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A Characterization of Q-Matrices

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Abstract. Let K(M) denote the set of all $q \in R^n$ such that the linear complementarity problem LCP(q,M) has a complementary solution. We show that (a) M is an S-matrix iff there is a $q^0 \in K(M)$ such that $q^0 < 0$ and (b) M is a Q-matrix iff M is a Q_0-matrix and an S-matrix.

Key Words. Linear complementarity problem, matrices, separating hyperplane theorem.

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1. Introduction.

Given an $n \times n$ matrix M with real entries and a vector $q \in R^n$, the linear complementarity problem LCP(q,M) is the problem of finding vectors $w, z \in R^n$ such that

$$Iw - Mz = q \tag{1}$$

$$w \geq 0, \; z \geq 0 \tag{2}$$

$$w^T z = 0. \tag{3}$$

A pair (w;z) is called a feasible solution if it satisfies (1) and (2); it is called a complementary solution if it satisfies (1), (2), and (3). The set of all $q \in R^n$ for which the LCP(q,M) has a complementary solution is denoted by K(M).

The linear complementarity problem arises in mathematical programming (Eaves (Ref. 1), game theory (Lemke (Ref. 2)), and economic equilibrium theory (Mathiesen (Ref. 3)).

For a certain class of matrices M, the existence of a feasible solution implies the existence of a complementary solution. Following Cottle (Ref. 4) we shall call these matrices the Q_0-matrices. (These matrices are also called K-matrices). They include the copositive plus matrices (which include the positive semidefinite matrices) (Lemke, Ref. 2), adequate matrices (Ingleton, Ref. 5), and Z-matrices (Chandrasekaran, Ref. 6).

Q_0-matrices are characterized by the convexity of $K(M)$ (Eaves, Ref. 7). It follows that Q-matrices (those for which $K(M) = R^n$) are Q_0-matrices. These include the L_*-matrices (Eaves, Ref. 1), P-matrices (Murty, Ref. 8), and regular matrices (Karamardian, Ref. 9).

It is natural to ask what additional conditions are required for a Q_0-matrix to be a Q-matrix. We show that a necessary and sufficient condition for a Q_0-matrix M to be a Q-matrix is that M be an S-matrix. Thus the intersection of the Q_0-matrices and the S-matrices consists of the Q-matrices. A characterization of Q-matrices within the class of P_0-matrices is given in Anagagic and Cottle (Ref. 10) where it is shown that among the P_0-matrices the Q-matrices are precisely the regular matrices.

2. Notations and Preliminaries

Let Pos[A] denote the cone generated by the column vectors of a matrix A, i.e.,

$$\mathrm{Pos}[A] = \{q: q = Ax, x \geq 0\}.$$

Consider the complementarity matrix [I,-M]. If A is a matrix whose jth column A_j is either I_j (the jth column of I) or $-M_j$ (the jth column of -M), then Pos[A] is called a complementary cone. The LCP(q,M) has a complementary solution iff q belongs to a complementary cone. Thus, K(M) is the union of all complementary cones. In the rest of this note, M is an n×n matrix. The interior of a set C is denoted by int(C).

The following results will be used in the proof of the main theorem.

Lemma 1. (Eaves, Ref. 7) The following statements are equivalent: (i) M is a Q_0-matrix;

(ii) K(M) is convex;

(iii) K(M) = Pos[I,-M].

Lemma 2. Let C_1 and C_2 be nonempty disjoint convex sets in R^n. Then there exists a hyperplane that separates them.

Proof: Mangasarian (Ref. 11).

Definition 1. M is an S-matrix iff there exists a $z^0 \geq 0$ such that $Mz^0 > 0$.

Remark 1. In literature, S-matrices are defined for any rectangular matrix. The next lemma characterizes square S-matrices in terms of the linear complementarity problem.

Lemma 3. M is an S-matrix iff there exists a $q^0 \in K(M)$ such that $q^0 < 0$.

Proof: ($\Rightarrow$) If M is an S-matrix, then there is a $z^0 \geq 0$ such that $Mz^0 > 0$. Define $q^0 = -Mz^0$. Then $q^0 < 0$ and $q^0 \in \text{Pos}[-M] \subseteq K(M)$.

($\Leftarrow$) If M is not an S-matrix, then the system $Mz > 0$, $z \geq 0$ has no solution, i.e., the system $-Mz < 0$, $z \geq 0$ has no solution. This implies that the complementary cone Pos[-M] has no point in the interior of the nonpositive orthant Pos[-I], i.e.,

$$\text{Pos}[-M] \cap \text{int}(\text{Pos}[-I]) = \phi.$$

By Lemma 2, there exists a hyperplane H separating Pos[-M] and int(Pos[-I]); hence, Pos[-M] and int(Pos[I]) are contained in the same closed half-space H^+. Since the closure of int(Pos[I]) is Pos[I], then $\text{Pos}[I] \subseteq H^+$. Hence, Pos[-M] and Pos[I] are contained in H^+ which implies that all the complementary cones and, therefore, $K(M)$, are contained in H^+. Hence, $K(M)$ has no point in the interior of Pos[-I], contrary to the hypothesis. □

3. The Main Result

Theorem 1. M is a Q-matrix iff M is a Q_0-matrix and an S-matrix.

Proof: ($\Rightarrow$) Since M is a Q-matrix, then $K(M) = R^n$; hence, it is a Q_0-matrix by Lemma 1 and an S-matrix by Lemma 3.

($\Leftarrow$) Let $q \in R^n$. We wish to show that $q \in K(M)$. Since M is an S-matrix, then, by Lemma 3, there is a $q^0 \in K(M)$ such that $q^0 < 0$. We have

$$q^0 = \sum_{j=1}^{n} q_j^0 I_j \tag{4}$$

and
$$q = \sum_{j=1}^{n} q_j I_j . \tag{5}$$

Now,
$$q = \lambda q^0 + q - \lambda q^0, \quad (\lambda > 0) \tag{6}$$

$$= \lambda q^0 + \sum_{j=1}^{n} (q_j - \lambda q_j^0) I_j \tag{7}$$

Since $q_j^0 < 0$ $(j=1,2,\ldots,n)$, we can choose λ large enough such that $(q_j - \lambda q_j^0) > 0$ $(j=1,2,\ldots,n)$. Hence, q can be expressed as a nonnegative linear combination of points in K(M). Since M is a Q_0-matrix, then K(M) is a convex cone (Lemma 1); hence, $q \in K(M)$. It follows that $K(M) = R^n$ and M is a Q-matrix.

□

4. A Remark on P-matrices

Among the Q-matrices, the P-matrices have been widely studied. It is well-known that the LCP(q,M) has a unique complementary solution for each $q \in R^n$ iff M is a P-matrix (Ref. 8). It is natural to ask what class of matrices M has the property that the LCP(q,M) has a unique complementary solution for each $q \in K(M)$.

Suppose that the LCP(q,M) has a unique complementary solution for each $q \in K(M)$. For each $q \ge 0$, the LCP(q,M) has a complementary solution ($(w;z) = (q;0)$) which, by hypothesis, is unique. Eaves (Ref. 1) showed that in this case, M is an L_*-matrix. (M is an L_*-matrix iff for every $z \in R^n$ such that $0 \ne z \ge 0$, there is a j such that $z_j > 0$ and $(Mz)_j > 0$.) The class of L_*-matrices coincides with the class of matrices M, defined by Cottle and Dantzig (Ref. 12), having the property that for every principal submatrix M_{JJ} of M, the system $M_{JJ}z_J \le 0$, $0 \ne z_J \ge 0$ has no solution. Cottle and Dantzig showed that this class of matrices are Q-matrices. Therefore, the LCP(q,M) must have $K(M) = R^n$ and M is a P-matrix. We thus have the following result.

Theorem 2. If the LCP(q,M) has a unique complementary solution for each $q \in K(M)$, then $K(M) = R^n$ and M is a P-matrix.

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