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GEOMETRIC PROPERTIES OF EXPLICITLY
QUASICONCAVE FUNCTIONS

by

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Geometric Properties of Explicitly Quasiconcave Functions

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Abstract

Quasiconcave functions are characterized by the convexity of the upper level sets. It is natural to ask what additional properties are required to characterize explicitly quasiconcave functions (which contain the strictly quasiconcave functions). We show that these additional properties can be expressed in terms of the properties of and relationships between the level set, the upper level set, the boundary and the profile of the upper level set.

Geometric Properties of Explicitly

Quasiconcave Functions

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1. Introduction

Explicitly quasiconcave functions are semistrictly quasiconcave functions that are quasiconcave. (Semistrictly quasiconcave functions need not be quasiconcave [Karamardian (1967)]). This class of functions includes the strictly quasiconcave functions and are useful in economic theory [Diewert (1981)] and operations research [Greenberg and Pierskalla (1971)].

Quasiconcave functions are characterized by the convexity of their upper level sets [Mangasarian (1969)] and it is natural to ask what additional properties are required to characterize explicitly quasiconcave and strictly quasiconcave functions. We show that these additional properties can be expressed in terms of the properties of and the relationships between the level set, the upper level set, the boundary and the profile of the upper level set.

Notation. R^n denotes the Euclidean n -space with the usual topology. The boundary of a set C is denoted by $b[C]$ and the interior of C by $i[C]$. For every pair of

distinct points $x, y \in \mathbb{R}^n$, the line segment joining x and y is denoted by $[x, y]$. If the points x, y are deleted from $[x, y]$, the line segment is denoted by (x, y) . The range of a function f is denoted by $\text{Ran}(f)$.

2. Semistrictly Quasiconcave Functions

Definition 2.1. A real-valued function f defined on a convex subset C of \mathbb{R}^n is quasiconcave on C iff $x, y \in C$, $\theta \in [0, 1]$, $f(x) \leq f(y) \implies f(x) \leq f[(1-\theta)x + \theta y]$.

Definition 2.2. A real-valued function f defined on a convex subset C of \mathbb{R}^n is semistrictly quasiconcave on C iff $x, y \in C$, $x \neq y$, $\theta \in (0, 1)$, $f(x) < f(y) \implies f(x) < f[(1-\theta)x + \theta y]$.

Definition 2.3. Let f be a real-valued function defined on a convex subset C of \mathbb{R}^n and let $\alpha \in \text{Ran}(f)$. The upper level set of f at α is defined as

$$UL_f(\alpha) = \{x \in C: f(x) \geq \alpha\}$$

and the level set of f at α is defined as

$$L_f(\alpha) = \{x \in C: f(x) = \alpha\}.$$

Definition 2.4. Let C be a convex subset of \mathbb{R}^n and let $x, y \in C$. The extension of $[x, y]$, denoted by $\text{ext}[x, y]$, is the line containing $[x, y]$. The extension of $[x, y]$ on C is defined as $\text{ext}_C[x, y] = \text{ext}[x, y] \cap C$.

Theorem 2.1. Let f be a real-valued semistrictly quasiconcave function defined on a convex subset C of \mathbb{R}^n and let $\alpha \in \text{Ran}(f)$.

(a) If $L_f(\alpha)$ contains a line segment $[x,y]$, then

$$\alpha = \max \{f(u) : u \in \text{ext}_C[x,y]\}.$$

(b) If $L_f(\alpha)$ contains a neighborhood, then

$$\alpha = \max \{f(x) : x \in C\}.$$

Proof: (a) Let $[x,y] \subseteq L_f(\alpha)$ and let $u \in \text{ext}_C[x,y]$. If $u \in [x,y]$, then $f(u) = \alpha$. If $u \notin [x,y]$, assume that $y \in (x,u)$ (The case $x \in (u,y)$ is treated similarly). If $\alpha < f(u)$, then $f(x) < f(u)$. By semistrict quasiconcavity, $f(x) < f(y)$ contradicting $f(x) = f(y)$. Hence, $f(u) \leq \alpha$.

(b) Let N be a neighborhood contained in $L_f(\alpha)$ and let $y \in N$. If $\alpha \neq \max \{f(x) : x \in C\}$, then there is a point $z \in C$ such that $\alpha = f(y) < f(z)$. By semistrict quasiconcavity, $f(y) < f[(1-\theta)y + \theta z]$ for each $\theta \in (0,1)$. By choosing θ small enough, the point $[(1-\theta)y + \theta z] \in N$, contrary to the fact that f is constant on N . ■

Remark 2.1. Theorem 2.1(b) says that if $\alpha \in \text{Ran}(f)$ and is not the maximum value of f , then the level set $L_f(\alpha)$ does not contain neighborhoods. In the language of utility theory, the indifference set $L_f(\alpha)$ is not thick. Hence, a semistrictly quasiconcave utility function that has no maximum does not have thick indifference sets.

3. Explicitly Quasiconcave Functions

Definition 3.1. A real-valued function defined on a convex subset C of \mathbb{R}^n is explicitly quasiconcave on C iff it is semistrictly quasiconcave and quasiconcave on C .

Theorem 3.1. Let f be explicitly quasiconcave on a convex subset C of \mathbb{R}^n . If f achieves its minimum in the interior of C , then f is constant on C .

Proof: Let $f(x^*) = \min \{f(x) : x \in C\}$. (1)

where $x^* \in i[C]$. If f is not constant on C , then there is a $y \in C$ such that

$$f(x^*) < f(y). \quad (2)$$

Since $x^* \in i[C]$, then there is a $z \in C$ and a $\theta \in (0,1)$ such that $x^* = (1-\theta)z + \theta y$. If $f(y) \leq f(z)$, then, by quasiconcavity, $f(y) \leq f(x^*)$, contrary to (2). Hence, $f(z) < f(y)$. This would then imply, by semistrict quasiconcavity, that $f(z) < f(x^*)$, contrary to (1). Hence, f is constant on C . ■

Lemma 3.1. Let f be a real-valued quasiconcave function defined on a convex subset C of \mathbb{R}^n . Let $x, y \in C$ and suppose that $f(x) < f(y)$. If $z \in (x, y)$ and $f(z) = f(x)$, then f is constant on $[x, z]$.

Proof: Let $u \in (x, z)$. By the quasiconcavity of f ,

$$f(x) \leq f(u). \quad (3)$$

If $f(y) < f(u)$, then by the quasiconcavity of f ,

$f(y) \leq f(z) = f(x)$, contrary to the hypothesis.

Hence, $f(u) \leq f(y)$. Again, the quasiconcavity of f implies that

$$f(u) \leq f(z) = f(x). \quad (4)$$

It follows from (3) and (4) that $f(u) = f(x) = f(z)$.

Since u is arbitrary, the conclusion follows. ■

Theorem 3.2. A real-valued function f defined on a convex subset C of \mathbb{R}^n is explicitly quasiconcave on C iff, for every $\alpha \in \text{Ran}(f)$,

(a) $UL_f(\alpha)$ is convex;

(b) $[x, y] \subseteq L_f(\alpha) \implies \alpha = \max \{f(u) : u \in \text{ext}_C[x, y]\}$.

Proof: (\implies) Part (a) follows from the quasiconcavity of f . Part (b) follows from Theorem 2.1.

(\impliedby) The quasiconcavity of f follows from the convexity of the upper level sets. Let $x, y \in C$, $x \neq y$, $\theta \in (0, 1)$, and $f(x) < f(y)$. Let $z = (1-\theta)x + \theta y$ and let $f(x) = \alpha$. By the quasiconcavity of f , $f(x) \leq f(z)$. Suppose that $f(x) = f(z)$. By Lemma 3.1, f is constant on $[x, z]$; hence, $[x, z] \subseteq L_f(\alpha)$. It follows from the hypothesis that $\alpha = \max \{f(u) : u \in \text{ext}_C[x, z]\}$. This implies that $f(y) \leq \alpha = f(x)$, contradicting the assumption. Hence, $f(x) < f(z)$, showing that f is semistrictly quasiconcave on C . ■

Remark 3.1. Martos (1975) gives the following line segment characterization of explicitly quasiconcave functions that does not involve the convexity of the upper level sets: A real-valued function defined on a convex subset C of R^n is explicitly quasiconcave on C iff every line segment $[x,y] \subseteq C$ can be partitioned into three (possibly empty) line segments $[x,u]$, $[u,v]$, $[v,y]$ such that f increases on $[x,u]$, is constant on $[u,v]$, and decreases on $[v,y]$.

Theorem 3.3. Let f be a real-valued explicitly quasiconcave function defined on a convex subset C of R^n and let $\alpha \in \text{Ran}(f)$. Then

- (a) $UL_f(\alpha)$ is convex;
- (b) $L_f(\alpha) \subseteq b[UL_f(\alpha)]$, if $\alpha < \sup \{f(x) : x \in C\}$.

Proof: Part (a) follows from the quasiconcavity of f . To prove part (b), let $y \in L_f(\alpha)$. Then $y \in UL_f(\alpha)$. Suppose that $y \in i[UL_f(\alpha)]$ and let $N(y)$ be a neighborhood of y contained in $UL_f(\alpha)$. We note that $\alpha = f(y) = \min \{f(x) : x \in UL_f(\alpha)\}$. By Theorem 3.1, f is constant on $UL_f(\alpha)$; therefore, $f(x) = \alpha$ for all $x \in UL_f(\alpha)$. Hence, $UL_f(\alpha) = L_f(\alpha)$ which implies that $L_f(\alpha)$ contains the neighborhood $N(y)$. By Theorem 2.1, $\alpha = \max \{f(x) : x \in C\}$, contrary to the hypothesis. Hence, y is a boundary point of $UL_f(\alpha)$. ■

Remark 3.2. If $\alpha = \sup \{f(x) : x \in C\}$, where $\alpha \in \text{Ran}(f)$, then $\alpha = \max \{f(x) : x \in C\}$ and we have $L_f(\alpha) = UL_f(\alpha)$. Conversely, let $L_f(\alpha) = UL_f(\alpha)$ and let $x \in C$. If $x \notin UL_f(\alpha)$, then $f(x) < \alpha$. If $x \in UL_f(\alpha)$, then $x \in L_f(\alpha)$; hence, $f(x) = \alpha$. It follows that $\alpha = \max \{f(x) : x \in C\}$.

Example 3.1. Consider the explicitly quasiconcave function f depicted in Figure 1, where $C = [0,1]$. For each $\alpha < \alpha_0 = \max \{f(x) : x \in C\}$, $L_f(\alpha)$ is a subset of (actually, equal to) $b[UL_f(\alpha)]$. But at $\alpha = \alpha_0$, $L_f(\alpha_0)$ is not a subset of $b[UL_f(\alpha)]$; in fact, $L_f(\alpha_0) = UL_f(\alpha_0)$. This example also illustrates Theorem 2.1.

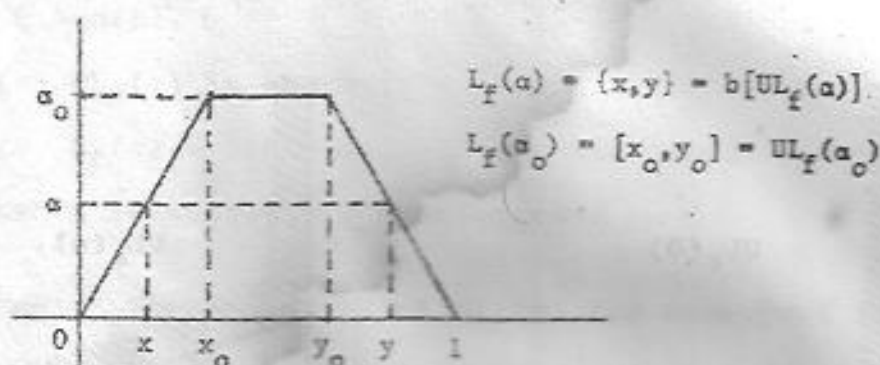


Figure 1

Example 3.2. The following function, given by Diewert (1981), shows that the necessary conditions in Theorem 3.3 are not sufficient for explicit quasiconcavity. Let $f: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be defined by

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } 0 \leq x_1 \leq 1, x_2 = 0 \\ \alpha > 0, & \text{if } \alpha x_1 + (1+\alpha)x_2 = \alpha(1+\alpha) \end{cases}$$

where $R_+^2 = \{x \in R^2: x \geq 0\}$. The function f is quasiconcave on R_+^2 since its upper level sets are convex (Figure 2). For every $\alpha \in \text{Ran}(f)$, $L_f(\alpha) \subsetneq b[UL_f(\alpha)]$ but f is not explicitly quasiconcave since it is not semistrictly quasiconcave on the nonnegative x_1 -axis. Note that $L_f(\alpha)$ is a proper subset of $b[UL_f(\alpha)]$. In Theorem 3.4 we show that when equality holds, the conditions become sufficient.

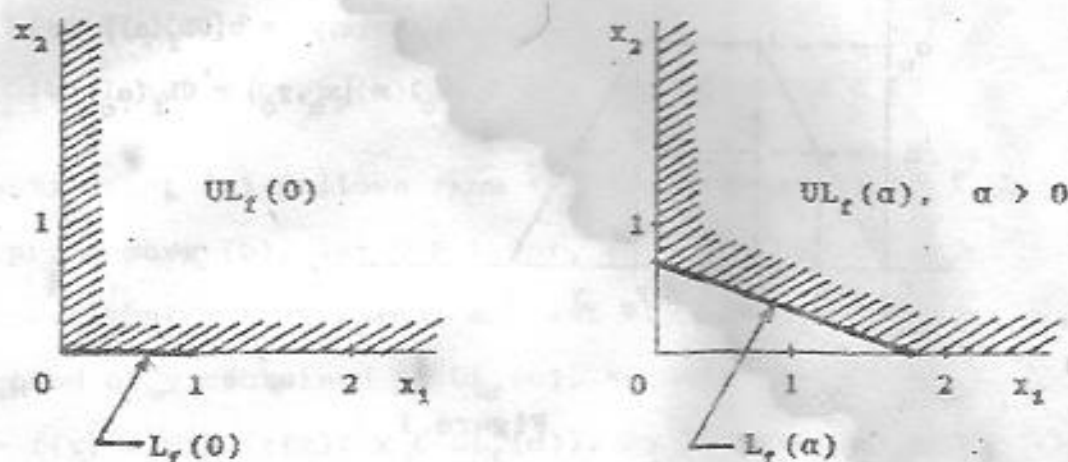


Figure 2

Lemma 3.2. Let C be a convex subset of R^n and let $[x, y] \subsetneq b[C]$. If $z \in \text{ext}_C[x, y]$, then $z \in b[C]$.

Proof: If $z \in [x, y]$, there is nothing to prove. If $z \notin [x, y]$, assume, without loss of generality, that $y \in (x, z)$ and let $y = (1-\theta)x + \theta z$, where $\theta \in (0, 1)$.

If $z \notin b[C]$, then $z \in i[C]$ since $z \in C$. Let $N(z)$ be a neighborhood of z such that $N(z) \subseteq C$. Then the set

$$N(y) = \{v: v = (1-\theta)x + \theta u, u \in N(z)\}$$

is a neighborhood of y and is contained in C . This shows that $y \in i[C]$, contrary to the hypothesis.

Hence, $z \in b[C]$. \square

Theorem 3.4. Let f be a real-valued function defined on a convex subset C of R^n . Suppose that, for every $\alpha \in \text{Ran}(f)$,

$$(a) \text{ } UL_f(\alpha) \text{ is convex;} \quad (5)$$

$$(b) \text{ } L_f(\alpha) = b[UL_f(\alpha)], \text{ if } \alpha < \sup \{f(u): u \in C\}. \quad (6)$$

Then f is explicitly quasiconcave on C .

Proof: The convexity of $UL_f(\alpha)$ for every $\alpha \in \text{Ran}(f)$ implies that f is quasiconcave on C . Let $x, y \in C$, $x \neq y$, $\theta \in (0, 1)$, and $f(x) < f(y)$. Let $z = (1-\theta)x + \theta y$. By the quasiconcavity of f , $f(x) \leq f(z)$. Suppose that $f(x) = f(z)$. Then, by Lemma 3.1, f is constant on $[x, z]$.

Let $f(x) = \alpha$. Then

$$\alpha = f(x) < f(y) \leq \sup \{f(u): u \in C\}.$$

Since $L_f(\alpha) = b[UL_f(\alpha)]$ and $[x, z] \subseteq L_f(\alpha)$, then $[x, z] \subseteq b[UL_f(\alpha)]$. Note that y is in the extension

of $[x, z]$ on $UL_f(\alpha)$. By Lemma 3.2, $y \in b[UL_f(\alpha)]$; hence, $y \in L_f(\alpha)$ and so, $f(y) = \alpha = f(x)$, contrary to the assumption. Consequently, $f(x) < f(z)$, showing that f is semistrictly quasiconcave on C . ■

The most useful property of explicitly quasiconcave functions for optimization is that every local maximizer is a global maximizer [Mangasarian (1969)]. Moreover, they are also characterized by the fact that their restrictions to line segments are also explicitly quasiconcave [Martos (1975)]. It follows that if a function f is explicitly quasiconcave, then on every line segment $[x, y]$ in its domain, every local maximizer on $[x, y]$ is a global maximizer on $[x, y]$.

Suppose that a quasiconcave function f defined on a convex set C has the property that on every line segment in C every local maximizer is a global maximizer. Let $x, y \in C$, $x \neq y$, $\theta \in (0, 1)$, and $f(x) < f(y)$. Consider the line segment $[x, y]$ and the point $z = (1-\theta)x + \theta y$. By quasiconcavity, $f(x) \leq f(z)$. If $f(x) = f(z)$, then, by Lemma 3.1, f is constant on $[x, z]$. This implies that a point $u \in (x, z)$ is a local maximizer on $[x, y]$ but not a global maximizer on $[x, y]$, contrary to the assumption. Hence, $f(x) < f(z)$, showing that f is explicitly quasiconcave. We thus have the following theorem:

Theorem 3.5. A real-valued function defined on a convex subset C of R^n is explicitly quasiconcave on C iff it is quasiconcave on C and on every line segment $[x,y]$ in C , every local maximizer on $[x,y]$ is a global maximizer on $[x,y]$.

4. Strictly Quasiconcave Functions

Definition 4.1. A real-valued function f defined on a convex subset C of R^n is strictly quasiconcave on C iff $x, y \in C$, $x \neq y$, $\theta \in (0,1)$, $f(x) \leq f(y) \implies f(x) < f[(1-\theta)x + \theta y]$.

Remark 4.1. It is clear from the definitions that a strictly quasiconcave function is semistrictly quasiconcave and quasiconcave, hence, explicitly quasiconcave.

Remark 4.2. A strictly quasiconcave function cannot be constant on a line segment.

Remark 4.3. If a strictly quasiconcave function has a maximizer, then it is unique [Avriel (1976)].

Definition 4.2. A convex subset C of R^n is said to be strictly convex iff $x, y \in C$, $x \neq y$, $\theta \in (0,1) \implies [(1-\theta)x + \theta y] \in i[C]$.

Corresponding to Theorem 3.3 we have the following theorem.

Theorem 4.1. Let f be a real-valued strictly quasiconcave function defined on a convex subset C of R^n . Then, for every $\alpha \in \text{Ran}(f)$,

$$(a) \text{ } UL_f(\alpha) \text{ is convex;} \quad (7)$$

$$(b) \text{ } L_f(\alpha) \subseteq b[UL_f(\alpha)]. \quad (8)$$

Proof: The proof of Theorem 3.3 applies here except when $\alpha = \sup \{f(x) : x \in C\}$. In this case, there is a maximizer $x^* \in C$ which is unique (Remark 4.3). Hence, $UL_f(\alpha) = \{x^*\} = b[UL_f(\alpha)] = L_f(\alpha)$. ■

Remark 4.4. As in Theorem 3.3, the conditions in Theorem 4.1 are not sufficient. In fact, even if condition (8) is replaced, as in Theorem 3.4, by the condition that $L_f(\alpha) = b[UL_f(\alpha)]$ for all $\alpha \in \text{Ran}(f)$, they would still be insufficient for strict quasiconcavity. For example, consider $f: R_+^2 \rightarrow R$ defined by

$$f(x_1, x_2) = x_1 x_2,$$

where $R_+^2 = \{x \in R^2 : x \geq 0\}$. The upper level sets are convex and $L_f(\alpha) = b[UL_f(\alpha)]$ for all $\alpha \in \text{Ran}(f)$. Hence, f is explicitly quasiconcave. But it is not strictly quasiconcave since it is constant on the nonnegative axes. However, if we replace (7) by strict convexity and (8) by equality, then the conditions become sufficient for strict quasiconcavity.

Theorem 4.2. Let f be a real-valued function defined on a convex subset C of R^n . Suppose that, for every $\alpha \in \text{Ran}(f)$,

$$(a) \text{ } UL_f(\alpha) \text{ is strictly convex;} \quad (9)$$

$$(b) \text{ } L_f(\alpha) = b[UL_f(\alpha)]. \quad (10)$$

Then f is strictly quasiconcave on C .

Proof: Let $x, y \in C$, $x \neq y$, $\theta \in (0,1)$, and $f(x) < f(y)$. Let $f(x) = \alpha$. Then $x \in L_f(\alpha)$ and $x, y \in UL_f(\alpha)$. Since $UL_f(\alpha)$ is convex for every $\alpha \in \text{Ran}(f)$, then f is quasiconcave on C . Hence, letting $z = (1-\theta)x + \theta y$, we have $f(x) \leq f(z)$ and so, $z \in UL_f(\alpha)$. Since $UL_f(\alpha)$ is strictly convex, then $z \in i[UL_f(\alpha)]$; hence, $z \notin b[UL_f(\alpha)]$ and so $z \notin L_f(\alpha)$. It follows that $f(z) > \alpha = f(x)$; hence, f is strictly quasiconcave on C . ■

Remark 4.5. Actually, the sufficiency conditions in Theorem 4.2 give more than what is required for strict quasiconcavity. In the next theorem we show a necessary and sufficient condition for strict quasiconcavity where the strict convexity of the upper level set is relaxed to a "local strict convexity" property, namely, the points of the level set are extreme points of the associated upper level set.

Definition 4.3. A point x of a convex subset C of R^n is called an extreme point of C iff there exist no points $y, z \in C$, both distinct from x , such that

$x = (1-\theta)y + \theta z$ for some $\theta \in (0,1)$. The set of extreme points of C is called the profile of C and is denoted by $p[C]$.

Theorem 4.3. A real-valued function f defined on a convex subset C of R^n is strictly quasiconcave on C , iff, for every $\alpha \in \text{Ran}(f)$,

(a) $UL_f(\alpha)$ is convex;

(b) $L_f(\alpha) \subseteq p[UL_f(\alpha)]$.

Proof: (\Rightarrow) Part (a) follows from the quasiconcavity of f . To prove part (b), let $x \in L_f(\alpha)$. Then $x \in UL_f(\alpha)$ and, by Theorem 4.1(b), $x \in b[UL_f(\alpha)]$. If x is not an extreme point of $UL_f(\alpha)$, then there are points y, z in $UL_f(\alpha)$, both distinct from x , such that $x = (1-\theta)y + \theta z$ for some $\theta \in (0,1)$. Note that

$$\alpha = f(x) \leq f(y) \quad (11)$$

$$\alpha = f(x) \leq f(z). \quad (12)$$

If $f(y) \leq f(z)$, then, by the strict quasiconcavity of f ,

$$f(y) < f[(1-\theta)y + \theta z] = f(x)$$

contrary to (11). If we suppose that $f(z) \leq f(y)$,

then, by the strict quasiconcavity of f ,

$$f(z) < f[(1-\theta)y + \theta z] = f(x)$$

contrary to (12). Hence, x is an extreme point of $UL_f(\alpha)$, i.e., $x \in p[UL_f(\alpha)]$.

(\Leftarrow) Let $x, y \in C$, $x \neq y$, $\theta \in (0,1)$, and $f(x) \leq f(y)$. Let $f(x) = \alpha$. Then $x, y \in UL_f(\alpha)$. Since

$UL_f(\alpha)$ is convex, then $z \equiv (1-\theta)x + \theta y \in UL_f(\alpha)$;
 hence, $f(z) \geq \alpha = f(x)$. If $f(z) = f(x)$, then $z \in L_f(\alpha)$,
 which, by hypothesis, implies that $z \in p[UL_f(\alpha)]$,
 i.e., z is an extreme point of $UL_f(\alpha)$, a contradiction.
 Hence, $f(x) < f(z) = f[(1-\theta)x + \theta y]$ showing that f is
 strictly quasiconcave on C . ■

Remark 4.6. In most economic applications, the upper
 level sets are required to be closed. When a set C is
 closed, the strict convexity of C is equivalent to the
 condition that every boundary point of C is an extreme
 point of C . Now, if f is strictly quasiconcave on C
 and $L_f(\alpha) = b[UL_f(\alpha)]$, then $UL_f(\alpha)$ is closed. From
 Theorem 4.3, it follows that every boundary point of
 $UL_f(\alpha)$ is an extreme point of $UL_f(\alpha)$. Hence, $UL_f(\alpha)$
 is strictly convex.

Corresponding to Theorem 3.5 we also have the
 following theorem whose proof is analogous to that
 of Theorem 3.5.

Theorem 4.4. A real-valued function f defined on a
 convex subset C of R^n is strictly quasiconcave on C iff
 it is quasiconcave on C and on every line segment $[x,y]$
 in C , a local maximizer on $[x,y]$ is the unique global
 maximizer on $[x,y]$.

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