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A Note on Isotone Solutions of the Parametric Linear Complementarity Problem

by

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Abstract. This paper shows that the parametric linear complementarity problem $w = Mz + q + \alpha p$, $w \geq 0$, $z \geq 0$, $w^T z = 0$, $\alpha \geq 0$ has isotone complementary solutions for $q = 0$ and every p iff M is a P-matrix. Thus, isotonicity for every $q \geq 0$ and every p reduces to monotonicity where M is a P-matrix. By excluding $q = 0$, it is shown that isotonicity is possible for every $0 \neq q \geq 0$ and every p where M is not a P-matrix.

Keywords. Parametric linear complementarity problem; isotone complementary solutions; matrices

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1. Introduction

For a given matrix $M \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem LCP(q,M) is that of finding $w, z \in \mathbb{R}^n$ such that

$$w = Mz + q, \quad w \geq 0, \quad z \geq 0, \quad w^T z = 0. \tag{1}$$

A pair (w;z) that satisfies (1) is called a complementary solution and the set of complementary solutions of the LCP(q,M) is denoted by C(q,M). The set of all $q \in \mathbb{R}^n$ for which the LCP(q,M) has a complementary solution is denoted by K(M).

For a given matrix $M \in \mathbb{R}^{n\times n}$ and vectors $q, p \in \mathbb{R}^n$, the parametric linear complementarity problem PLCP($q+\alpha p$,M) consists of the family of linear complementarity problems $\{LCP(q+\alpha p,M) \mid \alpha \geq 0\}$, where the parameter $\alpha \in \mathbb{R}$. The PLCP arose in the analysis of elastoplastic structures (Maier [7]) and has also found applications in other areas such as the computation of economic equilibria (Benveniste [1]; Pang and Lee [11]), portfolio selection (Pang [9]), and actuarial graduations (Pang, Kaneko, and Hallman [10]).

As proposed by Maier [7], the PLCP($q+\alpha p$,M) assumes $q > 0$ and is concerned with determining conditions under which the z-component of the complementary solution ($w(\alpha);z(\alpha)$) of the LCP($q+\alpha p$,M) has coordinates that are monotone nondecreasing with respect to α. When every coordinate of $z(\alpha)$ is monotone nondecreasing, the vector function $z(\alpha)$ is also said to be monotone nondecreasing.

When M is a P-matrix (i.e., M has positive principal minors), the LCP(q,M) has a unique complementary solution for each $q \in \mathbb{R}^n$ (Murty [8]). Thus, the monotonicity of $z(\alpha) = z(\alpha;q,p)$ is well-defined since $z(\alpha)$ is a point-to-point mapping. Under the assumption that M is a P-matrix and $q \geq 0$, Cottle [2] proved the following theorem:

Theorem 1.1. (Cottle [2]) Given the PLCP($q+\alpha p$,M) where M is a P-matrix. Then $z(\alpha) \equiv z(\alpha;q,p)$ is monotone nondecreasing for every $q \geq 0$ and every p iff M is a Minkowski matrix (i.e., a P-matrix with nonpositive off-diagonal entries).

In view of the importance of the uniqueness of complementary solutions for the monotonicity of $z(\alpha)$ to be well-defined the question arises: Are there matrices M other than the P-matrices for which the LCP(q,M) has a unique complementary solution for every

$q \in K(M)$? The answer is that there are none. For if the LCP(q,M) has a unique complementary solution for every $q \in K(M)$, then it has a unique complementary solution for every $q \geq 0$ since the nonnegative orthant of $\mathbb{R}^n$ is always a subset of $K(M)$. Consequently, M is an L_*-matrix (Eaves [5]) which implies that M is a Q-matrix or, equivalently, $K(M) = \mathbb{R}^n$ (Eaves [5]; Cottle and Dantzig [3]); hence, M is a P-matrix.

When M is not a P-matrix, the LCP($q+\alpha p$,M) may not have a complementary solution and when it has, the complementary solution may not be unique. Thus $z(\alpha)$ becomes a point-to-set mapping. In this case, Kaneko [6] proposed a more general definition of monotonicity. Let

$$T = \{\alpha \geq 0 \mid C(q+\alpha p, M) \neq \phi\}$$

and define the functions

$$z: T \to \mathbb{R}^n,$$

where $z(\alpha)$ is the z-component of an element of $C(q+\alpha p, M)$. Following Kaneko [6] we refer to these functions as complementary maps and adopt his generalized definition of monotonicity.

Definition 1.1. The PLCP($q+\alpha p$,M) is said to have an isotone complementary map iff there exists a complementary map $z(\alpha)$ such that $z_j(\alpha)$ is monotone nondecreasing with respect to $\alpha \in T$ for each $j = 1,2,\ldots,n$.

Definition 1.2. The PLCP($q+\alpha p$,M) is said to have isotone complementary solutions iff every complementary map $z(\alpha)$ is isotone with respect to $\alpha \in T$.

Remark 1.1. When M is a P-matrix, isotonicity coincides with monotonicity since there is only one complementary map.

Under the assumption that M is a Z-matrix (i.e., M has nonpositive off-diagonal entries), Kaneko [6] proved the following theorem:

Theorem 1.2 (Kaneko [6]) Let M be a Z-matrix. The PLCP($q+\alpha p$,M) has isotone complementary solutions for every $q \geq 0$ and every p iff M is a Minkowski matrix.

From Theorems 1.1 and 1.2, we see that isotonicity for every $q \geq 0$ and every p when M is a Z-matrix reduces to monotonicity for every $q \geq 0$ and every p with M being a P-matrix. This paper drops the assumption that M is a Z-matrix and proves that, at $q = 0$, the PLCP($0+\alpha p$,M) has isotone complementary solutions for every p iff M is a P-matrix. It follows that a necessary condition for isotonicity for every $q \geq 0$ and every p is that M be a P-matrix. (This is the necessary condition in Theorem 1.2 which was proved for Z-matrices in [6]). Thus the PLCP reduces to one with a P-matrix M as in Theorem 1.1. However, by excluding $q = 0$, it is possible to have a PLCP($q+\alpha p$,M) with isotone complementary solutions for every $0 \neq q \geq 0$ and every p where M is not a P-matrix. An example is presented in Section 4.

2. Further Definitions, Notations, and Previous Results

The cone generated by the columns of a matrix A is denoted by Pos[A], i.e., $\text{Pos}[A] = \{Ax \mid x \geq 0\}$. The j^{th} column of A is denoted by A_j. If A is an $n \times n$ matrix and if for each $j = 1,2,\ldots,n$, A_j is either I_j (the j^{th} column of the identity matrix I) or $-M_j$ (the j^{th} column of $-M$), then Pos[A] is called a complementary cone. The LCP(q,M) has a complementary solution iff q belongs to some complementary cone. Thus, K(M) is the union of all complementary cones. A complementary cone whose interior is nonempty is said to be nondegenerate; otherwise, it is said to be degenerate. The interior of Pos[A] is denoted by int(Pos[A]). If Pos[A] is an m-dimensional degenerate complementary cone in $\mathbb{R}^n$, then its relative interior is its interior in $\mathbb{R}^m$ and is denoted by relint(Pos[A]). The set of complementary cones forms a partition of $\mathbb{R}^n$ iff their union is $\mathbb{R}^n$ and they have nonempty interiors which are pairwise disjoint.

For a point q of a complementary cone Pos[A] we set

$$X(q,A) = \{x \mid Ax = q,\ x \geq 0\}.$$

For each $x \in X(q,A)$, the complementary solution of the LCP(q,M) obtained by setting the variables in (w;z) associated with A_j equal to x_j and the rest equal to zero is said to be induced by Pos[A].

The following theorems will be used to prove the main result.

Theorem 2.1. (Cottle and Stone [4]) Let Pos[A] be a degenerate complementary cone. For every q in relint(Pos[A]), the number of complementary solutions of the LCP(q,M) induced by Pos[A] is infinite.

Theorem 2.2 (Eaves [5]) M is an L_*-matrix iff the LCP(q,M) has a unique complementary solution for every $q \geq 0$.

Theorem 2.3. (Cottle and Dantzig [3]; Eaves [5]) If M is an L_*-matrix, then $K(M) = \mathbf{R}^n$.

Theorem 2.4. (Murty [8]; Samelson, Thrall & Wesler [12]) The set of complementary cones forms a partition of $\mathbf{R}^n$ iff M is a P-matrix.

Theorem 2.5. (Cottle [2]) Given the PLCP($q+\alpha p$,M) where M is a P-matrix and $q \geq 0$. Then $z(\alpha) = z(\alpha;q,p)$ is monotone nondecreasing for every p iff $(M^*)^{-1}q^* \geq 0$ for every principal submatrix M^* of M and corresponding subvector q^*.

3. The Main Result

The proof of the next lemma is straightforward:

Lemma 3.1. If $(w;z)$ is a complementary solution of the LCP(p,M), then $(\alpha w;\alpha z)$ is a complementary solution of the LCP(αp,M) for every $\alpha \geq 0$.

Lemma 3.2. Let p be a point in the relative interior of a degenerate complementary cone Pos[A]. Then the PLCP(0+αp,M) has a complementary map that is not isotone.

Proof: Let $0 < \alpha_1 < \alpha_2$. Since $p \in$ relint(Pos[A]), then $\alpha_1 p \in$ relint(Pos[A]). By Theorem 2.1, the number of complementary solutions of the LCP(α_1p,M) induced by Pos[A] is infinite. Let $(w(\alpha_1);z(\alpha_1))$ and $(w'(\alpha_1);z'(\alpha_1))$ be distinct complementary solutions of the LCP(α_1p,M) induced by Pos[A]. Then $z(\alpha_1) \neq z'(\alpha_1)$. Hence, there is an index k such that $z_k(\alpha_1) \neq z_k'(\alpha_1)$, say

$$z_k(\alpha_1) > z_k'(\alpha_1). \tag{2}$$

Let $x_j(\alpha_1)$ and $x_j'(\alpha_1)$ denote the variables in $(w(\alpha_1);z(\alpha_1))$ and $(w'(\alpha_1);z'(\alpha_1))$, respectively, associated with A_j. Then we have

$$Ax'(\alpha_1) = \alpha_1 p. \tag{3}$$

From (2) we have

$$x_k(\alpha_1) \equiv z_k(\alpha_1) > z_k'(\alpha_1) \equiv x_k'(\alpha_1). \qquad (4)$$

Let $(w(\alpha_2);z(\alpha_2))$ be a complementary solution of the LCP$(\alpha_2 p,M)$ induced by Pos[A] and let $x_j(\alpha_2)$ denote the variable in $(w(\alpha_2);z(\alpha_2))$ associated with A_j. Then

$$Ax(\alpha_2) = \alpha_2 p. \qquad (5)$$

Define α^* and $x(\alpha^*)$ by

$$\alpha^* = (1-\lambda)\alpha_1 + \lambda\alpha_2, \qquad 0 < \lambda < 1,$$

$$x(\alpha^*) = (1-\lambda)x'(\alpha_1) + \lambda x(\alpha_2). \qquad (6)$$

Then $\alpha_1 < \alpha^* < \alpha_2$, $x(\alpha^*) \geq 0$, and

$$\begin{aligned} Ax(\alpha^*) &= (1-\lambda)Ax'(\alpha_1) + \lambda Ax(\alpha_2) \\ &= (1-\lambda)\alpha_1 p + \lambda\alpha_2 p, \quad \text{from (3) and (5)}, \\ &= \alpha^* p. \end{aligned}$$

Hence, there is a complementary solution $(w(\alpha^*);z(\alpha^*))$ of the LCP$(\alpha^* p,M)$ induced by Pos[A] that is associated with $x(\alpha^*)$. Now, from (6),

$$\begin{aligned} x_k(\alpha^*) &= (1-\lambda)x_k'(\alpha_1) + \lambda x_k(\alpha_2) \\ &= x_k'(\alpha_1) + \lambda[x_k(\alpha_2) - x_k'(\alpha_1)]. \end{aligned}$$

Since $x_k(\alpha_1) > x_k'(\alpha_1)$, we can choose λ small enough such that

$$x_k(\alpha_1) > x_k'(\alpha_1) + \lambda[x_k(\alpha_2) - x_k'(\alpha_1)] = x_k(\alpha^*).$$

Thus, $z_k(\alpha_1) \equiv x_k(\alpha_1) > x_k(\alpha^*) \equiv z_k(\alpha^*)$

and this defines a complementary map that is not isotone. □

Lemma 3.3. If the PLCP(0+αp,M) has isotone complementary solutions for every p, then every complementary cone is nondegenerate.

Proof: Case 1. n = 1. If there is a degenerate complementary cone, then $M = M_{11} = 0$. Let $p > 0$. For $\alpha = 0$, $(0;z)$ is a complementary solution of the LCP(0p,M), where $z > 0$. For $\alpha = 1$, $(p;0)$ is a complementary solution of the LCP(1p,M), contrary to isotonicity.

Case 2. $n > 1$. If there is a degenerate complementary cone, then there is at least one with a nonempty relative interior. To prove this, we note that every degenerate complementary cone must have at least one column from $-M$ since Pos[I] is nondegenerate. If $-M$ has no zero column, then every degenerate complementary cone must have dimension $m \geq 1$; hence, its relative interior is nonempty. If $-M$ has a zero column, say $-M_j = 0$, then $\mathrm{Pos}[I_1,\ldots I_{j-1},-M_j,I_{j+1},\ldots I_n]$ is degenerate and of dimension n-1 and has a nonempty relative interior. By Lemma 3.2, there is a complementary map that is not isotone contrary to the hypothesis. □

Lemma 3.4. If the PLCP($0+\alpha p$,M) has isotone complementary solutions for every p, then M is an L_*-matrix.

Proof. Let $p \geq 0$. We show that the LCP(p,M) has a unique complementary solution. Since $p \geq 0$, then (p;0) is a complementary solution of the LCP(p,M). If the LCP(p,M) has another complementary solution (w;z), then $0 \neq z \geq 0$. For $\alpha > 1$, $\alpha p \geq 0$; hence, (αp;0) is a complementary solution of the LCP(αp,M) contrary to isotonicity. Thus, the LCP(p,M) has a unique complementary solution for each $p \geq 0$. By Theorem 2.2, M is an L_*-matrix. □

Theorem 3.1. The PLCP($0+\alpha p$,M) has isotone complementary solutions for every p iff M is a P-matrix.

Proof. (⇒) By Lemma 3.4, M is an L_*-matrix and so, by Theorem 2.3, $K(M) = \mathbb{R}^n$. Moreover, by Lemma 3.3, all the complementary cones are nondegenerate. We show that the complementary cones have pairwise disjoint interiors. Suppose not. Let Pos[A] and Pos[B] be distinct complementary cones whose interiors have a nonempty intersection and let

$$p \in \text{int}(\text{Pos}[A]) \cap \text{int}(\text{Pos}[B]).$$

Since Pos[A] is not identical to Pos[B], then there is a j such that $A_j \neq B_j$, say

$$A_j = -M_{j\cdot} \text{ and } B_j = I_j.$$

Let $(w^A(1);z^A(1))$ and $(w^B(1);z^B(1))$ denote the complementary solutions of the LCP(p,M) induced by Pos[A] and Pos[B], respectively. Since p is interior to both pos[A] and pos[B], we must have

$$z_j^A(1) > 0, \tag{7}$$

$$z_j^B(1) = 0 \quad (\text{since } w_j^B(1) > 0). \tag{8}$$

Let $\alpha > 1$ and let $(w^A(\alpha);z^A(\alpha))$ and $(w^B(\alpha);z^B(\alpha))$ be the complementary solutions induced by Pos[A] and Pos[B], respectively, of the LCP(αp,M). Then, by Lemma 3.1,

$$z_j^A(\alpha) = \alpha z_j^A(1) > 0 \tag{9}$$

$$z_j^B(\alpha) = \alpha z_j^B(1) = 0. \tag{10}$$

Conditions (7) and (10) violate isotonicity. Thus, the complementary cones form a partition of $\mathbb{R}^n$. By Theorem 2.4, M is a P-matrix.

($\Leftarrow$) If M is a P-matrix, then isotonicity coincides with monotonicity and the conclusion follows from Theorem 2.5 with q = 0. □

4. Conclusion

If the PLCP(q+αp,M) has isotone complementary solutions for every $q \geq 0$ and every p, then, in particular, it must have isotone complementary solutions for $q = 0$ and every p; hence, Theorem 3.1 shows that M must be a P-matrix. Thus, the PLCP is reduced to Cottle's [2] PLCP with a P-matrix M. As we have shown, the crucial point is isotonicity for $q = 0$ and every p. To look for other possibilities, we have to exclude $q = 0$. For example, consider the following matrix in [2]:

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The isotonicity property for every $0 \neq q \geq 0$ and every p can be seen from the complementary cones shown in Fig. 1.

Fig. 1

For example, let $q = [2\ 1]^T$ and $p = [-1\ -1]^T$. In this case, we have $T = [0, 3/2]$. For $0 \leq \alpha \leq 1$, the complementary solution is unique for each α and

$z_1(\alpha) = 0, \quad z_2(\alpha) = 0.$

For $1 \leq \alpha < 3/2$, the complementary solution is unique for each α and

$z_1(\alpha) = 0, \quad z_2(\alpha)$ increases with α; $z_2(\alpha) \to 1/2$ as $\alpha \to 3/2$.

For $\alpha = 3/2$, there are an infinite number of complementary solutions:

$$z_1(\alpha) = [0,\infty), \quad z_2(\alpha) = [1/2,\infty).$$

The graphs of $z_1(\alpha)$ and $z_2(\alpha)$ are shown in Fig. 2 which clearly shows the isotonicity of every complementary map.

Fig. 2

We note that M is not a P-matrix. Note, however, that M is a Z-matrix. Thus, Z-matrices may still be important in the PLCP($q+\alpha p$,M) where $0 \neq q \geq 0$.

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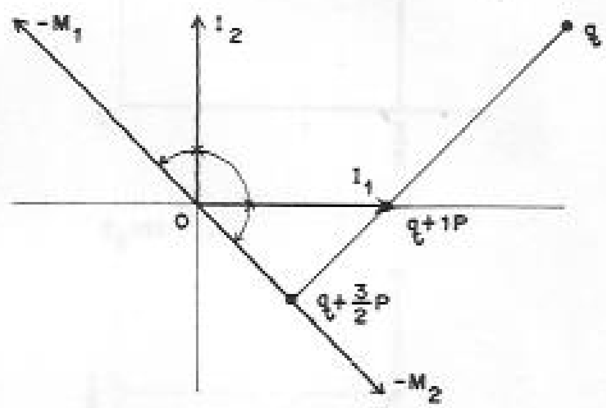
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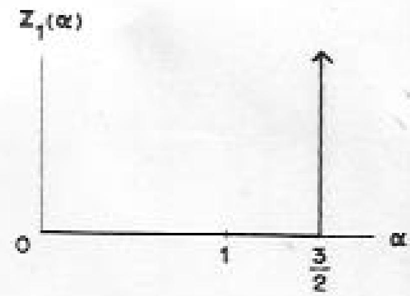
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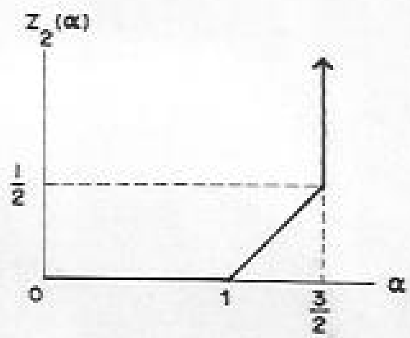
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3/2
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