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Production Under Uncertainty: Safety First and the Crop Mix

by

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Abstract

A safety first decision criterion suggests a secondary objective when the safety condition is satisfied. In the model of this paper, the product mix of a smallholder pursuing safety first is one where the crops produced are equally safe--he cannot afford to gamble and he does not. When the safety condition is satisfied, he chooses a riskier crop mix the larger his resources or the lower his degree of risk aversion.

PRODUCTION UNDER UNCERTAINTY: SAFETY FIRST AND THE CROP MIX

Introduction

This paper formulates a model of farm production decisions under uncertainty that suggests an interpretation of risk and risk aversion that is consistent with common usage. Under such an interpretation, a lower degree of risk aversion implies a riskier product mix when a safety constraint is met, and when it is not, safety first requires a mix of crops which are equally safe at the margin.

Preliminaries

Since farm outputs reach the market after a production lag, the prices at which they can be sold are uncertain at the time production decisions are made. Further uncertainty stems from the variability of realized outputs, given planned production levels, because of random factors (e.g. the weather) that affect yields. Let $q = (q_1, \ldots, q_n)$ denote the decision vector of production levels in a multi-product farm, and $y = (y_1, \ldots, y_n)$ the corresponding random outputs. Writing $p = (p_1, \ldots, p_n)$ for the random price vector, let $f(p, y| q)$ be the joint density of p and y given q. We can then write expected sales as

$$(1) \qquad V(q) = \int_0^\infty \cdots \int_0^\infty p \cdot y \; f(p, y| q) \; dp \; dy$$

and expected profit as

$$(2) \qquad \phi(q, \gamma) = V(q) - C(q, \gamma)$$

where $C(q, \gamma)$ is the cost of q given the cost parameter γ. We will assign a meaning to this parameter as needed. The usual procedure would maximize (2) subject say to a working capital constraint

(3) $C(q, \gamma) \leq A.$

The constraint constant A may be assumed to be higher if farm size is larger. We will, however, follow a different approach.

The Model

The starting point is the Cramér criterion as formulated by Roy which makes the minimization of the risk of a "disaster" the sole objective. While plausible in some circumstances, this criterion is clearly not general. Telser has proposed a modification by postulating a tolerable risk level such that if it is not exceeded, some objective function like (1) is maximized. Specifically, let

$$\begin{aligned} (4) \quad \pi(q, s) &= \Pr\{p.y \geq s \mid q\} \\ &= \int_{B(s)} \cdots \int f(p, y \mid q)\, dp\, dy \end{aligned}$$

where $B(s) = \{(p, y): p.y \geq s\}$ and s is a critical sales level. Then, following Telser, one would maximize (1) subject to (3) and

(5) $\pi(q, s) \geq \pi^*$

where the probability π^* is a decision parameter reflecting a person's attitude towards risk. The concept of an acceptable probability level π^* is familiar from the classical Neyman-Pearson rule which assumes that some specified probability of avoiding a Type I error is acceptable. The standard statistical practice of taking some probability level as good enough for the purpose, say, of detecting batches of items containing more than a certain fraction of defectives is similar. Terms like "reasonable risks" and "acceptable risks" carry the same idea. It is natural then to say that the value of π^* reflects a person's degree of risk aversion: the higher is π^*, the more he is risk averse. This sense of risk aversion, different from the usual Arrow-Pratt one, will be seen below to have a useful implication.

Although the Telser criterion is incomplete in that it is silent about the choice if no q satisfies both (3) and (5), it can be supplemented by the Cramér criterion which would simply maximize (4) subject to (3). A lexicographic combination of the two criteria thus defines a preference ordering: q will be preferred to q' if (i) $\min\{\pi(q, s), \pi^*\} > \min\{\pi(q', s), \pi^*\}$ or (ii) $\min\{\pi(q, s), \pi^*\} = \min\{\pi(q', s), \pi^*\}$ and $V(q) > V(q')$. Such a preference schema--an alternative to mean-variance analysis (Markowitz)--is essentially similar to Roumasset's safety-first model, the Telser criterion applying only if the choice is based on (ii).

The Telser problem

Suppose the Telser criterion applies. Writing $\pi_i = \partial\pi/\partial q_i$ and similarly for C_i and V_i, the Kuhn-Tucker conditions require that the optimal $q = q^0$ (denoting solution values by 0 superscripts which, however, will usually be omitted) satisfy

$$V_i - \alpha C_i + \beta\pi_i \leq 0 \qquad i = 1, \ldots, n \tag{6}$$

$$(V_i - \alpha C_i + \beta\pi_i)q_i = 0 \qquad i = 1, \ldots, n \tag{7}$$

$$A - C(q, \gamma) \geq 0 \tag{8}$$

$$(A - C(q, \gamma))\alpha = 0 \tag{9}$$

$$\pi(q, s) - \pi^* \geq 0 \tag{10}$$

$$(\pi(q, s) - \pi^*)\beta = 0 \tag{11}$$

and $q \geq 0$, $\alpha \geq 0$, $\beta \geq 0$. In (6) we must have the Lagrange multiplier

$\alpha > 0$ since $\beta \geq 0$ and V_i, C_i, π_i are all positive. If $\beta = 0$ the problem reduces simply to the maximization of $V(q)$ subject only to (3), which is relatively uninteresting, so we will focus on $\beta > 0$. We then have

$$(12) \quad V_i/C_i + \beta\pi_i/C_i = \alpha \quad \text{if} \quad q_i > 0$$

$$(13) \quad q_i = 0 \quad \text{if} \quad V_i/C_i + \beta\pi_i/C_i < \alpha.$$

Thus the optimal q will include a product with a low V_i/C_i if its π_i/C_i is high enough, as well as another with a low π_i/C_i if its V_i/C_i is high enough. One might then call the latter a high-risk high-return product, which is the usual case with a cash crop in low income countries, and the former a low-risk low-return one, like the food crop. We therefore propose to say that product i is less risky than j if $\pi_i/C_i > \pi_j/C_j$, and j has a higher return than i if $V_j/C_j > V_i/C_i$. From (13) a crop will be absent from the product mix if for the same return it is riskier than another, or if for the same risk its return is less.

Differentiating the equations in (6), (8) and (10) and writing $G_{ij} = V_{ij} - \alpha C_{ij} + \beta\pi_{ij}$, we have

$$\begin{bmatrix} G_{11} & \cdots & G_{1m} & -C_1 & \pi_1 \\ \vdots & & \vdots & \vdots & \vdots \\ G_{m1} & \cdots & G_{mm} & -C_m & \pi_m \\ -C_1 & \cdots & -C_m & 0 & 0 \\ \pi_1 & \cdots & \pi_m & 0 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ \vdots \\ dq_m \\ d\alpha \\ d\beta \end{bmatrix} = \begin{bmatrix} \alpha C_{1\gamma} d\gamma - \beta\pi_{1s} ds \\ \vdots \\ \alpha C_{m\gamma} d\gamma - \beta\pi_{ms} ds \\ C_\gamma d\gamma - dA \\ d\pi^* - \pi_s ds \end{bmatrix}.$$

by renumbering, m being the number of equations in (6). Let D denote the determinant of the coefficient matrix and D_{ij} the cofactor of its (i, j)th element. We will say that product i is "normal" if $\partial q_i/\partial A = -D_{m+1,i}/D > 0$.

Interpreting $d\gamma > 0$ to mean an increase in the cost of a particular i so $C_{i\gamma} > 0$ and $C_\gamma > 0$ but $C_{j\gamma} = 0$ for all $j \neq i$,

$$(14) \quad \partial q_i/\partial\gamma = \alpha C_{i\gamma} D_{ii}/D + C_\gamma D_{m+1,i}/D.$$

Since $D_{ii}/D < 0$ from the second-order conditions, $\partial q_i/\partial\gamma < 0$ if i is normal, as one should expect. We note the possibility that if i is not normal, a decrease in the cost of i may lead to a lower q_i.

Let C_0 denote fixed costs. If $d\gamma > 0$ is interpreted to mean an increase in C_0 only so $C_\gamma > 0$ but $C_{i\gamma} = 0$ for all i,

$$(15) \quad \partial q_i/\partial\gamma = -C_\gamma \partial q_i/\partial A.$$

Thus the qualitative effect of a decrease in C_0 is the same as that of an increase in A for a normal product.

To see the effects of a larger A on the product mix, suppose i and j are produced and j is riskier, so $V_j/V_i > C_j/C_i > \pi_j/\pi_i$. Greater production of i made possible by $dA > 0$ is obviously not optimal in the presence of j. Neither is the simple allocation of an extra dollar to j production, which gives only $V_j/C_j < \alpha = \partial V^0/\partial A$. Instead, consider putting $dq_i = -a\, dq_j$ where $a = \pi_j/\pi_i$, which just maintains the π^* constraint with $d\pi = \pi_i dq_i + \pi_j dq_j = 0$. With $dV = V_i dq_i + V_j dq_j = (V_j - aV_i)dq_j$ and $dC = (C_j - aC_i)dq_j = dA$, we find that since

$$V_j - aV_i = \alpha(C_j - aC_i)$$

from (12), $dV = \alpha\, dA$ as called for by $\partial V^0/\partial A = \alpha$. But $dq_j > 0$ if $dA > 0$ because $C_j - aC_i > 0$, so a larger A leads to a displacement of safer, lower return products by riskier, higher return ones.

To see the effects of a lower π^* on the same i and j, put $dq_i = -b\, dq_j$ where $b = C_j/C_i$ to maintain the A constraint. With $dV = (V_j - bV_i)dq_j$ and $d(-\pi^*) = -(\pi_j - b\pi_i)dq_j$, we also find that since

$$V_j - bV_i = -\beta(\pi_j - b\pi_i)$$

from (12), $dV = \beta\, d(-\pi^*)$ as required by $\partial V^0/\partial(-\pi^*) = \beta$. Again, $dq_j > 0$ if $d(-\pi^*) > 0$ because $\pi_j - b\pi_i < 0$. Thus a person with a lower degree of risk aversion chooses a riskier product mix. This implication of less risk aversion, which should be expected from the common meaning of the term, does not seem forthcoming from standard expected utility analysis without very strong assumptions (Nachman).

In the above discussion s is taken to be independent of A. The possibility that $s = s(A)$, $s'(A) > 0$, can be treated by requiring that

$$(16) \quad \partial V^0/\partial A + s'(A)\, \partial V^0/\partial s > 0$$

for otherwise a larger A would not increase the value of the objective function. We accordingly assume that (16) holds. Then, although an increase in A increases s, the net effect on V is that of A.

The Cramér problem

At sufficiently low values of A the Cramér criterion applies, and the optimal q would need to satisfy

(6') $\pi_i - \alpha C_i \overset{\leq}{=} 0 \qquad i = 1, \ldots, n$

(7') $(\pi_i - \alpha C_i)q_i = 0 \qquad i = 1, \ldots, n$

in addition to (8), (9), $q \overset{\geq}{=} 0$, and $\alpha \overset{\geq}{=} 0$. It follows from (6') that $\alpha > 0$ and

(12') $\pi_i/C_i = \alpha$ if $q_i > 0$

(13') $q_i = 0$ if $\pi_i/C_i < \alpha$.

Thus the optimal mix includes only products which are equally safe, riskier one being excluded.

Suppose it is optimal to produce only two, the cash crop q_1 and the food crop q_2. Writing $F_{ij} = \pi_{ij} - \alpha C_{ij}$, we have

$$\begin{bmatrix} F_{11} & F_{12} & -C_1 \\ F_{21} & F_{22} & -C_2 \\ -C_1 & -C_2 & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ d\alpha \end{bmatrix} = \begin{bmatrix} \alpha C_{1\gamma} d\gamma - \pi_{1s} ds \\ \alpha C_{2\gamma} d\gamma - \pi_{2s} ds \\ C_\gamma d\gamma - dA \end{bmatrix}$$

giving

(17) $\partial q_1/\partial A = - D_{31}/D = (C_2 F_{12} - C_1 F_{22})/D$

(18) $\partial q_2/\partial A = - D_{32}/D = (C_1 F_{21} - C_2 F_{11})/D$

and therefore

(19) $C_1 \partial q_1/\partial A + C_2 \partial q_2/\partial A = 1.$

The discussion surrounding eqs. (14) and (15), with $m = 2$, is also valid here. In particular, a decrease in the cost of a normal product has a production effect qualitatively the same as that of an increase in A. The latter will, by (19), always have a production effect.

If marginal costs are constant, it follows from (17) and (18) that $C_1 \partial q_1/\partial A < C_2 \partial q_2/\partial A$ if $\pi_{11}/C_1^2 < \pi_{22}/C_2^2$, in which case q_1/q_2 decreases as A increases.

The U-curve Phenomenon

Kunreuther and Wright have cited empirical evidence showing that as farm size increases in low-income regions, the ratio of q_1/q_2 falls before it rises. They explain the U-curve traced out by q_1/q_2 in terms of a lexicographic safety-first model, which accounts for the U-curve by relating the downward-sloping portion to the Cramér criterion and the upward-sloping portion to the Telser criterion. At the bottom of the U-curve, the farmer's decision can be formulated as the solution to a Cramér problem which solution has $\pi^0 = \pi^*$ or, equivalently, as the solution to a Telser problem where the feasible set $\{q: C(q, \gamma) \leqq A \ \& \ \pi(q, s) \geqq \pi^*\}$ is a singleton.

If A is decreased for a low-income farmer operating on the downward-sloping portion of the U-curve, he would choose a higher q_1/q_2 mix and produce less of the food crop. This does not mean, however, that the poorer farmer takes on more of a gambling attitude. Some authors (including Kunreuther and Wright, p. 220) have described him thusly, apparently because he is thought to be choosing a riskier product mix. Yet in ordinary speech one would say that such a person "cannot afford to gamble", making his behavior seem paradoxical. In fact, in the terminology of the preceding section, his choice is one where both products are equally safe. He is clearly not gambling when his decision criterion is safety first, and there is no paradox.

Concluding Comments

In the model of this paper, the first objective is to maximize the probability of surpassing some critical value of output unless that probability is satisfactory, in which case the expected value of output is maximized subject to the probability constraint. It follows in this case that a lower degree of risk aversion implies a more risky product mix, as one should require. Under the first objective, appropriately called safety first, the low income farmer always chooses a mix of equally safe products. He cannot afford to gamble, and he does not. But the probability of attaining the critical value of output being less than satisfactory from his viewpoint, the chances of failure are for him "too high". In contrast, the high income farmer whose probability constraint is satisfied chooses an even riskier product mix if he has more resources, so his expected return per dollar increases with farm size. If capital accumulation takes the form of land acquisition, this means that the larger the farm the faster it will grow. The model thus makes the strong prediction that over time, small farms will tend to be displaced by large farms which will get even larger.

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