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Consumer Choice of Qualities

by

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Abstract

This paper proposes a treatment of quality variations based on a lexicographic preferences assumption. The main result is that the average quality of goods consumed is higher at higher incomes which, though commonly observed, is not explained by standard theory.

1. Introduction

The aim of this paper is to present a new theory of consumer choice where there are quality differences among goods. Section 2 gives a quick review of lexicographic preferences, which provide a more general framework for the analysis of choice by viewing utility as a vector. Section 3 treats a special case that parallels standard theory in assuming strict quasi-concavity for each component of the utility vector. This restriction is relaxed in Section 4, and Section 5 considers the question of qualities and draws some implications under price and income changes. The main result is that the average quality of goods consumed is higher at higher incomes. It is also a consequence that almost always, higher quality goods will be substituted for lower quality ones if any of their prices is lowered, provided of course that they were among the consumer's purchases to begin with.

2. Lexicographic Utility

Let the utility of x, a point in consumption goods space, be written as $u(x) = (u_1(x), u_2(x), \ldots)$ where the ith component corresponds to the ith most important criterion or dimension of choice. The underlying idea is the existence of multiple needs and desires which form a hierarchy according to priority or importance. Those belonging to the same level in the hierarchy are subsumed under the same utility component, and $u_i(x) > u_i(y)$ if x is preferable to y on the basis of the ith criterion. Like the standard utility function, u_i is unique up to a positive monotone transformation.

The usual textbook statement of lexicographic preferences is that x is preferred to y if the first nonvanishing difference $u_i(x) - u_i(y)$,

$i = 1, 2, \ldots,$ is positive, which we will call L ordering. Most economists are correctly skeptical about the analytical usefulness of L ordering, for if u_1 is strictly quasiconcave (as is often assumed about the standard utility function), this suffices to determine choice and therefore the other components of utility beyond the first are redundant. Georgescu-Roegen (1954)[1/] has made clear, however, that a more useful representation of the preference relation is different.

Given the function u_i, let u_i^* denote a "satisficing" level[2/] such that if $u_i(x) \geq u_i^*$ and $u_i(y) \geq u_i^*$, then a person would not decide between them according to u_i (say food value) but on the basis of some other dimension (perhaps taste in the literal sense).[3/] Writing $v_i(x) = \min \{u_i(x), u_i^*\}$, the basic postulate of what we will call L* ordering is that x is preferred to y if the first nonvanishing difference $v_i(x) - v_i(y)$, $i = 1, 2, \ldots,$ is positive. The effective utility, so to speak, thus associated with x is $v(x) = (v_1(x), v_2(x), \ldots)$ instead of the $u(x)$ of L ordering.

Let $p > 0$ be the price vector and $Y > 0$ the budget so $S_0 = \{x \geq 0 \mid p'x \leq Y\}$ is the budget constraint set--it will be understood throughout that $x \geq 0$--and write

$$\theta_i = \max_x \{v_i(x) \mid x \in S_{i-1}\} \qquad i = 1, 2, \ldots \qquad (1)$$

$$S_i = \{x \in S_{i-1} \mid u_i(x) \geq \theta_i\} \qquad i = 1, 2, \ldots . \qquad (2)$$

Then S^*, the choice on S_0, is given by

$$S^* = S_0 \cap S_1 \cap S_2 \cap \ldots . \qquad (3)$$

One considers first only those x's whose $u_1(x) \geq \theta_1$ and drops everything

else as a possible choice. If S_1 is a many-element set, which is likely if $\theta_1 = u_1^*$, the second step selects only those x's whose $u_2(x) \geq \theta_2$, giving $S_2 \subset S_1$, and so on. Every succeeding dimension thus narrows the selection.

For simplicity it will be assumed that given any x there is an i such that $u_i(x) < u_i^*$, ruling out total satiation, and that given any $y \neq x$ there is an i such that $v_i(x) \neq v_i(y)$, ruling out complete indifference, so that S^* is a singleton set. Unless stated otherwise, the indices k and j will be used to denote θ_k as the first $\theta_i < u_i^*$ in (1) and S_j as the first singleton S_i in (2). Since all u_i^* $(i < k)$ are attainable, we will refer to u_k as the priority objective, or simply the objective, and since one will want to maximize $u_j(x)$ subject to $x \in S_{j-1}$, u_j will be called the maximand.

In the next section, standard results will be placed in a broader setting.

3. A Special Case

It is clear that if $\theta_i < u_i^*$ and the function u_i is strictly quasi-concave, then S_i is a singleton. Suppose each u_i is strictly quasi-concave. Then $k = j$, $\theta_i = u_i^*$ for all $i < j$, and the problem for the consumer is to maximize $u_j(x)$ subject to $p'x \leq Y$ and $u_i(x) \geq u_i^*$, $i = 1, \ldots, j-1$. The Kuhn-Tucker conditions below, suppressing the superscript in the optimal x^0 except when needed for greater clarity, are necessary and sufficient to solve the problem (Arrow and Enthoven 1961, Theorems 1 and 2):

$$Y - p'x \geqq 0 \tag{4}$$

$$(Y - p'x)\lambda_0 = 0 \tag{5}$$

$$u_i(x) - u_i^* \geqq 0 \qquad i = 1, \ldots, j-1 \tag{6}$$

$$(u_i(x) - u_i^*)\lambda_i = 0 \qquad i = 1, \ldots, j-1 \tag{7}$$

$$U_r - p_r\lambda_0 \leqq 0 \qquad r = 1, \ldots, n \tag{8}$$

$$(U_r - p_r\lambda_0)x_r = 0 \qquad r = 1, \ldots, n \tag{9}$$

$$x \geqq 0 \tag{10}$$

$$\lambda \geqq 0 \tag{11}$$

where $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{j-1})$ is a vector of Lagrange multipliers, $u_{ir} = \partial u_i/\partial x_r$, and

$$U_r = u_{jr} + \sum_{i=1}^{j-1} \lambda_i u_{ir}. \tag{12}$$

Some implications are straightforward. Since $U_r > 0$ for some r (otherwise the consumer would be at a point of complete satiety), $\lambda_0 > 0$ from (8) so $Y - p'x = 0$ in (4) because of (5). From (8) and (9),

$$x_r = 0 \quad \text{if} \quad U_r/p_r < \lambda_0 \tag{13}$$

$$U_r/p_r = \lambda_0 \quad \text{if} \quad x_r > 0 \tag{14}$$

and therefore

$$U_r/U_s = p_r/p_s \quad \text{if} \quad x_r, x_s > 0 \tag{15}$$

which have simpler analogues in standard theory. With utility as a vector, λ_0 is the "marginal utility" of income. Looking at (4)-(12), multiplication of (p, Y) by a positive scalar will change λ_0 but not x^0; i.e. the demand function f, $x^0 = f(p, Y)$, is homogeneous of degree zero.

In order to see the effects of variations in p and Y similar to familiar ones, consider the case where such variations leave the sets $J = \{i \mid \lambda_i > 0\}$ and $K = \{r \mid x_r > 0\}$ unchanged. As Appendix A shows, one obtains the Slutsky-type equation

$$\partial x_s/\partial p_r = \partial x_s/\partial p_r\big|_{V=\text{const}} - x_r\, \partial x_s/\partial Y \tag{16}$$

where V is a vector whose components are u_i $(i \in J)$ and u_j. From McKenzie (1957) or Hurwicz and Uzawa (1971, Theorem 1), which do not require the preference ordering to be representable by a real valued utility function, the matrix of substitution terms given by (16) is symmetric and negative semidefinite.

Observing that nothing in Section 2 precludes u_j being a function of $u_1, \ldots, u_{j-1}$, thus permitting tradeoffs among different criteria within the component u_j, the standard theory of consumer choice can be thought of as the special case where each u_i is strictly quasiconcave and all of (6) are satisfied as strict inequalities. Then, every $\lambda_i = 0$ in (7), (15) reduces to the usual $u_{jr}/u_{js} = p_r/p_s$, and (16) reduces to the standard Slutsky equation where only the value of the maximand u_j is held constant. In effect, the implicit assumption of standard theory is that every u_i^* $(i < j)$ is surpassed at x^0, the standard utility function being some composite u_j, which holds in some but evidently not in all cases.

The assumption of strict quasiconcavity rules out linear segments of u_i indifference surfaces, which can be relaxed.

4. A More General Case

Since a particular want can be accommodated by some goods but not by others, corresponding to each u_i let

$$G_i = \{r \mid \forall x: u_i(x) \overset{\leq}{=} u_i^* \rightarrow u_{ir} > 0\}$$

denoting goods by their indices. If r does not belong to G_i, then $u_{ir} = 0$ identically (e.g. no amount of clothing does anything to fill the need for food). Thus from (14), $x_r > 0$ only if $r \in G_i$ for some $i \overset{\leq}{=} j$: a good will not be present in the consumer's choice unless it is a G_i good, $i \overset{\leq}{=} j$.

Consider two G_i goods r and s. We will say that they are perfect substitutes with respect to u_i--they are ϕ_i substitutes for short--if for some constant a_{rs}, $u_{ir} = a_{rs} u_{is}$ everywhere (at every point x) and therefore the u_i indifference curves relating r and s are parallel straight lines. There is then a set of ϕ_i substitutes, possibly empty, associated with each u_i. If nonempty, u_i cannot be strictly quasiconcave.

We relax the assumption in Section 3 by now assuming that each u_i is quasiconcave but satisfies the following

Condition ϕ. If there are ϕ_i substitutes and $x_{(i)}$ is the corresponding subvector of x, then, given any constant $\bar{x}_{(i)}$, u_i is strictly quasiconcave on the set $\{x \mid x_{(i)} = \mu \bar{x}_{(i)}, \mu > 0\}$. If there are no ϕ_i substitutes, u_i is strictly quasiconcave.

Thus if the ϕ_i substitutes are held to fixed proportions--this includes the case of zero quantities for all except one of them--then strict quasi-

concavity holds on the set of x's so restricted, which does not seem unreasonable.

Two cases will need to be distinguished in the maximization problem when we consider the implications of a larger S_0 (due to $dY > 0$, or to $dp_r < 0$ provided $x_r > 0$): Case I, where S_k is a singleton; and Case II, where it is not, and therefore $k < j$. It follows from Condition ϕ that S_k is a many-element set in Case II only because of some ϕ_k substitutes which, at the given prices, permit u_k to be maximized at more than one point $x \in S_{k-1}$. (Suppose r and s are ϕ_k substitutes and $u_{kr}/p_r = u_{ks}/p_s$. Then $du_k = u_{kr}dx_r + u_{ks}dx_s = 0$ for all dx_r and dx_s satisfying $dY = p_r dx_r + p_s dx_s = 0$, and S_k cannot be a singleton.) We will say that these ϕ_k substitutes "form" S_k--they make it a many-element set--or that these goods "generate" a maximand different from the objective. Clearly, some of them must also be G_j goods, or u_j could not pick out the singleton S_j.

Noting that in Case II, $u_k(x) \geq \theta_k$ is a binding constraint in the problem, S_k is a subset of the budget (hyper)plane $\{x \mid p'x = Y\}$. Consequently a parallel shift of the budget plane due to $dY > 0$ gives a new S_k which is a many-element set again, because of the parallel u_k indifference surfaces of ϕ_k substitutes. The objective is unchanged by $dY > 0$, since $\theta_k < u_k^*$ to begin with. A large enough increase in Y will, however, make some utility component with a higher index become the objective. As one might expect, this will involve qualitative changes.

5. Quality Variations

In the classic formulations of Theil (1952) and Houthakker (1952), a commodity is a set of related goods differing only in quality level. Judgment of quality is that of the consumer, who can always ignore a higher priced good unless its quality is also higher. Quality being included as a separate argument in a real valued utility function, a commodity is represented by two dimensions: its quantity--all the goods belonging to a commodity are measured in the same physical units--and its quality level. This approach is unnecessarily restrictive however. First, it requires a complete ordering of the goods belonging to a commodity, which is often not possible. Second, the consumer cannot choose to have two or more qualities of the same commodity (e.g. butter and margarine). If the consumer does so, Theil considers only their average quality while Houthakker treats them as belonging to different commodities. Third, the implication is that the consumer would always be willing to give up some quality for a sufficiently larger quantity, which seems contrary to fact. Unless a person is very poor, he may well choose not to sacrifice any quality for more quantity (as in the case of food, or children). More important, it is left unexplained why quality levels may be expected to rise with income, which is the empirical observation. In order to have an explanation, we have to look at "the reasons why consumers want certain goods" (Houthakker 1952, p. 163).

Lancaster (1971, p. 6) goes in the right direction by arguing that it is the "characteristics" of goods, "those objective properties that are relevant to choice", that people want. In his approach there is a mapping from consumption goods space to characteristics space, and defining the utility function on the latter, preferences over goods are derived from

preferences over characteristics. Lancaster considers a possible correspondence between wants and characteristics, hence hierarchical and satiation possibilities, but does not pursue the matter. As in standard theory, his utility function is real valued and limited in scope. In the approach that we will follow, characteristics are bypassed and the dimensions of utility are directly the reasons for choice.

Since any particular good r does not contribute towards satisfying more than a relatively few wants, $u_{ir} = 0$ identically for some i. In addition to the physical properties that goods belonging to a commodity C must have, C can be described by a set of indices $I(C)$ such that $C = \bigcap_{i \in I(C)} G_i$. Accordingly, what the goods belonging to C have in common is their value for some components of utility and their irrelevance elsewhere: if $r \in C$, then $u_{ir} > 0$ only if $i \in I(C)$.

Denote by $\alpha(C)$ the good belonging to C whose quality level is α. For convenience simply write $\alpha = \alpha(C)$, $\beta = \beta(C)$, etc., which will be referred to as C goods. We propose to say that

$\alpha Q \beta$ (α has higher quality than β) if $u_{i\alpha} \geq u_{i\beta}$ for all i and $u_{h\alpha} > u_{h\beta}$ for some h, (17)

$\alpha Q_h \beta$ (α has u_h higher quality than β) if $\alpha Q \beta$ and $u_{h\alpha} > u_{h\beta}$, (18)

$\alpha R_h \beta$ (α is u_h better than β) if $\alpha Q_h \beta$ and $u_{h\alpha}/p_\alpha > u_{h\beta}/p_\beta$, (19)

$\alpha \bar{R}_h \beta$ (α is u_h superior to β) if $\alpha R_h \beta$ and for all $\beta' \neq \alpha$, $\alpha R_h \beta'$ if $\beta' R_h \beta$, (20)

the inequalities holding everywhere. Q is transitive and a partial ordering of C goods, in accordance with the fact that in some cases neither α nor β can be said to have higher quality than the other. If $\alpha Q \beta$ we can take it that $p_\alpha > p_\beta$ for otherwise, β could not be considered for purchase. Definition (19) seems in conformity with common usage: one product is said to be a better buy than another, even though it costs more, if it gives greater value for the price in some sense. Notice that if $\alpha R_h \beta$, it may be that α and β are ϕ_i substitutes for some $i \in I(C)$, but if α and β form S_k in Case II, then $u_{k\alpha}/p_\alpha = u_{k\beta}/p_\beta$ and one does not have $\alpha R_k \beta$. Unless stated otherwise, it will be understood that quality comparisons pertain to the maximand u_j, which is the utility component that determines the optimal choice.

Let x_C be the vector of C goods quantities. We will say that the commodity C is u_k normal--recall that the priority objective is u_k--if given $dY > 0$ and constant prices, more is spent on it, and if given $dp_r < 0$ (provided $x_r > 0$) and the same income, the expenditure on C is greater than the cost of the old x_C at the new prices.[4/] Clearly, at least one C is u_k normal. Unless stated otherwise, a u_k normal commodity will be understood when we speak of a normal C.

Income changes

Let $dY > 0$ and assume the following hypothesis, which will be referred to as H: C is normal, $x_\beta > 0$, and $\alpha \bar{R}_j \beta$ (i.e., β of a normal commodity is present in the choice but there is a superior good α, possibly absent before the change in income). Then,

$$u_{j\alpha}/p_\alpha > u_{j\beta}/p_\beta. \tag{21}$$

We will consider the effects of a larger outlay on C in the two possible cases.

Case I. With the u_i^* $(i < j)$ constraints satisfied, one would be more interested in α because of (21). An extra dollar spent on α makes u_j larger by $u_{j\alpha}/p_\alpha$. On the other hand, if the consumer substitutes α for β by buying one unit more of α and $b = \min_i \{u_{i\alpha}/u_{i\beta} > 0 \mid i < j\}$ units less of β, expenditure on C is increased by $p_\alpha - bp_\beta$ but u_j is raised by $u_{j\alpha} - bu_{j\beta}$ without violating the u_i^* $(i < j)$ constraints. From (21) it follows directly that

$$(u_{j\alpha} - bu_{j\beta}) > (p_\alpha - bp_\beta)\, u_{j\alpha}/p_\alpha. \tag{22}$$

Thus if $p_\alpha > bp_\beta$

$$(u_{j\alpha} - bu_{j\beta})/(p_\alpha - bp_\beta) > u_{j\alpha}/p_\alpha \tag{23}$$

and therefore the substitution gives a higher u_j per dollar. If $p_\alpha = bp_\beta$ hence $u_{j\alpha} > bu_{j\beta}$, the result is the same. If $p_\alpha < bp_\beta$ which means there is even some "savings" the inequality in (23) is reversed: any reduction in the u_j level due to the substitution can be more than made up by using the savings to buy more α. In short, an optimal allocation implies less β and more α than before. (Note that the argument does not require $x_\alpha > 0$ in the choice before or optimality of the latter at the then existing p and Y; cf. Appendix B.)

Case II. θ_k is automatically raised by the increase in Y. As noted in Section 4 however, the corresponding S_k is still a many-element set hence the maximand, which determines the optimal allocation, remains the same. The result is therefore the same, and we have

Proposition 1. Given H, a larger income implies a substitution of the higher quality good α for the lower quality β, hence an increase in the average quality consumed.

The reason for buying β to begin with is its greater u_i value per dollar's worth for some $i < k$, or β would have never appeared in the choice. It is even possible that $u_{i\beta}/p_\beta > u_{i\alpha}/p_\alpha$ for all $i < k$, $i \in I(C)$, in which case α must be absent from the choice at lower incomes where the objective has a lower index. This leads us to the next point.

If Y increases enough for θ_k to reach u_k^*, the objective will be a new $u_{k'}$, $k' > k$. (It need not be u_{k+1}, for u_{k+1}^* may have been attained in the course of reaching some u_i^*, $i \leq k$.) Consider the set of $G_{k'}$ goods: some of them are in the present choice, or else none. If the latter, at least one will be bought at a sufficiently higher income Y' where $u_{k'}$ is the objective. If the former, suppose that a $u_{k'}$ better good is absent from the present choice. It is clear that some such good would then be bought at Y', giving

Proposition 2. At a sufficiently higher income, a "new" good will be purchased unless no better good is available, and if any is available, at least one will enter the choice.

Suppose income rises from Y to Y'' and a higher quality good--the particular utility component u_h is unspecified as in (17)--enters the choice. There is no reason to have a higher value of any u_i with $\theta_i = u_i^*$, so the good must be of a higher quality as regards some u_h with $h \geq k$. If that good is u_k better at Y'', certainly it is also u_k better at Y since R_k is independent of income, in which case it should have been

bought earlier. Therefore $h > k$, and it is bought now only because u_k^* has been reached and there is a new objective with a higher index hence less important. Thus

Proposition 3. Prices being constant, the entry of a higher quality good in the choice at a higher income implies a less important objective.

This may have possibilities for welfare comparisons but our present interest lies in Propositions 1 and 2. As income rises to higher levels, new goods appear in the consumer's choice and higher quality goods displace inferior ones.[5/] We would thus expect the average quality of goods consumed to be higher at higher incomes. However, this empirical fact does not result only from one-one displacements, so to speak, where for each β there is a higher quality α. The more general case involves displacements where a group of goods is substituted for another group.

Let A and B be subsets of C--A and B are not necessarily mutually exclusive--and write $u_{iA} = \sum_{r \in A} u_{ir}$ and $p_A = \sum_{r \in A} p_r$. In (17)-(20), put A and B in place of α and β throughout (so $u_{i\alpha}$ becomes u_{iA}, etc.), and replace Q, Q_h, R_h, and $\bar{R}_h$ by Q', Q_h', R_h', and $\bar{R}_h'$, respectively, to define the latter.

Consider the hypothesis H': C is normal, $x_s > 0$ for all $s \in B$, and $A \bar{R}_j' B$. No particular $r \in A$ need have higher quality than any particular $s \in B$, so H' is a weaker hypothesis than H. Let us say that A is increased by one unit if every $r \in A$ is increased by one unit--the quantities x_r, $r \in A$, are not necessarily equal--and that A is substituted for B if A is increased and B - A is decreased. Exactly the same reasoning for Proposition 1, replacing α and β by A and B throughout,

then yields the more general

Proposition 1'. Given H', a larger income implies a substitution of the higher quality group of goods A for the lower quality B, hence an increase in the average quality consumed.

Proposition 1 is only the special case where A and B are one-element sets. Another special case, where only B is a one-element set, explains the common phenomenon of a variety of goods displacing a single good. For example, "If a woman's income is so low that she can afford only one pair of shoes, she will buy a pair that can be used on a great variety of occasions. If she can afford two pairs, she will not buy two of the same kind as before, but two pairs that complement each other by being useful on different occasions".[6/] Neither of the two new pairs may have higher quality than the original pair, but as a group they are better and they therefore displace the all-purpose shoes.

Simple changes in the arguments for Propositions 2 and 3 similarly give stronger versions where "available group of goods" and "higher quality group of goods" replace "available good" and "higher quality good", respectively.

Price changes

Suppose $x_\alpha > 0$ in addition to H, and let $dp_\alpha < 0$. Given the new prices, the old x_C can be purchased with a smaller outlay, and more will be spent on C.

Case I. The Case I argument leading to Proposition 1 is fully applicable--the prices there should be read as the new prices--and α will displace β.

Case II. There are two possibilities. (i) If α is not among the ϕ_k substitutes that form S_k, the fact remains that S_k is a many-element set under the price change, so the allocation is determined by u_j with the same result. (ii) If α is one of the goods which generate a maximand other than the objective, then β is also a G_k good. Consider the situation prior to the price change: since S_k is a subset of the budget plane where u_k is maximized, $du_k = u_{k\alpha}dx_\alpha + u_{k\beta}dx_\beta = 0$ and $dY = p_\alpha dx_\alpha + p_\beta dx_\beta = 0$ both hold at x^0, hence $u_{k\alpha}/p_\alpha = u_{k\beta}/p_\beta$ at x^0 whether or not α and β are ϕ_k substitutes. With the new price p'_α, we now have $u_{k\alpha}/p'_\alpha > u_{k\beta}/p_\beta$. Thus the same argument in Case I, putting k in place of j, gives

Proposition 4. If H holds, α will displace β if α is present in the choice and its price is decreased.

Let β have a lower price p'_β instead, but maintain $\alpha \bar{R}_j \beta$. Case I and Case II(i) of Proposition 4 can be repeated almost word for word, with the result that α will displace β. However, Case II(ii) is different: with β now more attractive at u_k which has a higher priority than u_j, using exactly the same reasoning gives the opposite result. (Interchange α and β throughout, and put $j = k$ in Case I of Proposition 1.) Thus

Proposition 5. Suppose H holds and the price of β is decreased but α, which is present in the choice, remains better. Then α will displace β unless the latter is one of the goods which generate a maximand different from the objective, in which case β will be substituted for α.

Such a case rests on the fortuitous circumstance that relative prices are just right in order for a lower-priced β to be one of the goods that form S_k, so we would conclude from Propositions 4 and 5 that α will displace β almost always under the stated conditions. A similar conclusion applies to groups A and B if any price p_s ($s \in A \cup B$) is decreased, exceptions arising only where $s \in B - A$ is one of the goods which form S_k.

6. Concluding Remark

It is generally observed that different income classes buy different goods and that higher income classes consume higher quality goods. This observation is not deducible from the extension of standard theory to quality choice but it follows from the model of this paper, which assumes that the consumer has lexicographic preferences "not because it would constitute a more convenient approach or lead to a simpler schema, but because it offers a more adequate interpretation of the structure of our wants" (Georgescu-Roegen 1954, p. 519).

Appendix A

Consider the equations in (4), (6) and (8) determined by the solution. Taking total differentials, one has the following system that includes only the results from those equations:

$$\begin{bmatrix} W & \nabla u_1 & \cdots & \nabla u_{j-1} & -p \\ \nabla u_1' & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \nabla u_{j-1}' & 0 & \cdots & 0 & 0 \\ -p' & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\lambda_1 \\ \vdots \\ d\lambda_{j-1} \\ d\lambda_0 \end{bmatrix} = \begin{bmatrix} \lambda_0\, dp \\ 0 \\ \vdots \\ 0 \\ E \end{bmatrix} \tag{A1}$$

where $W = (w_{rs})$, $w_{rs} = \partial U_r/\partial x_s$, $\nabla u_i = (u_{i1}, \ldots, u_{in})'$, and $E = -dY + \sum_s x_s\, dp_s$. Noting that $w_{rs} = w_{sr}$, the coefficient matrix is symmetric. Let D be the determinant of this matrix and D_{rs} the cofactor of its (r, s) element. Then, analogous to standard results, one has

$$\partial x_s/\partial p_r = (\lambda_0 D_{rs} + x_r D_{n+j,s})/D \tag{A2}$$

$$\partial x_s/\partial B = -D_{n+j,s}/D. \tag{A3}$$

Suppose $dp_r \neq 0$ for a particular r and $dp_s = 0$ for all $s \neq r$, and choose dY so that $du_j = \sum_s u_{js}\, dx_s = 0$. Then from (12) and (14),

$$\lambda_0 \sum_s p_s\, dx_s = \sum_s \sum_i \lambda_i\, u_{is}\, dx_s \tag{A4}$$

after putting $du_j = 0$. But from each equation in (6), $\sum_s u_{is}\, dx_s = 0$ and therefore both sides of (A4) vanish. Hence $E = 0$ in (A1) so

$$\partial x_s/\partial p_r\Big|_{V=\text{const}} = \lambda_0 D_{rs}/D. \tag{A5}$$

Finally, substituting (A5) and (A3) in (A2) gives (16).

Appendix B

Suppose $x_\alpha = 0$ to begin with. Notice that the very first unit of α substituted for b units of β means a nonbinding constraint in (6) if $u_{i\alpha}/u_{i\beta} > b$, making the corresponding $\lambda_i = 0$. Writing $c_\alpha = \sum_{i=1}^{j-1} \lambda_i u_{i\alpha}$ and $c = c_\alpha/c_\beta$, we therefore have $c = b$. From (14),

$$u_{j\alpha} + c_\alpha - p_\alpha \lambda_0 = 0$$

$$cu_{j\beta} + c_\alpha - cp_\beta \lambda_0 = 0$$

hence

$$(u_{j\alpha} - cu_{j\beta}) = (p_\alpha - cp_\beta)\lambda_0.$$

Since $b = c$, and $c < p_\alpha/p_\beta < u_{j\alpha}/u_{j\beta}$ from (14) and (21),

$$(u_{j\alpha} - bu_{j\beta})/(p_\alpha - bp_\beta) = \lambda_0.$$

This shows that if the choice before $dY > 0$ contains α and is optimal, the u_j value per dollar yielded by the substitution is equal to λ_0, which as usual gives the sensitivity of the solution value of the maximand to the constraint constant Y.

The argument in the text is couched so that it can be used repeatedly in the later discussion, particularly for Proposition 5.

Notes

1. This important paper has been relatively neglected in the literature, but see e.g. Chipman (1960), Encarnación (1964, 1983), Day and Robinson (1973), and the survey article by Fishburn (1974).

2. Cf. Simon (1955) who suggested the interpretation of choice as a vector but did not indicate how vectors would be ordered.

3. Although we take it as given, the u_i^* level corresponding to u_i is explainable in a more complete framework by biological and individual factors or by socially determined norms. One speaks of enough food, a dish with enough seasoning, a large enough variety of dishes for guests at a feast, etc.

4. The idea is that the real outlay on C is larger at a higher real income, using the new prices for measuring changes in real terms.

5. The displacements may be less than total, so it is possible for the choice to include three or more C goods of different qualities, e.g. an ordinary wine for the usual supper, a better one for Sunday dinner, and an even better one for special occasions.

6. Houthakker (1961, p. 724) refers to Norris (1952).

References

Arrow, K.J. and Enthoven, A.C. (1961). 'Quasi-concave programming.' Econometrica, vol. 29, pp. 779-800.

Chipman, J.S. (1960). 'The foundations of utility.' Econometrica, vol. 28, pp. 193-224.

Day, R.H. and Robinson, S.M. (1973). 'Economic decisions with L** utility.' In Multiple Criteria Decision Making (ed. J.L. Cochrane and M. Zeleny). Columbia, S.C.: University of South Carolina Press, pp. 84-92.

Encarnación, J. (1964). 'A note on lexicographical preferences.' Econometrica, vol. 32, pp. 215-217.

_____ (1983). 'Social values and individual choices.' Journal of Economic Behavior and Organization, vol. 4, pp. 265-275.

Fishburn, P.C. (1974). 'Lexicographic orders, utilities and decision rules: a survey.' Management Science, vol. 20, pp. 1442-1471.

Georgescu-Roegen, N. (1954). 'Choice, expectations and measurability.' Quarterly Journal of Economics, vol. 68, pp. 503-534.

Houthakker, H.S. (1952). 'Compensated changes in quantities and qualities consumed.' Review of Economic Studies, vol. 19, pp. 155-164.

_____(1961). 'The present state of consumption theory.' Econometrica, vol. 29, pp. 704-740.

Hurwicz, L. and Uzawa, H. (1971). 'On the integrability of demand functions.' In Preferences, Utility and Demand (ed. J.S. Chipman et al.). New York: Harcourt Brace, pp. 114.148.

Lancaster, K. (1971). Consumer Demand: A New Approach. New York: Columbia University Press.

McKenzie, L. (1957). 'Demand theory without a utility index.' Review of Economic Studies, vol. 24, pp. 185-189.

Norris, R.T. (1952). The Theory of Consumer's Demand, rev. ed. New Haven: Yale University Press.

Simon, H.A. (1955). 'A behavioral model of rational choice.' Quarterly Journal of Economics, vol. 69, pp. 99-118.

Theil, H. (1952). 'Qualities, prices and budget enquiries.' Review of Economic Studies, vol. 19, pp. 129-147.