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On Q-Matrices and the Boundedness of Solutions to Linear Complementarity Problems

by

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Abstract. This paper is concerned with the existence and boundedness of the solutions to the linear complementarity problem $w = Mz + q$, $w \geq 0$, $z \geq 0$, $w^T z = 0$, for each $q \in R^n$. It has been previously established that if M is copositive plus, then the solution set is nonempty and bounded for each $q \in R^n$ iff M is a Q-matrix. This result is shown to be valid also for L_2-matrices, P_0-matrices, nonnegative matrices and Z-matrices.

Key Words. Linear complementarity problem, matrices, bounded solutions, Q-matrices.

1. Introduction

For a given matrix $M \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem LCP(q,M) is that of finding $w, z \in \mathbb{R}^n$ such that

$$w = Mz + q, \quad w \geq 0, \quad z \geq 0, \quad w^T z = 0. \qquad (1)$$

A pair (w;z) that satisfies (1) is called a complementary solution and the set of complementary solutions of the LCP(q,M) is denoted by C(q,M). The set of all $q \in \mathbb{R}^n$ for which the LCP(q,M) has a complementary solution is denoted by K(M). When $K(M) = \mathbb{R}^n$, M is called a Q-matrix (or $M \in Q$).

The number of complementary solutions has been the subject of many investigations. Murty (Ref. 1) showed that the number of complementary solutions of the LCP(q,M) is finite for each $q \in \mathbb{R}^n$ iff M is a nondegenerate matrix (i.e., M has no zero principal minor). Hence, when M is degenerate, there exists a vector $q \in \mathbb{R}^n$ such that the LCP(q,M) has infinitely many complementary solutions. Murty (Ref. 1) showed that the set of such vectors is contained in the union of the degenerate complementary cones (i.e., cones with empty interiors). In fact we show that if a degenerate complementary cone is strictly pointed, then C(q,M) is infinite for each q in the relative interior of the cone but C(q,M) may be infinite or a singleton on the relative boundary of the cone; if the complementary cone is not strictly pointed, then C(q,M) is infinite and unbounded for each q in the cone.

The interest in the boundedness of C(q,M) stems from its implication on the stability of the LCP(q,M) with respect to perturbations to q and M (Ref. 2). A necessary and sufficient condition for C(q,M) to be bounded is that every complementary cone containing q is strictly pointed (Refs. 2 and 3). Thus C(q,M) is bounded for each $q \in \mathbb{R}^n$ iff all the complementary cones are strictly pointed. This is equivalent to the condition that the LCP(0,M) has a unique complementary solution, i.e., M is an R_0-matrix.

Mangasarian (Ref. 4) considered the problem of the nonemptiness and boundedness of C(q,M) for each $q \in \mathbb{R}^n$ and showed that in the class of copositive plus matrices, C(q,M) is nonempty and bounded for each $q \in \mathbb{R}^n$ iff $M \in Q$. This result is shown to be valid also for L_2-matrices (which contains the L-matrices and the copositive plus matrices), P_0-matrices, nonnegative matrices, and Z-matrices.

2. Notations and Further Definitions.

$\mathbb{R}^n$ denotes the n-dimensional real Euclidean space with the usual topology. $\mathbb{R}^{n\times n}$ denotes the class of n×n matrices with real entries. The ith row of a matrix A is denoted by $A_{i\cdot}$ and the jth column is denoted by $A_{\cdot j}$. The entry in the ith row and the jth column of A is denoted by A_{ij}. The identity matrix is denoted by I. For a given matrix A, the cone generated by the columns of A is denoted by Pos[A], i.e., Pos[A] = $\{Ax | x \geq 0\}$. The ray generated by a vector v is denoted by Pos[v].

For convenience, we list down the definitions of the classes of matrices used in this paper. The matrix M in each definition belongs to $R^{n \times n}$. If Y is a class of matrices and M ϵ Y, then M is called a Y-matrix.

Definition 2.1. M ϵ R iff the system

$$(Mz)_i + t = 0, \quad z_i > 0 \tag{2}$$

$$(Mz)_i + t \geq 0, \quad z_i = 0 \tag{3}$$

$$0 \neq z \geq 0, \quad t \geq 0. \tag{4}$$

has no solution. M ϵ R_0 iff the system (2)-(4) has no solution for t = 0. Thus, R $\subseteq$ R_0.

Definition 2.2. M ϵ $E^*(0)$ iff the LCP(0,M) has a unique complementary solution.

Definition 2.3. $M \in Q$ iff the LCP(q,M) has a complementary solution for each $q \in R^n$.

Definition 2.4. $M \in L_1$ (or M is semimonotone) iff for every $0 \neq z \geq 0$, there exists an index j such that $z_j > 0$ and $(Mz)_j \geq 0$.

Definition 2.5. $M \in L_*$ (or M is strictly semimonotone) iff for every $0 \neq z \geq 0$, there exists an index j such that $z_j > 0$ and $(Mz)_j > 0$.

Definition 2.6. $M \in L_2$ iff for every $0 \neq z \geq 0$ with $Mz \geq 0$ and $z^T Mz = 0$, there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2 z \neq 0$ and $(D_1 M + M^T D_2)z = 0$.

Definition 2.7. $M \in L$ iff $M \in L_1 \cap L_2$.

Definition 2.8. $M \in CP^+$ (or M is copositive plus) iff (i) $z \geq 0$ implies $z^TMz \geq 0$ and (ii) $z \geq 0$, $z^TMz = 0$ imply $(M + M^T)z = 0$.

Definition 2.9. $M \in P_0$ (P, N) iff all its principal minors are nonnegative (positive, negative).

Definition 2.10. $M \in S$ iff there exists a $0 \neq z \geq 0$ such that $Mz > 0$.

Definition 2.11. $M \in Z$ iff $M_{ij} \leq 0$ for $i \neq j$.

3. Boundedness of Complementary Solutions Induced by Degenerate Complementary Cones

Consider the convex polyhedral cone Pos[A] generated by the columns of a matrix A. Let

$$X(q,A) = \{x \mid Ax = q,\ x \geq 0,\ q \in Pos[A]\}$$

and $U(0,A) = \{u \mid Au = 0,\ 0 \neq u \geq 0\}$.

The following theorem gives a necessary and sufficient condition for $X(q,A)$ to be bounded.

Theorem 3.1. (Ref. 5) $X(q,A)$ is bounded iff $U(0,A) = \phi$.

Definition 3.1. A convex polyhedral cone $Pos[A]$ is said to be pointed iff the only subspace contained in it is the subspace consisting of the zero vector. It is strictly pointed iff $U(0,A) = \phi$. (In Ref. 2, the matrix A is called strictly pointed).

Remark 3.1. A strictly pointed cone is pointed but not conversely. For example, let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Pos[A] is the ray generated by $I_{\cdot 2}$ which is clearly pointed but Pos[A] is not strictly pointed since $U(0,A)$ contains the vector $u = [1\ 0]^T$.

Definition 3.2. A complementary cone is a cone Pos[A] where $A \in \mathbb{R}^{n \times n}$ and $A_{\cdot j} \in \{I_{\cdot j}, -M_{\cdot j}\}$ for each $j=1,2,\ldots,n$. A complementary cone is said to be nondegenerate iff its interior is nonempty; otherwise, it is said to be degenerate.

Definition 3.3. Let Pos[A] be a complementary cone and let $q \in$ Pos[A]. For each $x \in X(q,A)$, the complementary solution obtained by setting the variables in $(w;z)$ associated with $A_{\cdot j}$ equal to x_j and the rest equal to zero is called a complementary solution induced by Pos[A]. The set of all complementary solutions of the LCP(q,M) induced by Pos[A] is denoted by $C_A(q,M)$.

Remark 3.2. Since $X(q,A)$ is convex, then so is $C_A(q,M)$ for each q in the complementary cone $\mathrm{Pos}[A]$. Thus, $C_A(q,M)$ has either one or infinitely many elements.

Since $C_A(q,M)$ is bounded iff $X(q,A)$ is bounded, it follows that $C_A(q,M)$ is bounded iff $\mathrm{Pos}[A]$ is strictly pointed. We state this as a corollary to Theorem 3.1.

Corollary 3.1. Let $\mathrm{Pos}[A]$ be a complementary cone and let $q \in \mathrm{Pos}[A]$. Then $C_A(q,M)$ is bounded iff $\mathrm{Pos}[A]$ is strictly pointed.

$C(q,M)$ is the union of all the $C_A(q,M)$ such that $q \in \mathrm{Pos}[A]$. Thus, $C(q,M)$ is bounded iff $C_A(q,M)$ is bounded for each complementary cone $\mathrm{Pos}[A]$ containing q, i.e., iff every complementary cone containing q is strictly pointed. We thus have the following theorem (see also Refs. 2 and 3):

Theorem 3.2. $C(q,M)$ is bounded iff every complementary cone containing q is strictly pointed.

Corollary 3.1 implies that if Pos[A] is not strictly pointed, then $C_A(q,M)$ is unbounded (therefore, infinite) for each $q \in$ Pos[A]. If Pos[A] is strictly pointed and nondegenerate, then $C_A(q,M)$ is a singleton for each $q \in$ Pos[A]. If Pos[A] is strictly pointed and degenerate, then we show that $C_A(q,M)$ is infinite for each q in the relative interior of Pos[A]; on the relative boundary of Pos[A] it is possible for $C_A(q,M)$ to be infinite or a singleton.

Definition 3.4. Let C be an m-dimensional cone in $\mathbb{R}^n$ where $m < n$. The relative interior of C, denoted by relint(C), is the interior of C in the relative topology of $\mathbb{R}^m$.

Definition 3.5. A frame of a convex polyhedral cone is a finite set of rays which generate the cone such that no ray in the set is in the convex hull of the others.

Theorem 3.3. (Ref. 6) A pointed convex polyhedral cone is generated by its extreme rays. Its frame is unique and consists of the extreme rays of the cone.

Theorem 3.4. (Ref. 6) If C is a convex polyhedral cone and {Pos[v_1], ..., Pos[v_m]} is a frame of C, then

$$relint(C) = \left\{\sum_{j=1}^{m} \gamma_j v_j \,\middle|\, \gamma_j > 0,\ j=,1,2,\ldots,m\right\}.$$

Theorem 3.5. Let Pos[A] be a degenerate complementary cone.

(i) If Pos[A] is strictly pointed, then $C_A(q,M)$ is infinite for each $q \in$ relint(Pos[A]).

(ii) If Pos[A] is not strictly pointed, then $C_A(q,M)$ is infinite for each $q \in$ Pos[A].

Proof: (i) Let $q \in \text{relint}(\text{Pos}[A])$. Since Pos[A] is strictly pointed, then it is pointed and, by Theorem 3.3, must have a unique frame consisting of its extreme rays. For convenience, let the frame of Pos[A] be $\{\text{Pos}[A_{\cdot 1}], \ldots, \text{Pos}[A_{\cdot m}]\}$, where $m < n$. By Theorem 3.4,

$$q = \sum_{j=1}^{m} x_j A_{\cdot j}, \quad x_j > 0 \quad (j=1,\ldots,m).$$

Define
$$q^0 = \sum_{j=m+1}^{n} A_{\cdot j}.$$

We note that $q^0 \neq 0$ for, otherwise, the vector u defined by
$$u_j = \begin{cases} 0, & j=1,\ldots,m \\ 1, & j=m+1,\ldots,n. \end{cases}$$

would be an element of U(0,A). Since $q^0 \in \text{Pos}[A]$, then

$$q^0 = \sum_{j=1}^{m} \gamma_j A_{\cdot j}, \quad \gamma_j \geq 0, \quad j=1,\ldots,m.$$

Define
$$\theta_k = x_k/\gamma_k = \min\{x_j/\gamma_j \mid \gamma_j > 0\}.$$

Since $q^0 \neq 0$, there exists a $j \in \{1,\ldots,m\}$ such that $\gamma_j > 0$. Hence, θ_k is well-defined. Moreover, $\theta_k > 0$. For $\theta \in [0,\theta_k]$, define $x(\theta)$ by

$$x_j(\theta) = \begin{cases} x_j - \theta\gamma_j, & j=1,\ldots,m \\ \theta, & j=m+1,\ldots,n. \end{cases}$$

Then $x(\theta) \geq 0$ and

$$Ax(\theta) = \sum_{j=1}^{m}(x_j - \theta\gamma_j)A_{\cdot j} + \sum_{j=m+1}^{n}\theta A_{\cdot j}$$

$$= \sum_{j=1}^{m}x_jA_{\cdot j} - \theta\sum_{j=1}^{m}\gamma_jA_{\cdot j} + \theta\sum_{j=m+1}^{n}A_{\cdot j}$$

$$= q.$$

Hence, $x(\theta) \in X(q,A)$ for each $\theta \in [0,\theta_k]$; therefore, $X(q,A)$ is infinite and so is $C_k(q,M)$.

(ii) This follows from Corollary 3.1. □

Remark 3.3. The following example shows that on the relative boundary of a strictly pointed degenerate complementary cone, $C(q,M)$ may be infinite or a singleton.

Example 3.1.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad q^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}, \qquad q^2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

The points q^1 and q^2 are on the relative boundary of Pos[A] (*Fig. 1*).

Fig. 1

The only solution of the system

$$Ax = q^1, \; x \geq 0$$

is $x = [1\ 0\ 0]^T$. Thus, $C_A(q^1,M)$ is a singleton. In fact, it is easy to check that $C(q^1,M)$ is a singleton. On the other hand, the system

$$Ax = q^2, \; x \geq 0$$

is satisfied by $x = [0\ \lambda\ 1-\lambda]^T$ where $\lambda \in [0,1]$. Each choice of λ induces a complementary solution of the LCP(q^2,M). Hence, $C_A(q^2,M)$ is infinite and so is $C(q^2,M)$.

4. The Q-Matrix Property and the Boundedness of Complementary Solutions

This section examines the nonemptiness and the boundedness of C(q,M) for each $q \in \mathbb{R}^n$. We begin with the boundedness of C(q,M) for each $q \in \mathbb{R}^n$ and its equivalent conditions. The equivalences in the following theorem are straightforward.

Theorem 4.1. The following statements are equivalent:

(i) C(q,M) is bounded for each $q \in \mathbb{R}^n$.

(ii) All the complementary cones are strictly pointed.

(iii) $M \in E^*(0)$.

(iv) $M \in R_0$.

Remark 4.1. In view of Theorem 4.1, the nonemptiness and boundedness of C(q,M) for each $q \in \mathbb{R}^n$ require that $M \in Q \cap R_0$. The R-matrices satisfy this condition since

$R \subseteq R_0$ and $R \subseteq Q$ (Ref. 7). We note that the R-matrices include the strictly semimonotone matrices L_* (Ref. 7) which, in turn, include the P-matrices (Ref. 8). The R-matrices, however, do not exhaust the Q-matrices in R_0. For example,

$$M = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

is a Q-matrix in R_0 but $M \notin R$ since the system (2)-(4) has a solution $z = [1\ 0]^T$, $t = 1$. The Q-nature of some R_0-matrices like M may be determined by using the following theorem (Ref. 9): If $M \in R_0$ and there exists a vector q nondegenerate with respect to M such that the LCP(q,M) has an odd number of complementary solutions, then $M \in Q$. (A vector $q \in \mathbb{R}^n$ is said to be nondegenerate with respect to M iff it does not lie in any subspace generated by (n-1) or less column vectors of $[I,-M]$.)

There is a simpler way of determining the Q-nature of the above matrix M. We note that M belongs to the class of N-matrices introduced by Saigal (Ref. 10). For this class of R_0-matrices, Kojima and Saigal (Ref. 11) showed that if the entries of M are not all negative, then $M \in Q$.

We now consider some classes of square matrices that are not contained in R_0. We begin with the copositive plus matrices and a theorem established by Mangasarian (Ref. 4).

Theorem 4.2. (Ref. 4) Let M be copositive plus. Then C(q,M) is nonempty and bounded for each $q \in R^n$ iff $M \in Q$.

Remark 4.2. Pang (Ref. 12) showed that in the class CP^+, the Q-matrices coincide with the R_0-matrices, i.e.,

$$CP^+ \cap Q = CP^+ \cap R_0. \qquad (5)$$

This result and Theorem 4.1 imply Theorem 4.2.

Remark 4.3. In general, if a class Y of square matrices satisfies the condition

$$Y \cap Q \subseteq Y \cap R_0, \qquad (6)$$

then, for $M \in Y$, $C(q,M)$ is nonempty and bounded for each $q \in R^n$ iff $M \in Q$. As seen in Remark 4.2, condition (6) is an equality for $Y = CP^+$. It is also an equality for L-matrices (Ref. 12) and P_0-matrices (Ref. 13). We show that it is also an equality for nonnegative matrices and a proper inclusion for L_2-matrices and Z-matrices.

The next two theorems, due to Pang (Ref. 12), establish condition (6) for the classes L and L_2.

Theorem 4.3. (Ref. 12) Let $M \in L$. Then $M \in R_0$ iff $M \in Q$.

Theorem 4.4. (Ref. 12) Let $M \in L_2 \cap Q$. Then $M \in R_0$.

Remark 4.4. Theorems 4.3 and 4.4 imply, respectively, that

$$L \cap Q = L \cap R_0 \tag{7}$$

and

$$L_2 \cap Q \subsetneq L_2 \cap R_0. \tag{8}$$

Since $CP^+ \subsetneq L \subseteq L_2$ (Ref. 8), we see that condition (6) extends from CP^+ to L_2 with equality holding also for the class L. The following example shows that the inclusion in (8) is proper:

$$M = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

$M \in L_2 \cap R_0$ but $M \notin L_2 \cap Q$.

The following theorem, due to Aganagic and Cottle (Ref. 13), establishes condition (6) as an equality for the P_0-matrices:

Theorem 4.5. (Ref. 13) Let $M \in P_0$. Then $M \in R_0$ iff $M \in Q$.

We now show that condition (6) holds as an equality for nonnegative matrices by showing that if $M \geq 0$, then $M \in R_0$ iff $M \in Q$. We use the following theorem established by Murty (Ref. 1):

Theorem 4.6. (Ref. 1) Let $M \in R^{nxn}$ such that $M \geq 0$. $M \in Q$ iff $M_{jj} > 0$ for each $j = 1,\ldots,n$.

Theorem 4.7. Let $M \in R^{nxn}$ such that $M \geq 0$. Then $M \in R_0$ iff $M \in Q$.

Proof: ($\Rightarrow$) Suppose $M \notin Q$. Then, by Theorem 4.6, there exists a j such that $M_{jj} = 0$. It is easy to check that $(w;z) = (M_{\cdot j};I_{\cdot j})$ is a complementary solution of the LCP(0,M), contradicting the uniqueness of (0;0) as the complementary solution of the LCP(0,M).

($\Leftarrow$) Suppose that $M \notin R_0$. Then the LCP(0,M) has a complementary solution $(w^0;z^0)$ such that $0 \neq z^0 \geq 0$, say $z_j^0 > 0$. Then $w_j^0 = 0$. Since $M \in Q$, then $M_{jj} > 0$ by Theorem 4.6. We then have

$$0 = w_j^0 = M_{j1}z_1^0 + \ldots + M_{jj}z_j^0 + \ldots + M_{jn}z_n^0 > 0,$$

a contradiction. □

We now show condition (6) for Z-matrices. We use the following result due to Fiedler and Ptak (Ref. 14):

Theorem 4.8. (Ref. 14). If $M \in Z \cap S$, then $M \in P$.

It is known that $Q \subseteq S$ (Ref. 15). Hence, if $M \in Z \cap Q$, then $M \in P$. Since $P \subseteq Q$ (Ref. 1), it follows that, in the class of Z-matrices, the Q-matrices coincide with the P-matrices. We thus have the following theorem:

Theorem 4.9. Let $M \in Z$. Then $M \in Q$ iff $M \in P$.

Remark 4.5. Theorem 4.9 states that $Z \cap Q = Z \cap P$. Since $P \subseteq R_0$, we have

$$Z \cap Q \subseteq Z \cap R_0. \tag{9}$$

The following example shows that condition (9) is a proper inclusion:

$$M = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}.$$

M is both a Z-matrix and an R_0-matrix but not a Q-matrix.

In view of the preceding results, Mangasarian's theorem for copositive plus matrices (Theorem 4.2) also holds for L_2-matrices, P_0-matrices, nonnegative matrices, and Z-matrices.

Theorem 4.10. Let M belong to any of the following classes of matrices: L_2, P_0, Z, and nonnegative matrices. Then $C(q,M)$ is nonempty and bounded for each $q \in \mathbb{R}^n$ iff $M \in Q$.

Remark 4.6. It is interesting to note that Theorem 4.10, which holds for subclasses of L_1 such as the nonnegative matrices, CP^*, L, and P_0, does not hold for L_1. Jeter and Pye (Ref. 16) gave an example of a Q-matrix in L_1 that does not belong to R_0.

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Special Symbols

$\mathbb{R}$	set of real numbers
ϕ	empty set
ϵ	is an element of
$\notin$	is not an element of
$\cap$	intersection
$\subseteq$	set inclusion
$(\Rightarrow)$	proof of necessity
$(\Leftarrow)$	proof of sufficiency
	sigma (summation)
γ Σ	gamma
θ	theta
λ	lambda

Caption

Fig. 1. Illustration of Example 3.1