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A CLASS OF REDUCIBLE DYNAMIC CONTROL PROBLEMS

by

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Abstract

We show that analysis of control problems with a "reduction set," i.e., a subset of the control vector that satisfies certain strong conditions, avoids the "curse of dimensionality." The problem boils down to solving n static equations in n unknowns (the control plus the state variables) and leapfrogs the 2-point boundary value problem that besets more general control problems.

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The application of optimal control to modelling of economic and other phenomena is heavily hamstrung by the "curse of dimensionality." The latter refers to the humungus and almost always hopeless analytical enterprise connected with the solution of the generated Hamiltonian equations. If k is small, say exactly 1, i.e., there is exactly one state variable, which by design characterizes many an application of optimal control, the problem can lend itself to doable but still fairly difficult analysis. Problems with more than one state variable ordinarily lend themselves only to numerical analysis. This naturally involves numerically specifying the relevant parameters and hoping that the set of specifications allows convergence to an equilibrium solution.

In this note, we consider a class of control problems that do not entail tremendous analytical difficulty. We call these problems, reducible dynamic control problems, because the optimal control variables are solved from a system of static equations without confronting the Hamiltonian equation system.

The Optimal Control Approach

Optimal control approach to dynamic optimization problems reduces to the specification and maximization of the Hamiltonian function:

$$H(x, u, \lambda, t) = I(x, u, t) + \lambda f(x, u, t)$$

x being the k-vector state variable, u is the m-vector control variable and λ is the k-vector costate variable, $I(x, u, t)$ is the objective function and $f(x, u, t) = \dot{x}$ is the system of k equations of motion, one for each state variable. For a maximum, it is necessary that (Intriligator, 1971)

$$\text{(i)} \quad \partial H/\partial u = 0$$

$$\text{(ii)} \quad \dot{\lambda} = -\partial H/\partial x$$

all within the relevant time interval. There are m equations in (i), and k equations in (ii). Note that (i) is a function of x, u and λ so that from (i), u can be solved as a function of x and λ. Eq. (ii), together with the k-vector $\dot{x} = f(x, u, t)$, forms a 2k system of differential equations. The u satisfying (i) will behave according as the elements of λ and x behave and their behavior in time $(x(t), \lambda(t))$ forms a solution of the 2-point boundary value problem with $2k$ canonical equations. This is tremendously difficult if not ultimately impossible in its most general form. In the next few pages, we define a class of problems which avoids the "curse of dimensionality."

Reducible Dynamic Control Problems

Definition 1:

A $u^o \subset u$ is called "reduction set" if it satisfies the following properties:

(i) u^o is a k-element set. We now renumber the set u such that numbers 1, 2, ..., k refers to elements of u^o.

(ii) Every $u_i \in u^o$ maps one to one into the elements of the k-vector x. We now let x_i be the x element mapped into by u_i.

(iii) $\dot{x}_i = f_i(x, u, t) = g_i(u_i) f^{io}(x, u/u^o, t) + f^{il}(x, u/u^o, t)$ where u/u^o is the complement of u^o in u.

(iv) $I(x, u, t) = \sum_{i=1}^{k} h(u_i)\, I_i(x, u/u^o, t) \quad u_i \in u^o$

(v) $f^{io}(x, u/u^o, t) = I_i(x, u/u^o, t)$

(vi) $g', h' \neq 0, g'' = h'' = 0$

Definition 2:

A control problem is "reducible" if the set $u^* \cup x^*$ (the union of the optimal control vector and the optimal state vector) solves the system of $m + k$ single period equations: m equations from 1^o maximizing conditions and k from the set of equations of motion with known previous period's state vector.

Remark:

In essence, reducibility chops the multiperiod dynamic problem to a series of single period "static" problems requiring at each time only the knowledge of the previous period's state vector. This latter is always available even in period 1 since the initial period state vector is one boundary constraint.

Proposition:

A dynamic control problem with a "reduction set," u^o, is reducible.

Proof:

The relevant Hamiltonian for this problem if u^o exists is:

$$H = \sum_{i=1}^{k} h_i(u_i)\ I_i(x, u/u^o, t) + \sum_{i=1}^{k} \lambda_i [g_i(u_i)\ I_i\ (x, u/u^o, t) + f^{i1}\ (x, u/u^o, t)]$$

The 1^o conditions are

(i) $\partial H/\partial u_i = h_i' I_i + \lambda_i I_i g_i' = 0 \qquad u_i \in u^o$

(ii) $\partial H/\partial u_k = 0$ $\qquad\qquad u_k \in u/u^o$

(iii) $-\partial H/\partial x = \dot{\lambda}$

Note that in (i) there are k equations; in (ii) there are $m-k$ equations and in (iii) there are again k equations. From (i) we have

$$\lambda_i = -h_i'/g_i' \qquad\qquad \forall_i = 1 \ldots, k$$

Thus $\dot{\lambda}_i = 0 \qquad\qquad \forall_i = 1 \ldots, k$

from property (vi) of a reduction set u^o. Thus all λ_i in (ii) and (iii) are determined and (iii) becomes:

(iii') $\qquad -\partial H/\partial x = 0$

Now (ii) and (iii) are both functions of $(x, u^o, u/u^o)$ or $m+k$ unknowns. $\dot{x}$ is a function of $(x, u^o, u/u^o)$. Define $\dot{x}_i = x_i - x_i^o$ where x_i is the value of state variable x_i at current period and x_i^o is its value at the immediately preceding period. Naturally, x_i^o is always known even as $x(o)$, the initial state vector, is also known at initial period. Now there are k equations in $\dot{x}$ so we have

(iv) $\qquad x_i - x_i^o = q_i(u_i)\, I_i(x, u/u^o, t) + f^{i1}(x, u/u^o, t)$

$$i = 1 \ldots, k$$

The system consisting of (ii), (iii) and (iv) has $m+k$ static equations in $m+k$ static unknowns. These equations are static in the sense that x and u do not need to be solved from a system of differential equations. The elements of the set $(u^* \cup x^*)$ form the $m+k$ unknowns. Q.E.D.

The central idea of this class of reducible optimal control problems is the structure that allows the costate variable to be constant. This structure rests on the existence the reduction subset u^o. With the costate variables constant, we need not solve the two-point boundary value problem with $2k$ differential equations. Instead, by knowing the last period's state vector (or the initial state vector when period 1 is in question), we can solve the $m+k$ static equations in $m+k$ variables (u^*, x^*). The "curse of dimensionality" is avoided.

Reinvestment and Rent Dues: An Example

Consider a firm maximizing accumulated dividends defined as $(1-s)\pi^*$ where π^* is the profit at any given point in time. π^* is a concave function of operating capital C. Now operating capital grows through reinvestment (for the moment, no borrowing is allowed), i.e.,

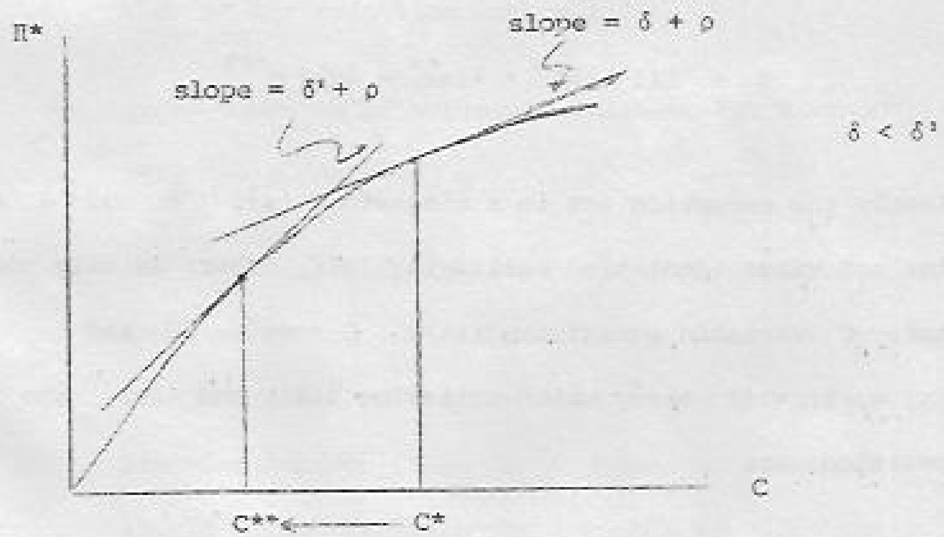
$$\dot{C} = s\pi^* - \delta C$$

(3) $C - C^o = s\pi^* - \delta C$

Since $\partial\pi^*/\partial C$ is not a function of s, we get optimal C^* from here. With optimal C^* we get optimal s^* from (3).

From (2) it is clear that optimal operating capital C^* falls with a rise in δ since π^* is a nondecreasing concave function of C. This is shown in Fig. 1.

Fig. 1



The decrease in C^* forces a drop in s^*. Thus a rise in rent exactions (by say the political bosses) decreases the internal savings rate by firms. Firms would then tend to grow slower under a situation of heavier rent dues. (For application of the result to aggressive rent-seeking see Fabella, 1985).

where δ is the proportion of operating capital used to service rent dues. Note that rent-dues δC does not improve the profit outlook. The problem for the firm is

$$\max \int (1-s)\pi^* e^{-\rho t} dt$$

$$\text{s.t.} \quad \dot{C} = s\pi^* - \delta C$$

The Hamiltonian is

$$H = \{(1-s)\pi^* + \lambda(s\pi^* - \delta C)\} e^{-\rho t}$$

Clearly the reduction set is a singleton $\{s\}$. h and g are all plus and minus identities satisfying (vi). There is only one state C variable satisfying (ii). $\dot{C} = s\pi^* - \delta C$ and $I(x, u, t) = (1-s)\pi^*$ which satisfies (iii) and (iv). The 1^o conditions are

$$(1) \quad \partial H/\partial s = -\pi^* + \lambda\pi^* = 0 \rightarrow \lambda = 1, \ \dot{\lambda} = 0$$

$$(2) \quad -\partial H/\partial C = \left((1-s)\frac{\partial\pi^*}{\partial C} + s\frac{\partial\pi^*}{\partial C} - \delta\right) e^{-\rho t} = \frac{d}{dt} e^{-\rho t}\lambda = -e^{-\rho t}\rho$$

$$\text{or} \qquad \partial\pi^*/\partial C = \rho + \delta$$

With previous period's state variable C^o known, let $\dot{C} = C - C^o$, so that

Summary

In the preceding pages we have formally defined the structure of a class of control problem that allows reducibility, i.e., allows the optimal control vector and the state vector to be solved from a system of static equations once the last period's state vector is known. The peculiarity is generated by the existence of a reduction subset u^o of the control vector u that satisfies certain simplifying assumptions:

(a) the cardinality of the state variable set x equals that of the reduction set u^o;

(b) every equation of motion is a linear function of only one element of u^o;

(c) the objective function is a linear function of all the elements of u^o. This allows the shadow prices to be constant across the plan period and allows the planning horizon to be chopped down to a sequence of single period decision problems as we allow the equation of motion $\dot{x} = x - x^o$ where x^o is last period's state vector.

It may be difficult to force certain problems into the mold that allows reducibility. We give an example of one involving rent dues and reinvestment. Still the analytical difficulty posed by a full blown non-reducible control problem is immense and may allow only numerical simulation.

References

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