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On a Class of Semimonotone Qo-Matrices In the Linear Complementarity Problem

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On a Class of Semimonotone Q_0-Matrices in the Linear Complementarity Problem

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Abstract. This paper is concerned with the class L^* of nxn real matrices M for which the linear complementarity problem, $w = Mz + q$, $w \geq 0$, $z \geq 0$, $w^Tz = 0$, has a unique complementary solution for each q such that $0 \neq q \geq 0$. It is shown that (a) L^* lies strictly between L_* and L_1, the classes of strictly semimonotone and semimonotone matrices, respectively, (b) L^*-matrices are Q_0-matrices, and (c) L_* is the largest class of Q-matrices in L^*.

Keywords. Linear complementarity problem, Q_0-matrices, semimonotone matrices.

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1. Introduction

For a given $M \in R^{nxn}$ and $q \in R^n$, the linear complementarity problem LCP(q,M) is that of finding $w, z \in R^n$ such that

$$Iw - Mz = q \quad (1)$$

$$w \geq 0,\ z \geq 0 \quad (2)$$

$$w^t z = 0. \quad (3)$$

The linear complementarity problem has been shown to be a fundamental unifying mathematical form for linear programming, quadratic programming, and bimatrix games (Cottle and Dantzig [2]; Murty [9]). Other applications of the LCP include engineering problems (Maier [7]) and the computation of economic equilibria (Mathiesen [8]; Murty [9]).

A pair (w;z) that satisfies (1), (2), and (3) is called a complementary solution and the set of complementary solutions of the LCP(q,M) is denoted by C(q,M). The set of all $q \in R^n$ for which the LCP(q,M) has a complementary solution is denoted by K(M). An important problem in linear complementarity theory is the identification of square matrices M for which K(M) is convex. It is known (Eaves [5]) that the convexity of

K(M) is equivalent to the condition that (1), (2) and (3) has a solution whenever (1) and (2) has a solution. When K(M) is convex, M is called a Q_0-matrix (or $M \in Q_0$) and when $K(M) = R^n$, M is called a Q-matrix (or $M \in Q$). Thus, $Q \subset Q_0$.

The class Q_0 is known to be large. It contains, for example, the matrices M obtained when convex quadratic programming problems (which include the linear programming problems) are transformed into linear complementarity problems. These matrices turn out to be positive semidefinite matrices which have been shown to be Q_0-matrices (Murty [9]).

A large class of Q-matrices was considered by Cottle and Dantzig [2] and later characterized by Eaves [5] as the class L_* of matrices M for which the LCP(q,M) has a unique complementary solution for every $q \geq 0$. L_*-matrices are also called strictly semimonotone matrices (Karamardian [6]). A larger class of matrices containing L_* was defined and denoted by Eaves [5] as L_1 which consists of the square matrices M such that the LCP(q,M) has a unique complementary solution for every $q > 0$. L_1-matrices are also called semimonotone matrices (Karamardian [6]). L_1-matrices are not contained in Q; in fact, they are not contained in Q_0.

This paper defines a class L^* of matrices that is intermediate between L_* and L_1. We say that $M \in L^*$ iff the

LCP(q,M) has a unique complementary solution for each q such that $0 \neq q \geq 0$. Thus, $L_* \subset L^* \subset L_1$. We give examples to show that these inclusions are proper. We also show that every principal submatrix of an L^*-matrix is an L^*-matrix and that $L^* \subset Q_0$.

Among the L^*-matrices, we distinguish between those that are in L_* and those that are not in L_*. We refer to the latter class of matrices as L_*'-matrices. We show that an L_*'-matrix is a Q_0-matrix but not a Q-matrix. Hence, $L^* \subseteq Q_0$ and L_* is the largest subclass of Q-matrices contained in L^*.

2. Further Definitions and Notations

R^n denotes the n-dimensional real Euclidean space with the usual topology. R^n_+ denotes the nonnegative orthant of R^n. R^{nxn} is the class of nxn matrices with real entries. If $M \in R^{nxn}$ and $J \subseteq \{1,2, \ldots, n\}$, the principal submatrix of M obtained by deleting the rows and columns of M corresponding to indices not in J is denoted by M_{JJ} and the corresponding subvectors of w, z, and q are denoted by w_J, z_J, and q_J, respectively.

The cone generated by the columns of a matrix A is denoted by Pos[A], i.e., Pos[A] = $\{Ax \mid x \geq 0\}$. The jth column of a matrix A is denoted by $A_{.j}$. If $A \in R^{nxn}$ and if for all j = 1,2, ..., n, $A_{.j}$ is either $I_{.j}$ (the jth column of the identity matrix) or $-M_{.j}$ (the jth column of $-M$), then Pos[A] is called a complementary cone. The LCP(q,M) has a complementary solution iff q belongs to some complementary cone. Thus, K(M) is the union of all complementary cones.

A complementary cone whose interior is nonempty is said to be nondegenerate; otherwise, it is said to be degenerate. Equivalently, Pos[A] is nondegenerate iff the columns of A are linearly independent.

For any two sets S and T, the set of elements in S that are not in T is denoted by S\T.

For easy reference we list down the characterizations of the classes of matrices used in this paper:

$Q_0 = \{M \in R^{nxn} \mid K(M) \text{ is convex}\}$

$Q = \{M \in R^{nxn} \mid K(M) = R^n\}$

$L_1 = \{M \in R^{nxn} \mid$ LCP(q,M) has a unique complementary solution for all $q > 0\}$

$L_* = \{M \in R^{nxn} \mid$ LCP(q,M) has a unique complementary solution for all $q \geq 0\}$

$L^* = \{M \in R^{nxn} \mid$ LCP(q,M) has a unique complementary solution for all $0 \neq q \geq 0\}$

$L_*' = L^* \setminus L_*$

3. The L°-Matrices

Definition 3.1. $M \in L^\circ$ iff the LCP(q,M) has a unique complementary solution for each nonzero $q \in R^n_+$.

Remark 3.1. It is clear from the definitions that $L_1 \subseteq L^\circ \subseteq L_2$. The following examples show that these inclusions are proper.

Example 3.1.

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The complementary cones are shown in Figure 1. (In the figures, a nondegenerate complementary cone is indicated by a two-headed curved arrow touching the generators of the cone while a degenerate complementary cone is indicated by a straight two-headed arrow coinciding with the generators of the cone.)

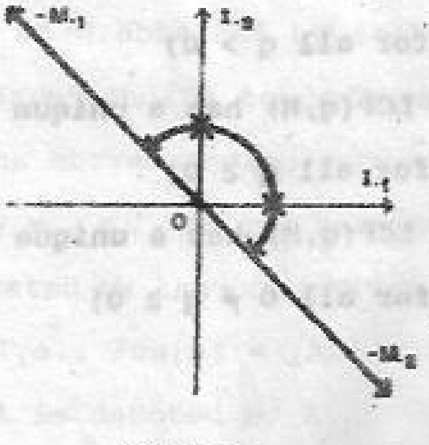

Figure 1

Note that if $q > 0$, the only complementary solution is $w = q$, $z = 0$. If q lies on the ray generated by $I_{.1}$ or $I_{.2}$, say $q = [q_1, 0]^T$, where $q_1 > 0$, then the only complementary solution is $w = [q_1, 0]^T$, $z = 0$. Thus, $M \in L^*$. If $q = 0$, we get an infinite number of complementary solutions given by $w = 0$, $z = [z_1, z_2]^T$ where $z_1 = z_2$ and z_2 is any nonnegative number. Thus $M \in L_*$.

Example 3.2.

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The complementary cones are shown in Figure 2.

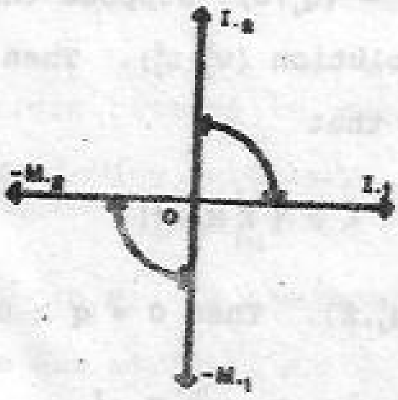

Figure 2

It is clear that if $q > 0$, the only complementary solution of the LCP(q,M) is $w = q$, $z = 0$. Thus $M \in L_1$. If q lies on the ray generated by either $I_{.1}$ or $I_{.2}$, say $q = [q_1, 0]^T$, where $q_1 > 0$, then we obtain an infinite number of complementary solutions given by $w = [z_2 + q_1, 0]$, $z = [0, z_2]$ where z_2 is any nonnegative number. Thus, $M \in L^*$.

Theorem 3.1. Every principal submatrix of an L^*-matrix is an L^*-matrix.

Proof: Let $M \in L^* \cap R^{n \times n}$. We prove the theorem for principal submatrices of order n-1. For principal submatrices of order $r = 1, \ldots, n-2$, the proof is used repeatedly.

Without loss of generality, let the principal submatrix M_{JJ} be the one obtained by deleting the nth row and the nth column of M. Let $q_J \in R^{n-1}$ such that $0 \neq q_J \geq 0$. Then the LCP(q_J, M_{JJ}) has a complementary solution $(w_J^1; z_J^1) = (q_J; 0)$. Suppose that there is another complementary solution $(w_J^2; z_J^2)$. Then $0 \neq z_J^2 \geq 0$. Choose a number λ such that

$$\lambda \geq \left|\sum_{j=1}^{n-1} M_{nj} z_j^2\right|$$

and set $q^T = [q_J^T, \lambda]$. Then $0 \neq q \geq 0$. Define

$$w_n^2 = \lambda + \sum_{j=1}^{n-1} M_{nj} z_j^2.$$

Then $w_n^2 \geq 0$ and the LCP(q,M) has the following complementary solutions:

$$(w^1; z^1) = (q; 0); \qquad (w^2; z^2) = (w_J^2, w_n^2; z_J^2, 0).$$

Since $z_J^2 \neq 0$, these two complementary solutions are not equal. This is not possible since $M \in L^*$. Hence, the LCP(q_J, M_{JJ}) has a unique complementary solution and, therefore, $M_{JJ} \in L^*$. □

4. The L_4'-Matrices

Definition 4.1. $M \in L_4'$ iff $M \in L^* \backslash L_4$.

The following results on the boundedness of $C(q,M)$ will be needed in the proofs.

Definition 4.2. A cone Pos[A] is said to be strictly pointed iff $X(0,A) \triangleq \{x \mid Ax = 0,\ 0 \neq x \geq 0\} = \phi$.

Theorem 4.1. (Cottle [1]). $C(q,M)$ is bounded iff every complementary cone containing q is strictly pointed.

Since the origin belongs to every complementary cone, we have the following corollary:

Corollary 4.1. $C(0,M)$ is bounded iff all the complementary cones are strictly pointed.

If all the complementary cones are strictly pointed, then $X(0,A) = \phi$ for each complementary cone Pos[A]; hence, the only complementary solution of the LCP(0,M) is $(w;z) = (0;0)$. We thus have

Corollary 4.2. $C(0,M)$ is bounded iff the LCP(0,M) has a unique complementary solution.

Theorem 4.3. Let $M \in L_*' \cap R^{n \times n}$. Then

(i) Pos[-M] is a hyperplane which supports R_+^n only at the origin;

(ii) Pos[I,-M] is a closed halfspace bounded by Pos[-M].

Proof: (i) By Theorem 4.2, rank(M) = n-1; hence, the columns of -M span a hyperplane H. We show that Pos[-M] = H. Clearly, Pos[-M] $\subseteq$ H.

Let $q \in H$. Then $q = -Mx$ for some $x \in R^n$. By Lemma 4.1, there exists an $x^0 > 0$ such that $-Mx^0 = 0$. Hence, there exists a $\lambda > 0$ such that $\lambda x^0 + x > 0$. Now,

$$-M(\lambda x^0 + x) = \lambda(-Mx^0) - Mx = q;$$

hence, $q \in$ Pos[-M] and H $\subseteq$ Pos[-M].

To prove that Pos[-M] supports R_+^n only at the origin, we note that Pos[-M] contains the origin which is an extreme point of R_+^n. Moreover, if $0 \neq q \geq 0$, then $q \notin$ Pos[-M]; for, otherwise, there would exist an x such that $0 \neq x \geq 0$ and $q = -Mx$. Then the LCP(q,M) will have at least two complementary solutions

$$(w^1;z^1) = (q;0) \quad \text{and} \quad (w^2;z^2) = (0;x),$$

contradicting the fact that $M \in L^*$. It follows that

$\mathrm{Pos}[-M] \cap R^n_+ = \{0\}$.

(ii) Let the hyperplane Pos[-M] be given by

$$\mathrm{Pos}[-M] = \{x \mid p^T x = 0\}$$

and let $\mathrm{Pos}[-M]^+$ denote the closed halfspace bounded by Pos[-M] and containing R^n_+, i.e.,

$$\mathrm{Pos}[-M]^+ = \{x \mid p^T x \geq 0\},$$

so that $p > 0$. We prove that $\mathrm{Pos}[-M]^+ = \mathrm{Pos}[I,-M]$. Clearly, $\mathrm{Pos}[I,-M] \subseteq \mathrm{Pos}[-M]^+$.

Let $q \in \mathrm{Pos}[-M]^+$. Define

$$\alpha = (p^T q)/(p^T p)$$

and set $$q^0 = q - \alpha p.$$

Then $$p^T q^0 = p^T q - \alpha p^T p = 0;$$

hence, $q^0 \in \mathrm{Pos}[-M]$. It follows that q^0 is a nonnegative linear combination of the columns of $-M$. Since $q = q^0 + \alpha p$, $\alpha \geq 0$, and p is a positive linear combination of the columns of I, we conclude that q is a nonnegative linear combination of the columns of $[I,-M]$. Thus, $q \in \mathrm{Pos}[I,-M]$; hence, $\mathrm{Pos}[-M]^+ \subseteq \mathrm{Pos}[I,-M]$. □

Remark 4.2. Theorem 4.3(ii) implies that an L'_*-matrix M is not a Q-matrix since K(M) is a subset of Pos[I,-M] which is a halfspace. In fact K(M) = Pos[I,-M] which can be shown by proving that M is a Q_0-matrix. To do this we use the following theorems :

Theorem 4.4. (Eagambaram and Mohan [3]) Let $M \in R^{n \times n}$ such that rank(M) = n-1 and Mx = 0 and $p^T M = 0$ for some vectors x > 0 and p > 0. Then M is a Q_0-matrix.

Theorem 4.5. (Eaves [4]) $M \in Q_0$ iff K(M) = Pos[I,-M].

Theorem 4.6.

(i) If $M \in L'_*$, then $M \in Q_0 \backslash Q$.

(ii) $L^* \subseteq Q_0$.

(iii) L_* is the largest subclass of Q-matrices in L^*.

Proof: (i) Let $M \in R^{n \times n}$. By Theorem 4.2, rank(M) = n-1. By Lemma 4.1, there is an x > 0 such that Mx = 0. From the proof of Theorem 4.3(ii), we note that the normal p to Pos[-M] may be chosen to be a positive vector. Since $-M_{\cdot j} \in$ Pos[-M] for j = 1,...,n, then $p^T(-M_{\cdot j}) = 0$ for j = 1,...,n. Thus, $p^T(-M) = 0$ or $p^T M = 0$. By Theorem 4.4, M is a Q_0-matrix. By Theorem 4.5 K(M) = Pos[I,-M]. By Theorem 4.3(ii), K(M) is a halfspace; hence, $M \notin Q$.

(ii) Since $L_* \subseteq Q \subseteq Q_0$, $L'_* \subseteq Q_0$, and $L^* = L_* \cup L'_*$, then $L^* \subseteq Q_0$.

(iii) This immediately follows from (i). □

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