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Lexicographic Group Decision

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Abstract. Assuming lexicographic expected utilities, a solution to a group decision problem obtains from a repeated application of Pareto optimality. If unique, the solution is characterized by the Nash bargaining conditions suitably modified.

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1. Introduction

Consider a group of persons who must make a joint decision. Since each member of the group may have a different choice if he were to decide for the group, the problem is to define a group decision that an impartial arbiter, charged with the task of deciding for the group, might consider as reasonable. In a previous paper [2] that used a lexicographic preferences framework, a solution was proposed for an arbitration problem involving only two persons and no uncertainty. Assuming lexicographic expected utility functions [5, 7, 3, 4] the same solution is applicable to the case of group decision and a similar result holds: if the solution is unique, it satisfies the Nash conditions [6] suitably reformulated and is the only solution that does so.

2. The model

Let $x = (x^1, \ldots, x^n)$ be the group decision when person $h = 1, \ldots, n$ chooses the point x^h in his individual decision space. Write $u^h(x) = (u_1^h(x), u_2^h(x), \ldots)^T$ where $u_i^h(x)$ is the expected ith utility $(i = 1, 2, \ldots)$ of x as perceived by h, and T denotes the transpose so $u^h(x)$ is a column vector. Letting $xP_i^h y$ mean $u_i^h(x) > u_i^h(y)$ and $xI_i^h y$ mean $u_i^h(x) = u_i^h(y)$, h prefers x to y $(xP^h y)$ if $xP_j^h y$ for some j and $xI_i^h y$ for all $i < j$. That is, h's preference ordering of the x's is given by the lexicographic ordering of the $u^h(x)$'s. We write $u(x) = (u^1(x), \ldots, u^n(x))$ so that if S is a set of alternatives (possible group decisions), $x \in S$ can be represented as a point $u(x) \in U(S) = \{u(x) \mid x \in S\}$. We will also write $U_i(S) = \{u_i(x) \mid x \in S\}$ where $u_i(x) = (u_i^1(x), \ldots, u_i^n(x))$.

Denoting the admissible set by A, we assume that $U(A)$ is compact; $U(A)$ is also convex with mixed strategies. Let

$$A_i = \{x \in A_{i-1} \mid \forall y \in A_{i-1}: (\exists h: yP_i^h x) \rightarrow (\exists k: xP_i^k y)\},$$
$$i = 1, 2, \ldots$$

where $A_0 = A$. A_i would be the set of Pareto optimal elements in A_{i-1} if each h had only a real valued utility function u_i^h, and we will say that the elements of A_i are u_i-optimal. Putting

$$A^* = A \cap A_1 \cap A_2 \cap \ldots$$

as the group decision, A^* might be called the lexicographic Pareto solution. To get A^* the arbiter follows a straightforward procedure. First he uses the $u_1 = (u_1^1, \ldots, u_1^n)$ functions to determine the set A_1 of u_1-optimal points. He then looks at the u_2 functions to determine the set $A_2 \subset A_1$ of u_2-optimal as well as u_1-optimal points, and so on. The arbiter thus narrows down his choice by letting the successive u_i's do the work, so to speak, and he brings in no extraneous or arbitrary considerations to arrive at the group decision.

A^* will clearly be smaller than the usual Pareto set with real valued utilities, and it may happen that for some i the u_i functions will pick out the same choice on A_{i-1} in which case $U(A_i) = U(A^*)$ is a singleton. $U(A^*)$ might also be a singleton if utility is infinite-dimensional; see [2, p. 232]. The next section focuses on the singleton $U(A^*)$ case, which gives a Nash-type result.

3. Nash properties

Let $\pi(c)$ be a permutation of the components of $c = (c^1, \ldots, c^n)$, and say that $U(S)$ is symmetrical if for every x, $c = u(x) \in U(S)$ implies that for every $\pi(c)$, $\exists y$: $u(y) \in U(S)$ and $u(y) = \pi(c)$; similarly, $U_i(S)$ is symmetrical if for every x, $c = u_i(x) \in U_i(S)$ implies that for every $\pi(c)$, $\exists y$: $u_i(y) \in U_i(S)$ and $u_i(y) = \pi(c)$. Denoting a possible solution by $g(A)$, not necessarily the solution $A^* = g^*(A)$, consider the following requirements.

<u>Condition 1 (invariance)</u>: The solution $g(A)$ is unchanged by arbitrary positive linear transformations of u_i^h $(i = 1, 2, \ldots;\ h = 1, \ldots, n)$.

<u>Condition 2 (symmetry)</u>: If $U(A)$ is symmetrical, then $U(g(A)) = \{\bar{u}\}$ with $\bar{u}^1 = \ldots = \bar{u}^n$.

<u>Condition 3 (Pareto optimality)</u>: No element of $g(A)$ is Pareto inferior to any element of A. (As usual, x is Pareto inferior to y if someone prefers y to x and no one prefers x to y.)

<u>Condition 4 (rational choice)</u>: If $A \subset A'$ and $A \cap g(A') \neq \emptyset$, then $A \cap g(A') = g(A)$.

Conditions 1 and 2 are the same as those of Nash extended to n persons and multidimensional utilities. Conditions 3 and 4 are the same as Nash. It follows from Lemmas 1 and 2 below that if $U(A^*)$ is a singleton, $g = g^*$ if and only if g satisfies Conditions 1 to 4.

Lemma 1. With $U(g^*(A))$ a singleton, g^* satisfies Conditions 1 to 4.

Proof. Condition 1 is clearly satisfied.

Assume the hypothesis of Condition 2. Since $U_1(A_0)$ is symmetrical, so is $U_1(A_1)$ which is just the "northeast" boundary of $U_1(A_0)$. We can assert that if $U(A_0)$ and $U_1(A_1)$ are both symmetrical, so is $U(A_1)$. [For suppose $U(A_1)$ is not symmetrical. Then there is an x in A_1, say $\tilde{x}$, such that $c = u(\tilde{x}) \in U(A_1)$ and there is some $\pi(c)$, say $\tilde{\pi}(c)$, such that for all y in A_1, $u(y) \neq \tilde{\pi}(c)$. However, $U(A_0)$ is symmetrical and A_1 is a subset of A_0, so for every $\pi(c)$ there must be some y in $A_0 - A_1$ such that $u(y) \in U(A_0 - A_1)$ and $u(y) = \pi(c)$. We would therefore have, say, $u(\tilde{y}) = \tilde{\pi}(c)$ as well as $u(\tilde{x}) = c$. But these two equations imply that $u_1(\tilde{y})$ belongs to $U_1(A_1)$ since $U_1(A_1)$ is symmetrical and A_1 has all the u_1-optimal elements in A_0, and therefore $\tilde{y}$ belongs to A_1 contradicting $\tilde{y} \in A_0 - A_1$.] Thus $U(A_1)$ is symmetrical, hence also $U_2(A_1)$, $U_2(A_2)$ and $U(A_2)$ by the same reasoning. Repeating the argument, $U(A_i)$ is symmetrical for all i, and the conclusion of Condition 2 follows.

Condition 3 is satisfied since x is Pareto inferior to some y in A only if x does not belong to A_i for some i, which would contradict $x \in A^*$.

To establish Condition 4, let its hypothesis hold. With $U(A^*)$ a singleton, we need only show that (i) $A \cap g^*(A') \subset g^*(A)$, which is false only if there is a z such that that (ii) $z \in A \cap A_1' \cap A_2' \cap \ldots$ but (iii) $z \in A - A^*$. Suppose such a z. From $z \in A \cap A_1'$ in (ii) and the fact that $A \subset A'$, we have $z \in A_1$ directly. Thus $z \in A_1 \cap A_2'$ using (ii). Since z belongs to A_1 and is u_2-optimal in the larger set A_1', clearly it is u_2-optimal in A_1, i.e., $A_1 \cap A_2' \subset A_2$, and therefore $z \in A_2$. Repetition of the argument gives $z \in A_i$ for all i, which contradicts (iii) and proves (i).

<u>Lemma 2</u>. If g satisfies Conditions 1 to 4 and $U(g^*(A))$ is a singleton, $g = g^*$.

<u>Proof</u>. Using Condition 1 we can put $U(A^*) = \{\hat{u}\}$ where $\hat{u}^1 = \ldots = \hat{u}^n$ without changing $g(A)$, and we need to show that $U(g(A)) = \{\hat{u}\}$. Let us say that A' symmetrically contains A if for every x, $c = u(x) \in U(A') - U(A)$ implies that for some y and some $\pi(c)$, $u(y) \in U(A)$ and $u(y) = \pi(c)$. Choose A' so that A' symmetrically contains A and $U(A')$ is symmetrical. Then by Condition 2, $U(g(A)) = \{\bar{u}\}$ with $\bar{u}_1 = \ldots = \bar{u}^n$. Noting that $U(A)$ and $U(A')$ have exactly the same elements u of the form $u^1 = \ldots = u^n$, the hypothesis of Condition 4 is satisfied, and therefore $U(g(A)) = \{\bar{u}\}$. Since x is u_1-optimal if $u(x) = \hat{u}$ it is not possible that $\bar{u}_1 > \hat{u}_1$; on the other hand, $\bar{u}_1 < \hat{u}_1$ means that $g(A)$ violates Condition 3. Hence $\bar{u}_1 = \hat{u}_1$. The argument can be

repeated with regard to u_2 to get $\bar{u}_2 = \hat{u}_2$, etc., so that $\bar{u} = \hat{u}$ as required.

4. Independence of irrelevant alternatives

For the purpose of this section, which is to say a word about Arrow's [1] independence of irrelevant alternatives (IIA) condition, it will suffice to consider just two persons and simply $A = \{x, y\}$ such that (i) xP_1^1y and yP_1^2x and therefore (ii) xP^1y and yP^2x. IIA requires that $g(A)$ be invariant with respect to any changes that do not alter (ii). With g^*, one has $A_1 = \{x, y\}$ from (i) so $g^*(A) = \{x\}$ if $u_2(x) > u_2(y)$. If (i) hence (ii) remain the same but $u_2(y) > u_2(x)$, we would have $g^*(A) = \{y\}$ instead, which seems only reasonable. However, IIA is violated.

The motivation for IIA is to make $g(A)$ independent of non-available or irrelevant alternatives, i.e. alternatives outside A (see [1, p. 26]) which IIA does accomplish, but it goes farther by requiring $g(A)$ to be a function only of individual preferences (the P^h's) on A. It thus rules out any group decision function like g^* where the group decision depends on a finer structure of preferences (the P_1^h's). In the simple example above, a change in that finer structure changes $g^*(A)$ but not individual preferences, so IIA is violated and yet the group decision is obviously independent of irrelevant alternatives.

5. Conclusion

On the assumption that the members of a group have multidimensional expected utility functions, a repeated application of Pareto optimality yields the lexicographic Pareto solution, which has the attractive property that if a singleton, it satisfies the Nash conditions extended to the present case. Interestingly, it fails to satisfy Arrow's independence of irrelevant alternatives condition although the group decision is in fact independent of such alternatives. This Arrow requirement, which was designed merely to rule out dependence of the group decision on non-available alternatives, is thus more restrictive than was originally intended.

References

1. Arrow KJ (1963) Social values and individual choices, 2nd ed. Wiley, New York

2. Encarnación J (1986) Lexicographic arbitration. Econ Letters 21: 231-234

3. Fishburn PC (1971) A study of lexicographic expected utility. Management Sci 17: 672-678

4. Fishburn PC (1982) The foundations of expected utility, Chapter 4. Reidel, Dordrecht

5. Hausner M (1954) Multidimensional utilities. In: Thrall RM, Coombs CH and Davis RL (eds) Decision processes. Wiley, New York

6. Nash JF (1950) The bargaining problem. Econometrica 18: 155-162.

7. Thrall RM (1954) Applications of multidimensional utility theory. In: Thrall RM, Coombs CH and Davis, RL(eds) Decision processes. Wiley, New York