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Instability in the Basic Endogenous Growth Model

by

*José Encarnación, Jr.**

*Professor Emeritus, School of Economics, University of the Philippines

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José Encarnación, Jr.
University of the Philippines

Abstract

It is shown that the competitive equilibrium path is unstable in the basic model of endogenous growth with increasing returns. An extension of the model to include physical capital accumulation separately is also unstable.

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I. Introduction

Romer (1986) started what has since been called "endogenous growth theory" (Hammond and Rodriguez-Clare (1993, p. 392); see also Helpman (1992) and Lucas (1993) for other surveys of this literature, and Solow (1994) for a historical perspective). That paper presented "the basic model" (Romer (1986, p. 1034) of endogenous growth with increasing returns to knowledge (or knowledge and physical capital in fixed proportions). The present paper shows that the competitive equilibrium path is unstable in that model and also in a natural extension of it that has a separate physical good.

II. The Basic Model

Let there be a large number N of competitive firms in the economy. The representative firm has a production function $F(k, K, x)$ where k is its firm-specific stock of knowledge, K is aggregate knowledge defined as $K = Nk$, and x denotes other inputs specific to the firm. It is assumed that F is concave and homogeneous of degree one in k and x given K, and convex in K. Suppressing the fixed x, write $f(k, K) = F(k, K, x)$. The representative consumer, who owns the representative firm, can increase the firm's k by investing in research:

$$\dot{k}/k = g((f - c)/k) \tag{2.1}$$

where c is consumption. It is assumed that $g' > 0$, $g'' < 0$, g is bounded from below by $g(0) = 0$ and from above by $\alpha =$ const, and a normalization puts $g'(0) = 1$ to fix units of k.

The consumer has a concave utility function and a rate of time preference $\delta > \alpha$, and he wishes to maximize $\int_0^\infty u(c(t))e^{-\delta t}dt$ subject to (2.1), given the path $K(t)$, $t \geq 0$, and $k(0) > 0$. The current-value Hamiltonian is $H = u(c) + \lambda kg$, and in order to solve the consumer's problem it is necessary that there be a path $\lambda(t)$, $t \geq 0$, satisfying

$$\dot{\lambda} = \delta\lambda - \lambda[g + (f_k - (f - c)/k)g'] \tag{2.2}$$

where f_k denotes the partial of f with respect to its first argument, and the transversality condition

$$\lim_{t\to\infty} \lambda(t)k(t)e^{-\delta t} = 0. \tag{2.3}$$

Letting $k^*(t)$, $t > 0$, denote a solution, a competitive equilibrium (or CE) path requires the given $K(t)$ to be such that $K(t) = Nk^*(t)$, $t \geq 0$. Romer (1986) gives sufficient conditions for a CE path where k and c grow without bound, and we will take it that those conditions are satisfied. Among those conditions, one puts $\bar{k}$ as that value of k such that $f_k > \delta$ for all $k > \bar{k}$, a fact to be used shortly.

In the phase-plane with k on the horizontal axis and λ on the vertical, the upper boundary of the region where $\dot{k} = 0$ can be called the $\dot{k} = 0$ curve. At a point on this curve, $g(0) = 0$ and $g'(0) = 1$, so (2.2) reduces to $\dot{\lambda} = \lambda(\delta - f_k) < 0$ if $f_k > \delta$, and therefore the $\dot{\lambda} = 0$ locus must lie above the $\dot{k} = 0$ curve for all $k > \bar{k}$. The CE trajectory may be upward sloping (see Fig. 1a), in which case $\dot{\lambda} > 0$ along the trajectory and hence the latter lies above the $\dot{\lambda} = 0$ locus, or downward sloping (Fig. 1b), in which case it lies below. Our aim is to show that the CE path is unstable.

Suppose in Fig. 1a that some adventitious event puts the current point (k, λ) slightly below the CE path. Writing $\theta = \dot{\lambda}/\lambda$, instability is shown if at the lower value of λ (and the same value of k), g/θ is higher, i.e. θ/g is lower. Since by

$$\partial H/\partial c = u'(c) - \lambda g' = 0 \qquad (2.4)$$

a lower λ implies a higher c or a higher g', and the latter implies a lower g hence a higher c, a lower λ is equivalently a higher c. Thus we need to show that

$$\partial(\theta/g)/\partial c < 0 \qquad \text{(IC)}$$

holds. If a disturbance puts (k, λ) slightly above the CE path in Fig. 1a, instability follows if at the higher λ (lower c), θ/g is higher, which is the same instability condition (IC). In Fig. 1b, suppose (k, λ) is slightly below the CE path. The latter is unstable if at the lower λ, the absolute value of θ/g (i.e. $-\theta/g$) is higher. This is also (IC). Finally, if (k, λ) is slightly above the CE path, instability is shown with $-\theta/g$ lower at the higher λ, which is (IC) again.

From (2.2),

$$\theta/g = \delta/g - 1 - (f_k - (f - c)/k)g'/g$$

and straightforward calculations give

$$\frac{\partial(\theta/g)}{\partial c} = \frac{\delta g' - gg' - (f_k - (f - c)/k)(g'g' - gg'')}{g^2 k}. \quad (2.5)$$

In the Fig. 1a case where $\dot{\lambda} > 0$, (2.2) implies $\delta g' - gg' - (f_k - (f - c)/k)g'g' > 0$. Thus (2.5) will be negative if $(f_k - (f - c)/k)$ is positive and sufficiently large. To see that this will be so, observe that the investment $f - c$ increases k by the amount $\dot{k}$ which increases f by the amount $f_k\dot{k} = f_k gk$. In effect there is an investment-output relationship such that $(f - c)/k = f_k g$. Now since f_k is increasing with k and K along the CE path but g (which has an upper bound $\alpha < \delta$) can increase only fractionally, f_k must increase more than $(f - c)/k$. This means that $f_k - (f - c)/k$ will increase with k, making (2.5) negative for k large enough. In Fig. 1b case where $\dot{\lambda} < 0$, (2.2) implies $\delta g' - gg' - (f_k - (f - c)/k)g'g' < 0$ and it is immediate that

(2.5) is negative. This confirms (IC), and we conclude that the CE path is unstable.

III. An Extension

A simple extension can be made to include the accumulation of physical capital or "machines" separately. In order to have a minimum of changes from the basic model, assume that machines are measured in the same units as the consumption good, and with $m(0) > 0$, let

$$h = rmf(k, K) \tag{3.1}$$

where r is the fraction of labor allocated for the production of either the consumption good or knowledge, m is the number of machines, and f now excludes machines among its suppressed arguments. Instead of (2.1) we now have:

$$\dot{k} = kg((h - c)/k) \tag{3.2}$$

$$\dot{m} = (1 - r)mf(k, K). \tag{3.3}$$

Writing $\overline{H} = u(c) + \lambda_1 kg + \lambda_2(1 - r)mf$,

$$\partial\overline{H}/\partial c = u'(c) - \lambda_1 g' = 0 \tag{3.4}$$

$$\partial\overline{H}/\partial r = \lambda_1 mfg' - \lambda_2 mf \geq 0 \tag{3.5a}$$

$$= 0 \quad \text{if} \quad r < 1. \tag{3.5b}$$

If $r < 1$,

$$\begin{aligned}\dot{\lambda}_1 &= \delta\lambda_1 - \lambda_1[g + (h_k - (h - c)/k)g'] - \lambda_2(1 - r)mf_k \\ &= \delta\lambda_1 - \lambda_1[g + (mf_k - (rmf - c)/k)g'] \end{aligned} \tag{3.6}$$

using (3.5b) to eliminate λ_2, and

$$\dot{\lambda}_2 = \delta\lambda_2 - \lambda_2(1 - r)f - \lambda_1 rfg'$$
$$= \lambda_2(\delta - f) \tag{3.7}$$

using (3.5b) again. There is a CE path if, in addition to (3.1) - (3.7),

$$\lim_{t\to\infty} \lambda_1(t)k(t)e^{-\delta t} = 0 \tag{3.8}$$

$$\lim_{t\to\infty} \lambda_2(t)m(t)e^{-\delta t} = 0. \tag{3.9}$$

Condition (3.8) is essentially the same as (2.3), and in view of (3.7), which says that λ_2 falls to zero eventually, (3.9) is satisfied. The sufficiency conditions for a CE path in the basic model thus give a CE path in this extension. Notice that if $r = 1$, the λ_2 term in the Hamiltonian vanishes and we have the basic model with a constant m. We will therefore suppose that $r < 1$ in looking at the question of stability in what follows.

Consider a point on the $\dot{k} = 0$ curve in the (k, λ_1) phase plane. Referring to (3.6), one has $\dot{\lambda}_1 = \lambda_1(\delta - mf_k) < 0$ for k large enough, so the $\dot{\lambda} = 0$ locus lies above the $\dot{k} = 0$ curve. The earlier discussion about the CE path in the (k, λ) plane can then be repeated almost word for word. Writing $\theta_1 = \dot{\lambda}_1/\lambda_1$, we show first that $\partial(\theta_1/g)/\partial c < 0$. From (3.6),

$$\theta_1/g = \delta/g - 1 - (mf_k - (rmf - c)/k)g'/g$$

$$\frac{\partial(\theta_1/g)}{\partial c} = \frac{\delta g' - gg' - (mf_k - (rmf - c)/k)(g'g' - gg'')}{g^2 k} \tag{3.10}$$

which, as in the earlier discussion, is negative for k large enough. (Note that (3.10)) differs from (2.5) as regards the expression $(mf_k - (rmf - c)/k)$ only. Investing $rmf - c$ gives $\dot{k}$ which increases h by the amount $h_k\dot{k} = rmf_k\dot{k}$. Also, $\dot{k}$ increases the output of machines, $\dot{m}$, by the amount $(1 - r)mf_k\dot{k}$. The increases added up to $mf_k\dot{k}$, hence the relationship $(rmf - c)/k = mf_k g$.) Thus the CE path is unstable in the (k, λ_1) phase plane, which suffices to show instability unless (i) the CE path is stable in the (m, λ_2) phase plane and (ii) it is strongly stable in the sense that its stability counters the instability as regards (k, λ_1).

To examine (i) we observe that because of (3.7), the CE path in the (m, λ_2) plane is downward sloping and therefore the $\dot{\lambda}_2 = 0$ locus lies above the CE path. The earlier discussion of Fig. 1b can then be repeated since (3.4) and (3.5b) make a lower λ_2 equivalently a higher c. In short, (i) holds only if, writing $\theta_2 = \dot{\lambda}_2/\lambda_2$ and $q = \dot{m}/m$, $\partial(\theta_2/q)/\partial c > 0$. But from (3.7) and (3.3),

$$\theta_2/q = (\delta - f)/(1 - r)f$$
$$\partial(\theta_2/q)/\partial c = 0.$$

Thus (i) fails to hold, and we conclude that the CE path is unstable in this extension of the basic model.

IV. Concluding Remark

We have seen that the competitive equilibrium path is unstable in the basic model of endogenous growth and also in an extension of it that includes physical capital accumulation separately. It might be asked whether instability defeats the model as positive theory, for which it was intended. The matter does not seem to be entirely clear, since it could be argued that actual economies that have been growing steadily in the past can take a downward turn. Instability might then be considered a plus for the model.

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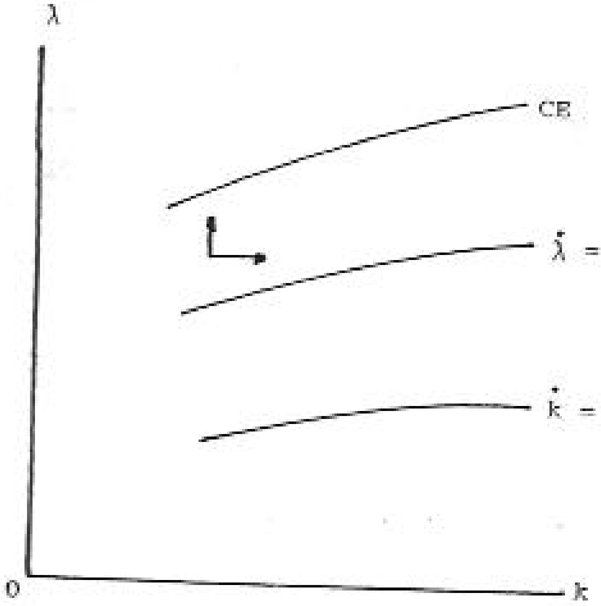

Fig. 1a

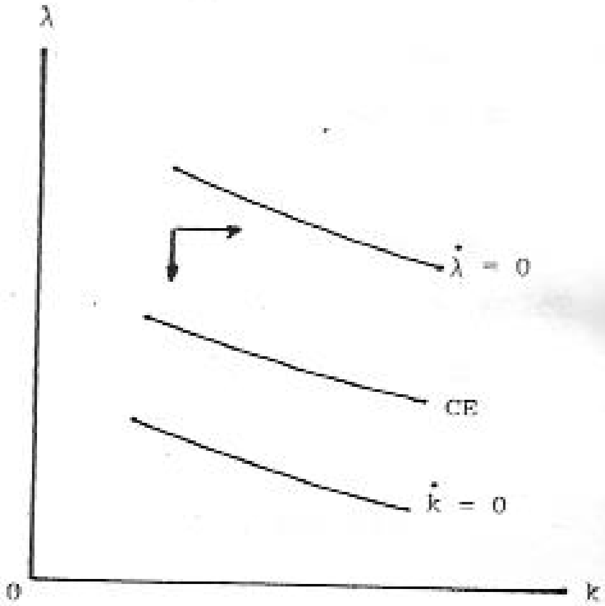

Fig. 1b