial \gamma_j^3}$, it is, in principle, feasible for (46) to be positive over a local or global range of γ_j. For the simplicity of avoiding having to make such specific assumption about $\lambda_j(\gamma_j)$, we assume aside this possibility and leave (46) as being in general negative in sign.

Consider now the last term on the right-hand side of (43.1). Whereas the first two terms focused upon the interplay between short and long-term loan rates in the current period, this last term can be shown to reflect the effect of loan pre-commitments that span beyond the current period since:

$$-\beta\left(r_A^L - r_A^S\right)\left(1 + \lambda_{j+1}\right)\left(\lambda_j'\alpha_j - \lambda_j\right) = \beta\left(r_A^L - r_A^S\right)\left(1 + \lambda_{j+1}\right)\left[\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j}\right] \qquad (47)$$

This term is ambiguous in sign, implying that marginal cost could either increase or decrease with loan instruments that span several periods depending upon the sign of the annualized spread $r_A^L - r_A^S$.

To better see the source of this ambiguity, it is helpful to view (47) in its two components. First, it is tautological that the decision to invest into reserves in period j is a simultaneous decision to give up period j+1 returns from period-j two-period loans. This is reflected in the expression:

$$-\beta r_A^L\left(\lambda_j'\alpha_j - \lambda_j\right) = \beta r_A^L\left[\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j}\right] > 0 \qquad (48)$$

which directly comes out of taking the derivative of the second period return, $\beta r_A^L\alpha_j^L$, with respect to the choice variable γ_j. This is plainly foregone revenues and therefore can only increase marginal cost and hence reduce optimal reserves.

Second, the same marginal increase in reserve will mean that a portion of this extra unit, $\frac{-\partial \alpha_j^L(\gamma_j)}{\partial \gamma_j}$, was <u>not</u> allocated as long-term loan in period j. This has the subsequent effect of expanding available uncommitted resources in period j+1. Using (42.2) and taking the derivative of discounted interest revenues in period j+1,

$$\frac{\partial \left[\beta r_A^L \alpha_{j+1}^L + \beta r_A^S \alpha_{j+1}^S\right]}{\partial \gamma_j} = -\beta\left[r_A^L \lambda_{j+1} + r_A^S(1-\lambda_{j+1})\right]\left[\frac{-\partial \alpha_j^L(\gamma_j)}{\partial \gamma_j}\right] \tag{49}$$

reflects the marginal income in period j+1 that would accrue because of the extra unit of uncommitted resource carried over into period j+1. Since (49) is a potential revenue flow, it must serve to reduce the marginal cost to the bank in period j, independent of actions taken in period j+1. *Ceteris paribus*, this implies an offsetting shift away from reserves and back into loans.

The expression in (47) is the sum of (48) and (49). With all other sub-expressions determinate in sign, the net direction of these two opposing effects will ultimately depend upon the sign of the spread $r_A^L - r_A^S$. This follows from the fact that available funds may either be pre-committed in the preceding period at the annualized rate r_A^L or carried over for allocation into the loan portfolio in the next period. The latter channel of allocation is, however, expected to provide a rate of return at an average of the two loan instruments. Which of these channels prove to be more profitable clearly depends upon the magnitude of the annualized rate r_A^L relative to be average rate $r_A^L\lambda_{j+1} + r_A^S(1-\lambda_{j+1})$. The spread $r_A^L - r_A^S$ completely summarizes this comparison since a positive (negative) spread would always mean that the annualized rate is greater (less) than the average rate, for $0 < \lambda_{j+1} < 1$.

For lack of better terminology, we refer to (48) and (49) as the commitment and option effect respectively of having multi-period loans. Depending on whether resources are committed as equity loans in the current period or freely carried over into the next period (by allocating them as current period reserves), optimal reserves will either be lower (commitment effect) or higher (option effect) than the intraperiod optima found when all loans were strictly of a one period maturity. Furthermore, as the bank is increasingly willing to offer annualized rates lower than fixed one-period rates, $r_A^L = r_A^L + \tau$ for $\tau > 0$, the concomitant increase in optimal reserve holdings can be interpreted not only from the perspective of a decline in the profitability of long-term loans but also reflective of the increased flexibility that the bank maintains in allowing resources to be uncommitted. This is an important extension, particularly in a volatile financial environment, because it can explain why a bank may choose to offer relatively lower annualized rates with longer term loan instruments.

The above analysis clearly holds for (43.2) as well. In fact, a general N-period model will lead to the same functional form for marginal cost over the first N-1 periods, each of which reflect the (1) foregone earnings from current-period, (2) the income-commitment effect and (3) the income-option effect. It should not be surprising to find that if long-term loans were defined to span over T-periods, the respective within-period expression for the marginal cost of reserves will account for the income-commitment and income-option effects over the succeeding T-1 periods as well. Finally, in the general model under N-periods with T-period long-term loans $\forall$ $N>T$, the terminal period optima reflects only the weighing of the short and long-term loan rates. This should be obvious since by definition there is no further period to consider to weigh the commitment and option effects.

3. Effect of Interest-Rate Shocks

Since the solution takes into account the possible effects of two-period loans in the succeeding period, interest rate changes will have a more pronounced dynamic dimension. One certainly expects that the timing of the interest change will be important in determining net results if only because across-period arbitrage possibilities are evaluated by the commitment and option effects. Given further the ambiguity in (47), it is not at all obvious then how optimal reserves respond to variously-timed interest rate shocks.

3.1 A Permanent Increase in Loan Rates

The particular revenue created by the extra unit of resource has been shown in (48) and (49) to be critical in determining the effect upon optimal reserves. Relative to the case were all loans had a fixed one-period term, the equilibria under multi-period loans could either be higher or lower, depending upon whether the substitution or income effect dominated. Defining $\gamma^*_{1,j}$ to be the solution in period j, for all $j<T-1$, for a portfolio made up of strictly one-period loans, the use of multi-period loans will imply two possible equilibria $\gamma^*_{C,j} < \gamma^*_{1,j} < \gamma^*_{O,j}$ where "C" and "O" indicate which of the two effects dominate.

Consider now an exogenous and permanent rise in loan rates, known to take effect beginning period j. In this period, the increase in both r_A^S and r_A^L will change marginal cost by the magnitude:

$$1 + \left[1+\beta\left((1-\lambda_{j+1}\right)\right]\left(\lambda'_j\alpha_j - \lambda_j\right) \quad = \quad 1 - \frac{-\,\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j} - \beta\left(1-\lambda_{j+1}\right)\frac{-\,\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j} \qquad (50)$$

$$-\left[1+\beta\left(1-\lambda_{j+1}\right)\right]\left(\lambda'_j\alpha_j - \lambda_j\right) \quad = \quad \frac{-\,\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j} + \beta\left(1-\lambda_{j+1}\right)\frac{-\,\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j} \qquad (51)$$

both of which are unambiguously positive. This simply states that the stream of future revenues will now be unambiguously higher at that present level of loan exposures, tantamount to a consistently larger foregone return should the allocation be made towards reserves rather than loans. Reserves can only be more costly at the margin with the higher loan rates, inducing a decrease in reserves.

If both loans had a fixed one-period term (but somehow differentiated in other meaningful respects), an increase in the loan rates would clearly generate an increase in the marginal cost of reserves:

$$\frac{\partial\ r_A^S - \left[r_A^L - r_A^S\right]\left(\lambda'_j\alpha_j - \lambda_j\right) - r_R}{\partial\ r_A^L} \quad = \quad \frac{-\,\partial\alpha_j^L}{\partial\ \gamma_j} \quad > \quad 0 \qquad (52)$$

$$\frac{\partial\ r_A^S - \left[r_A^L - r_A^S\right]\left(\lambda'_j\alpha_j - \lambda_j\right) - r_R}{\partial\ r_A^S} \quad = \quad \frac{-\,\partial\alpha_j^S}{\partial\ \gamma_j} \quad = \quad 1 - \frac{-\,\partial\alpha_j^L}{\partial\ \gamma_j} \quad > \quad 0 \qquad (53)$$

and therefore reduce optimal reserve holdings. Note, however, that while any exogenous and independent increase in either r_A^L or r_A^S would correspondingly increase marginal cost by $\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j}$ and $\frac{-\partial\ \alpha_j^S(\gamma_j)}{\partial\ \gamma_j}$ respectively, the relative magnitude of these derivatives will depend upon the level of $\lambda_j(\gamma_j)$ since $\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j}$ is itself positively correlated with $\lambda_j(\gamma_j)$. Thus, for higher (lower) values of $\lambda_j(\gamma_j)$, the effect upon the marginal cost of a change in r_A^L will be greater (less) than the any equivalent change in r_A^S.

The last point is worth some emphasis. Again, assuming only one-period loans exist, the optimality condition can again be written as:

$$\Gamma(\gamma_j) = r_\Gamma\omega_j(\gamma_j) - \Omega\omega_j'(\gamma_j) - \left\{r_A^S - \left(r_A^L - r_A^S\right)\left(\lambda_j'\alpha_j - \lambda_j\right) - r_R\right\} \tag{54}$$

where the implicit function theorem allows us to derive:

$$\frac{d\ r_A^L}{d\ r_A^S} = -\frac{\dfrac{\partial\ \Gamma(\gamma_j)}{\partial\ r_A^S}}{\dfrac{\partial\ \Gamma(\gamma_j)}{\partial\ r_A^L}} = -\frac{1 - \dfrac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j}}{\dfrac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j}} < 0 \tag{55}$$

This downward sloping function represents an array of ordered pairs in r_A^L-r_A^S space where inverse changes in these rates offset to keep the level of optimal reserves constant. This array will not be unique to $\lambda_j(\gamma_j)$ since $\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j}$ varies proportionally with the level of $\lambda_j(\gamma_j)$. In particular, (55) will be steeper for lower-values of $\lambda_j(\gamma_j)$ and conversely flatter when the loan portfolio is mainly made up of equity loans.

In a "good" state of the economy where banks can afford to be substantially exposed to the capital (long term loan) market, a decline in r_A^S need only be compensated with a less than proportional increase in r_A^L to avoid any change in the reserve position. A full increase in r_A^L that matches the decline in will serve to reduce reserves, increase loans, increase the proportion of funds in equity loans and subsequently increase the profitability of the portfolio.

However, the power of the rate r_A^S in the secondary reserve market becomes dominant when the economy is in a "bad" state and where banks are more concerned with liquidity than profitability. In this state, banks keep a high level of reserve and complement this with a loan portfolio that is predominantly in the form of secondary reserves. Any subsequent decrease in r_A^S that is fully matched by an increase in r_A^L should imply a shift away from loans and into reserves, relatively away from equity and more towards cash loans, increasing the bank's liquidity position but at the cost of a portfolio with lower profitability.

The presence of loan pre-commitments subsequently adds the respective last terms to **(50)** and **(51)**. Both of these last terms are clearly positive in sign, suggesting that the rise in marginal cost due to an increase in r_A^S would actually be <u>lower</u> with multi-period loans than with pure one-period loans. Increases in r_A^L, however, will now find that marginal cost will increase by a relatively <u>larger</u> amount than if no long-term loans exist.

To see why this is the case, it is again convenient to decompose these added terms into distinct option and commitment effects. The rise in r_A^S in particular will tend to increase the weighted loan rate in period j+1 taken from the perspective of period j. The decision not to commit resources in period j is made more attractive by the flexibility provided by the option effect described above. This is precisely why marginal cost rises by less with multi-period loans when r_A^S is increased, with the difference exactly measured by the term:

$$- \beta\left(1-\lambda_{j+1}\right)\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j} \quad < \quad 0 \tag{56}$$

The rise in r_A^L is a little more complex. On one hand, higher rates on long-term loans means higher second period return. This potential income has been shown to imply a shift away from reserves and into two-period loans. However, the same rise in r_A^L aids the option effect by increasing the weighted rate on loans. For all period j where r_A^L is higher, the higher weighted loan rate will be felt in both the current period as well as for the next period, discounted appropriately. Thus, the last term in **(51)** is an aggregate for the pure commitment effect:

$$\beta\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j} \quad > \quad 0 \tag{57}$$

and the result of the rise in r_A^L that helps fuel the option effect in the succeeding period:

$$-\beta\lambda_{j+1}\frac{-\partial\ \alpha_j^L(\gamma_j)}{\partial\ \gamma_j} \quad < \quad 0 \tag{58}$$

Since the (absolute value of) latter measures the rise in the weighted loan rate per unit increase in r^L_A, the appropriate weight $0 < \lambda_{j+1} < 1$ guarantees that the pure commitment effect will more than offset the indirect contribution of r^L_A to the option effect. Subsequently, one finds that increase in marginal cost will be higher as multi-period loans take effect.

3.2 A Temporary Increase in Loan Rates

Some ambiguity sets in if any increase in interest rates would be seen as temporary. Consider, for example, the case where interest rates jump in period j+1 but return to their pre-jump levels thereafter:

$$r^L_{A,j+1} > \left\{ r^L_{A,j} = r^L_{A,j+2} \cdots = r^L_{A,T} \right\} \text{ and } r^S_{A,j+1} > \left\{ r^S_{A,j} = r^S_{A,j+2} \cdots = r^S_{A,T} \right\}$$

If this is foreseen by the banks, this will be programmed into their decisions beginning in period j. Unlike the permanent increase in interest rates, however, the immediate effect in period j is only upon the potential revenue flows expected in period j+1. In short, only the terms defined in (56)-(58) are involved. Consequently, the one-time temporary increase to $r^L_{A,j+1}$ and $r^S_{A,j+1}$ will respectively affect marginal cost by:

$$\beta\left(1-\lambda_{j+1}\right)\frac{-\partial\ \alpha^L_j(\gamma_j)}{\partial\ \gamma_j} \quad > \quad 0 \tag{59}$$

$$-\ \beta\left(1-\lambda_{j+1}\right)\frac{-\partial\ \alpha^L_j(\gamma_j)}{\partial\ \gamma_j} \quad < \quad 0 \tag{60}$$

It is of some surprise to verify that unit changes in both loan rates will in fact fully offset each other, requiring no change in optimal reserves in period j. This follows from the fact that the increase in potential second-period income from period-j long-term loans is exactly equal to the increase in the weighted loan rate in period j+1:

$$\beta \frac{-\partial \alpha_j^L(\gamma_j)}{\partial \gamma_j} dr_A^L = \beta\left[dr_A^L \lambda_{j+1} + dr_A^S(1-\lambda_{j+1})\right]\left[\frac{-\partial \alpha_j^L(\gamma_j)}{\partial \gamma_j}\right] \tag{61}$$

for all $dr_A^L \equiv \left[r_{A,j+1}^L - r_{A,j}^L\right] = \left[r_{A,j+1}^S - r_{A,j}^S\right] \equiv dr_A^S$. Unequal increases in r_A^S and r_A^L will of course lead to a decrease (increase) in optimal reserves if the commitment effect is greater (less) than the option effect, that is, if dr_A^L is greater (less) than dr_A^S.

In period j+1, the higher rates will only affect the contemporaneous weighted loan rate. Rewriting (43.2) to explicitly denote time subscripts and using the relationships defined in (44):

$$\left\{r_{A,j+1}^L \frac{-\partial\alpha_{j+1}^L}{\partial \gamma_{j+1}} + r_{A,j+1}^S \frac{-\partial\alpha_{j+1}^S}{\partial \gamma_{j+1}}\right\}$$
$$+ \beta^2\left[r_{A,j+2}^L - r_{A,j+2}^S\right]\left(1 + \lambda_{j+2}\right)\left[\frac{-\partial \alpha_{j+1}^L(\gamma_{j+1})}{\partial \gamma_{j+1}}\right] \tag{62}$$

it is clear that the rise in at least one--if not both--loan rate(s) will always increase period j+1 marginal cost:

$$dr^{L}_{A,j+1} \frac{-\partial\alpha^{L}_{j+1}}{\partial \gamma_{j+1}} + dr^{S}_{A,j+1} \frac{-\partial\alpha^{S}_{j+1}}{\partial \gamma_{j+1}} \tag{63}$$

regardless of the absolute magnitude of the individual increments. Unambiguously, this means that period j+1 optimal reserves will always be lower than what would have been allocated without the temporary rise in rates. Beyond period j+1, reserves rise back to their level before the rate change.

By extension, a temporary one-period rise in interest rates is seemingly equivalent to separating the intraperiod effects of a permanent increase outlined in (50) and (51) into two adjascent periods. The key difference, however, is that the permanent increase will lead to a decline in reserves for all periods, irrespective of the magnitude of the rate change. A temporary increase, in contrast, will only have this effect in the second of the adjascent periods and may have no effect in the first. The interesting consequence of this difference is that judicious use of a policy that temporarily changes loan rates by the same increment allows banks to optimally decide on its reserve position as if it only carried strict one-period loans even when it maintains two-period equity loans in its portfolio. The focus is then limited (and simplified) to the effect of the rate shock on the contemporaneous weighted loan rate, not required to consider the expected future stream and the balance between the substitution and income effects.

4. A Summary of Key Results

The standard bank reserve model is extended by explicitly introducing the dimension of time. This feature has not been addressed in the literature which seems to be a very surprising omission given that the concensus view of this financial institution is that of a (differentiated) microeconomic firm maximizing over terminal wealth. To allow for direct comparability, the key features of the basic static model are all retained, subject to convenient, if not unavoidable, changes in notation. The bank continues to maximize current period profits but now in addition to the discounted stream of future flows.

The results indicate that the solution across periods is stationary. The same optimal rule will be used by banks to evaluate its best possible reserve position, dependent upon the same set of variables, structured in the same marginal condition for each of the periods considered. All the information needed by the bank to make its choice is embodied in the various interest rates as well as in the penalty structure that institutionally defines the cost of illiquidity. Neither the contemporaneous level of portfolio variables nor its comparative levels in previous periods become relevant to the optimal rule and this follows from the perspective that at the margin, prices fully reflect trading incentives and penalties.

The stationary solution is in fact qualified by an even stronger feature: the intraperiod optima will be exactly the same as that in the static case. This literally suggests that the optimization in a multi-period horizon can be taken as a series of repeated one-period models, leading to the same set of optima. This result was shown to be robust, particularly when the allocation constraints were modeled explicitly as a Markov process. Thus, irrespective of the length of time considered and regardless of the accuracy of choices made in previous periods, the problem of optimizing the reserve position of banks ultimately does become an issue of evaluating within period incentives.

The reason for this is quite straightforward and simple that it is easy to overlook. Financial statements must always balance *ipso facto* and therefore banks can only allocate what it actually has in resources, not what it expected to find. Subsequently, even though an array of optimal reserves can be conceptually be projected over N periods, it is a fact that the guess for a particular period will have to be either correct or incorrect, the consequences of both of which the bank must absorb into its financial position before it can proceed to the next period. It is the updated and balanced financial position that it takes into the beginning of the next planning period for allocation. Regardless of what had happened in previous periods, the actual holding of resources must still be allocated. Hence, the total asset position at the beginning of any period already fully summarizes the history and consequences of previous choices and the only new information for the current period are then reflected in the incentives the bank finds in the interest rates and the penalty structure for the period. It is this overlooked but binding process of financial balance, update and revision that induces the Markov stationary solution.

Against this backdrop of intertemporal optimization, the model looked further into the loan portfolio with better detail. The model is further extended to account for the possibility of loans with fixed terms beyond one planning period. This means that resources for these type of instruments will have to be pre-committed by the bank and therefore would not be free to be allocated over the period in which the multi-period loan is in effect. The results lead to an ambiguity as to whether optimal reserves would either increase or decrease. This was explained in terms of the competing influence of an income-commitment effect and an income-option effect. The former would reduce reserves because there multi-period loans provide multi-period revenues and thus more will be foregone with a decision to invest marginally with reserves. However, this brings about the option of arbitrage across periods which induces an increase in reserves. Every unit that is actually invested in reserves is a unit that can be freely allocated in the succeeding period. This lack of pre-commitment would then be priced at the margin at the average rate expected in the next period, lowering the current period's marginal cost for reserve holding. Subsequently, how the two competing effects balance on net depends exclusively on the magnitude of the rates, comparing the next-period flow from two-period loans versus the expected return from freely investing in loans next period.