University of the Philippines

Discussion Paper 8503 April 1985

On the Time Consistency of Optimal Plans

by

José Encarnación, Jr.

On the Time Consistency of Optimal Plans

José Encarnación, Jr.
School of Economics, University of the Philippines
Quezon City 3004, Philippines

Abstract

This paper shows that the Kydland and Prescott argument for rules versus discretion--to the effect that optimal policies are time inconsistent--is invalid, points out that the Pollak approach does not solve the time inconsistency problem as originally formulated by Strotz, and indicates the simple solution within the Strotz framework.

Introduction

A problem of time inconsistency is said to arise if in the course of time an optimal plan should need revision, because no longer optimal, although nothing unforseen has occurred. This problem was first analyzed by Strotz (1955-56) in an often cited paper. The "solution" given by Strotz was however considered invalid by Pollak (1968) who proposed an alternative. The inconsistency issue has been studied since then by a number of writers; see e.g. Hammond (1976). If the plan or policy revision is caused by a change in preferences, as in the Strotz model, there would be no reason to be surprised. What is surprising is the claim of Kydland and Prescott (1977) and also by Calvo (1978) that an optimal policy is in general time inconsistent even with unchanged preferences.[1/] We will take a closer look at the question and show that the claim is unwarranted. We also review Pollak's approach and observe that it does not solve the inconsistency problem as formulated by Strotz. Finally, staying within the Strotz framework, we indicate the simple solution.

For easier reference to the original papers we will usually follow their notation and sacrifice notational uniformity in our discussion. The time subscript will be omitted where there is no ambiguity. Dots indicate derivatives with respect to time, which will be understood as right-hand derivatives where necessary. Lastly, unidentified page references will pertain to whichever paper is being discussed.

1. Kydland and Prescott

Let the plan or policy $\pi = (\pi_1, \ldots, \pi_T)$ be a sequence of government policy actions in time periods 1 to T and $x = (x_1, \ldots, x_T)$ the decisions of the private sector. In their 1977 paper Kydland and Prescott (K & P henceforth) assume that

$$x_t = X_t(x_1, \ldots, x_{t-1}; \pi) \qquad t = 1, \ldots, T$$

and that there is a social welfare function $S(x, \pi)$ which is maximized by a plan if it is optimal. Suppose $T = 2$. Assuming an interior solution, for period 2 the optimal plan must satisfy $A + B = 0$ where

$$A = \frac{\partial S}{\partial x_2} \frac{\partial X_2}{\partial \pi_2} + \frac{\partial S}{\partial \pi_2}$$

$$B = \left(\frac{\partial S}{\partial x_1} + \frac{\partial S}{\partial x_2} \frac{\partial X_2}{\partial x_1} \right) \frac{\partial X_1}{\partial \pi_2} .$$

If the plan is also to be consistent, K & P require that it maximize $S(x, \pi)$ with x_1 and π_1 already given, in which case π_2 must satisfy $A = 0$. Accordingly, only if $B = 0$ "would the consistent policy be optimal", and with this observation K & P conclude that "the inconsistency of the optimal policy is easily demonstrated by a two-period example" (p. 476). Evidently K & P mean to say that $B = 0$ is an independent condition, not true in general, so that it would be fortuitous for a policy to be both optimal and consistent.

That the K & P argument is faulty is not hard to see. Suppose an optimal plan $\pi^0 = (\pi_1^0, \pi_2^0)$ is adopted so that the optimal values π_1^0 and x_1^0 are realized in period 1. Consider for period 2 the problem of maximizing $S(x_1^0, x_2, \pi_1^0, \pi_2)$ where $x_2 = X_2(x_1^0; \pi_1^0, \pi_2)$. Obviously π_2^0 satisfies $A = 0$ and, since $A + B = 0$ to begin with, also $B = 0$. It is thus incorrect to consider the latter as an independent requirement. (A similar error is made by Kydland 1977, pp. 312-313, in another context.)

2. Calvo

In Calvo's (1978) model of a monetary economy it is assumed that the demand for money in real terms, m^d, is given by

$$\log m^d = -a\pi^* \quad (a > 0) \tag{1}$$

where π^* is the expected rate of inflation at time t. Let p denote the price level. On the assumption of "perfect foresight", $\pi^* = \pi \equiv \dot{p}/p$ and $m^d = m \equiv M/p$, M being the nominal money stock which the government uses to influence m. Due to other assumptions that we need not go into,

$$M_t = M_0 - \int_0^t p_w x_w \, dw \tag{2}$$

$$x = (m \log m)/a - \dot{m} \tag{3}$$

where x is net real taxes. The time path of m is chosen so as to maximize

$$\int_0^\infty (u(c_t) + v(m_t))e^{-\delta t} \, dt$$

where c is output and δ the discount rate. It is assumed that there exists an amount m^F such that $v'(m^F) = 0$; one might therefore think of m^F as the optimum quantity of money (OQM).

An optimal m path in the model requires $x_t = 0$ and $v'(m_t) = 0$ at $t = 0$, and time consistency requires $m_t = \bar{m}$ (const) for all t. Accordingly from (3), $\log \bar{m} = 0$ or $\bar{m} = 1$, so $v'(1) = 0$ and therefore

$m^F = 1$, which Calvo says is "a condition that cannot be derived from the assumptions of the model. We can then assert that optimal policies will not generally be time consistent" (p. 1419). The suggestion is that "the special case where the OQM is attained at a specific value (=1 in our model)" is in the nature of a fluke (p. 1420).

Like K & P's, the argument is faulty. Suppose an optimal m path, so $m_0 = m^F$, which for consistency implies that $\bar{m} = m^F$. Then M is constant in (2) because $x = 0$ throughout from (3) and the fact that $x_0 = 0$. Thus $\pi^* = \pi = 0$. But (1) in effect makes the unit for measuring m the amount demanded when $\pi^* = 0$, and since that amount is m^F, $m^F = 1$. This supposedly special case is simply the result of an implicit normalization.

In another paper Calvo (1978a) formulates a model of seignorage from money creation where the optimal policy maximizes total discounted seignorage in real terms. Let θ be the growth rate of money supply and assume that θ is bounded by $\theta^- \leq \theta \leq \theta^+$. Time consistency in the model requires the optimal solution to be a constant θ, say θ^*. Calvo finds (p. 513) that if $\theta^- < \theta^* < \theta^+$, then θ^* must be equal to $\theta^S = 1/a$ where a is a parameter in the model. He goes on to find that θ^S cannot be the optimal solution because the latter has $\theta_t = \theta^+$ at $t = 0$. From this argument Calvo draws the conclusion that there "is no time consistent solution" to the optimization problem (p. 514). Here the error is quite transparent. Clearly the correct conclusion is only that θ^* cannot be an interior solution.

This review suggests that in the absence of uncertainty, time inconsistency is not a problem for an optimal plan unless there is a change in preferences, which was the original setting of the inconsistency theme.

3. Strotz

Let the "instantaneous utility function" $u(x)$ be given with $u'(x) > 0$ and $u''(x) < 0$. Denoting consumption at time t by $x(t)$, assume that a person who appraises the possibilities at time τ would decide to maximize the utility functional

$$\phi_\tau = \int_\tau^T w(t - \tau)\, u(x(t))\, dt \qquad 0 \leq \tau \leq T \tag{4}$$

subject to the conditions

$$\int_\tau^T x(t)\, dt = K(\tau) \tag{5}$$

$$K(\tau) = K(0) - \int_0^\tau x(t)\, dt \tag{6}$$

where the initial stock $K(0)$ is given and $\int_0^\tau x(t)dt$ is a historical fact at the decision point τ. It is the discount or weight function $w(t - \tau)$, which makes the discounting of the future shift with τ, that gives the Strotz (1955-56) model its distinctive properties.

Writing $v(x) = u'(x)$ we know that for a maximum it is necessary that

$$\frac{\dot{v}(x(t))}{v(x(t))} = -\frac{\dot{w}(t - \tau)}{w(t - \tau)} \qquad \tau \leq t \leq T \tag{7}$$

so at $t = \tau$ one must have

$$\frac{\dot{v}(x(\tau))}{v(x(\tau))} = -\frac{\dot{w}(0)}{w(0)} = -\dot{w}(0) \tag{8}$$

with the normalization $w(0) = 1$. Let $x^0 = f(t; \tau, K(\tau))$, $\tau \leq t \leq T$, denote the optimal consumption plan given τ and $K(\tau)$. A decision made at $\tau = 0$ determines the entire path $x^0 = f(t; 0, K(0))$. However, another appraisal at $\tau_1 > 0$ will usually have $f(\tau_1; \tau_1, K(\tau_1))$ conflicting with $f(\tau_1; 0, K(0))$, for the new plan satisfies (8) with $\tau = \tau_1$ there but the original one has $\tau = 0$ and $t = \tau_1$ in (7), and $\dot{w}(\tau_1)/w(\tau_1) \neq \dot{w}(0)$ in general.

Strotz observed that a consistent plan (i.e. one that will need no revision at any later decision point) requires the discount function to be of the form $e^{-r(t-\tau)}$, which he calls the "harmony" case. Instead of (4),

$$\phi_\tau^r = \int_\tau^T e^{-r(t-\tau)} u(x(t))\, dt \tag{4'}$$

should then become the maximand. Strotz reasoned further that because of (8), the optimal consistent plan must have $r = -w(0)$ in (4'). We will call this the H solution to the inconsistency problem, which Pollak (1968) has examined.

4. Pollak

Consider the case where decisions are made discretely at n specified points of time τ_i $(i = 0, 1, \ldots, n-1)$ with $\tau_0 = 0$. Replacing τ by τ_i throughout in (4)-(8), call the rewritten equations (4i)-(8i). In the discrete decision case, one has a sequence of n optimization problems $i = 0, 1, \ldots, n-1$ wherein problem i is to maximize (4i) subject to (5i) and (6i) with $\int_0^{\tau_i} x(t)dt$ in (6i) being given by the solutions to the problems preceding i. Thus problem 0 is to be solved first, then problem 1, and so on. We denote the path from 0 to T in the discrete decision case as $x^d = g(t; n)$, so $g(t; n) = f(t; 0, K(0))$ on the interval $[0, \tau_1)$, $g(t; n) = f(t; \tau_1, K(\tau_1))$ on $[\tau_1, \tau_2)$, and so on.

In general the function g is discontinuous at the points $\tau_i \neq 0$ but (8i) holds at all τ_i. We will refer to $x^d = g(t; n)$ as the N path (given n).

Let $\{x(t)\}_\tau^T$ denote a consumption path from τ to T. The S solution proposed by Pollak (1968) gives the path $\{x^s(t)\}_0^T$ that meets the following conditions:

(a) $\{x^s(t)\}_{\tau_i}^T$ is preferred at the decision point τ_i to all other paths $\{x(t)\}_{\tau_i}^T$ satisfying $\int_{\tau_i}^T x(t)\,dt = K(\tau_i)$, $i = 0, 1, \ldots, n-1$;

(b) $K(\tau_i) = K(\tau_{i-1}) - \int_{\tau_{i-1}}^{\tau_i} x^s(t)\,dt$, $i = 1, \ldots, n-1$.

Of course, $\{x^s(t)\}_{\tau_i}^T$ is a subpath of $\{x^s(t)\}_{\tau_{i-1}}^T$. One must therefore find that sequence $\{K(\tau_1), \ldots, K(\tau_{n-1})\}$ such that $\{x^s(t)\}_0^T$ is most preferred among all $\{x(t)\}_0^T$ satisfying (a) and (b) for $i = 1, \ldots, n-1$. The idea is that a "sophisticated" person would take the fact of different future preferences as a constraint in formulating his plan.

In general the S path would be different from the N path that is made by a "naive" person who at each τ_i maximizes ϕ_{τ_i} given $K(\tau_i)$. However in the particular case $u(x) = \log x$, Pollak has shown that the two paths coincide. Suppose now that decisions can be made continuously as τ goes from 0 to T, and let $x^c = h(t) = h(\tau)$ be the limit path of $x^d = g(t; n)$ as $n \to \infty$ and $\max\{\tau_i - \tau_{i-1}\} \to 0$. If $u(x) = \log x$, then $x^c = h(t)$ is the S as well as the N path, (7) becomes

$$\dot{x}(t)/x(t) = \dot{w}(t - \tau)/w(t - \tau), \tag{9}$$

and one gets

$$x(\tau) = K(\tau)/\int_\tau^T w(t - \tau)\,dt \tag{10}$$

on the h path. On the other hand, the H solution would have

$$x(\tau) = K(\tau) / \int_{\tau}^{T} e^{\dot{w}(0)(t-\tau)} dt. \tag{11}$$

Since (10) and (11) give different answers, Pollak concludes that the H solution does not yield the optimal consistent plan.

Explaining where the Strotz approach went wrong, Pollak rightly points out that (8) need not hold on the optimal path.[2/] If it is possible for decisions to be made continuously so that there is a new plan at every t, the h path would consist of only the left endpoints of $x^0 = f(t; \tau, K(\tau))$ as τ increases from 0 to T, i.e. $h(\tau) = f(\tau; \tau, K(\tau))$. However, Strotz's main point was that the optimal plan must use an exponential discount function if consistency is to be assured no matter when another decision is made, and the flaw in Strotz's argument does not of course make the S solution the correct one.

The S solution is consistent only if the decision points τ_i are specified in advance, which would be a gratuitous assumption. Notice that if the timing of decisions were known beforehand, the N path itself would qualify as a plan, there would be no conflict at later decision points, and there would be no inconsistency problem in the first place. There is such a problem precisely because a fresh appraisal can be made at <u>any</u> time. Though the S solution is analytically interesting in itself, it does not solve the problem as formulated by Strotz.[3/]

5. Strotz Simply Revised

As Strotz has shown (pp. 171-172), the only sure way of resolving the inconsistency problem of Section 3 is for a person to adopt, through education and "social pressure" (p. 177), an exponential function $e^{-r(t-\tau)}$ in place of the original $w(t - \tau)$. The H solution does go wrong however, not because it differs from S, but for a much simpler reason. Remembering that (8) is based on (4) as the maximand, there is no reason why one should have $r = - \dot{w}(0)$ after (4') has replaced (4). In fact, putting $r = - \dot{w}(0)$ fails to make use of the one property of w that, in the nature of the case, is most relevant.

A person who analyzes the inconsistency problem would see that his control over the consumption path is effective only until the next decision point, which could be any time after τ_0. His original w gives the specific distribution of weights that he assigns over the planning period, but that distribution has to be replaced by an exponential one if he is to have a consistent plan. Since he controls only the present and not the future, which is the subject of later decisions, the only reasonable thing that he can be expected to do is to give the right weight to the future in making his present decision. Let a satisfy $w(0)/\int_0^T w(t)dt = 1/\int_0^T e^{-at}dt$. Putting $r = a$ in (4'), which we will call the A solution, he gets a consistent plan by maximizing ϕ_0^a since (7) becomes simply

$$\dot{v}(x(t)/v(x(t)) = a \qquad 0 \leq t \leq T \tag{7'}$$

and therefore, as in the H solution, the new plan at any later time is merely a continuation of the original one at τ_0. The difference is that the A solution assigns the correct relative weights to the present and the future in accordance with the original discount function.

Notes

1. As Fischer (1980, p. 232, n. 38) has noted, "The remarkable feature of the Kydland-Prescott result is that it can apparently occur even if the policy authority is maximizing the expected utility of the representative individual and if individual tastes are consistent through time."

2. Curiously Pollak says, for large n in the discrete decision case, that $\dot{v}(x)/v(x)$ is "close to" $-\dot{w}(0)$ "at almost every point, and that the approximation becomes closer" as n gets larger (p. 208). The more precise statement is that for any n, (8i) is true at every τ_i and false almost everywhere.

3. There is an existence and uniqueness issue about the S solution but that is secondary to the more important question of whether it does solve Strotz's inconsistency problem--see Peleg and Yaari (1973) and Goldman (1980).

References

Calvo, G.A. (1978), "On the Time Consistency of Optimal Policy in a Monetary Economy", Econometrica, 46, 1411-1428.

Calvo, G.A. (1978a), "Optimal Seignorage from Money Creation", Journal of Monetary Economics, 4, 503-517.

Fischer, S. (1980), "On Activist Monetary Policy with Rational Expectations", in S. Fischer (ed.), Rational Expectations and Economic Policy (Chicago: University of Chicago Press) 211-235.

Goldman, S.M. (1980), "Consistent Plans", Review of Economic Studies, 47, 533-537.

Hammond, P.J. (1976), "Changing Tastes and Coherent Dynamic Choice", Review of Economic Studies, 43, 159-173.

Kydland, F. (1977), "Equilibrium Solutions in Dynamic Dominant-Player Models", Journal of Economic Theory, 15, 307-324.

Kydland, F.E. and Prescott, E.C. (1977), "Rules Rather than Discretion: The Inconsistency of Optimal Plans", Journal of Political Economy, 85, 473-491.

Peleg, B. and Yaari, M.E. (1973), "On the Existence of a Consistent Course of Action when Tastes are Changing", Review of Economic Studies, 40, 391-401.

Pollak, R.A. (1968), "Consistent Planning", Review of Economic Studies, 35, 201-208.

Strotz, R. H. (1955-56), "Myopia and Inconsistency in Dynamic Utility Maximization", Review of Economic Studies, 23, 165-180.