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AN EXTENSION OF FAIRNESS IN NASH: A REVISION

by

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Abstract

The Symmetry axiom which introduces a sense of fairness in the Nash axiom system for the 2-person allocation game does not apply to more general allowable utility spaces. We develop the concept of subsymmetric sets and replace the Symmetry axiom with the "Minimized Inequality axiom" which includes the Symmetry axiom. We show that the Nash bargaining solution is robust against this generalization and by Nash's own uniqueness result, still remains unique.

AN EXTENSION OF FAIRNESS IN NASH: A REVISION*

by R.V. Fabella

In an attempt to define a suitable choice rule which picks the solution to a two-person allocation game, Nash (1953) considered as desirable six axioms, viz., individual rationality, feasibility, Pareto optimality, independence of irrelevant alternatives á la Nash, independence of linear transformation, and symmetry. The first three are nondebatable. The fourth, the way Nash defined it (everything will be formally defined later) leaves little room for discussion. It just says that if an allocation solves the game with feasible set T, it also solves a game with a smaller feasible set $T' \subset T$ if the allocation is still in T'. The fifth axiom is a concession to the Von Neumann-Morgenstern result that linear transforms of VN-M utility functions numerically represent the same underlying preference relations (J. Von Neumann and O. Morgenstern, 1947). The sixth, the axiom of symmetry, is interesting but presents some problems. Nash wanted the idea of "fairness" imbedded in his axiom system. No one quarrels with "fairness" at least in principle. It is the way he introduced it that, we argue, can stand improvement. To properly discuss this, let us first set down the formal structure of the model.

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Let T be the choice set. Let S be the 2-dimensional utility space into which every point in T is mapped. If we endow each of the two players with a VN-M utility functions, S is a closed, bounded from above and convex subset of the two-dimensional Euclidean space (Owen, 1968). Our problem then is how to pick a point in S. In actual allocation games, it is commonplace to find that the more one player gets, the less the other will have. How little will a player accept as a price for cooperation? It is natural to assume that a player will set as a price of cooperation that amount that he can obtain by unilateral action regardless of the maneuvers of the other. In game theory, this is the maximin value of the game for the player. Let $U^* = (u^*, v^*) \in S$, $i = 1, 2$, be the vector of maximin values for the 2 players. Note that the maximin values are in utility terms. The version of the game with U^* and S is written as (U^*, S). A solution, $\bar{U} = (\bar{u}, \bar{v})$ to the allocation game is that picked by a choice rule r, i.e.,

$$r\ (S, U^*) = \bar{U}$$

The Symmetry axiom of Nash goes as follows:

If S satisfies the following:

(a) Symmetry, i.e., $(u, v) \in S \leftrightarrow (v, u) \in S$

(b) $u^* = v^*$

Then $\bar{u} = \bar{v}$

This is a conditional axiom. If S is symmetric, and the maximin values are identical for the two players, the solution on S should exhibit welfare equity, i.e., $\bar{u} = \bar{v}$. Thus fairness means that equal initial positions imply equal final positions. It is the applicability of the axiom to certain configurations of S that comes to mind.

Figures (1a) and (1b) give configurations to which the Symmetry axiom applies since they are symmetric around the 45° line. Figures (2a) and (2b) are nonsymmetric and thus do not allow the Symmetry axiom to apply.

Fig. 1

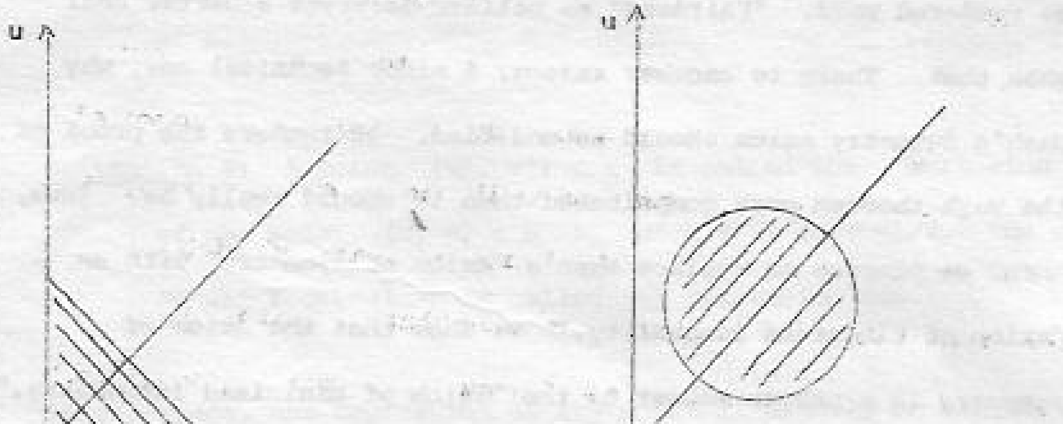
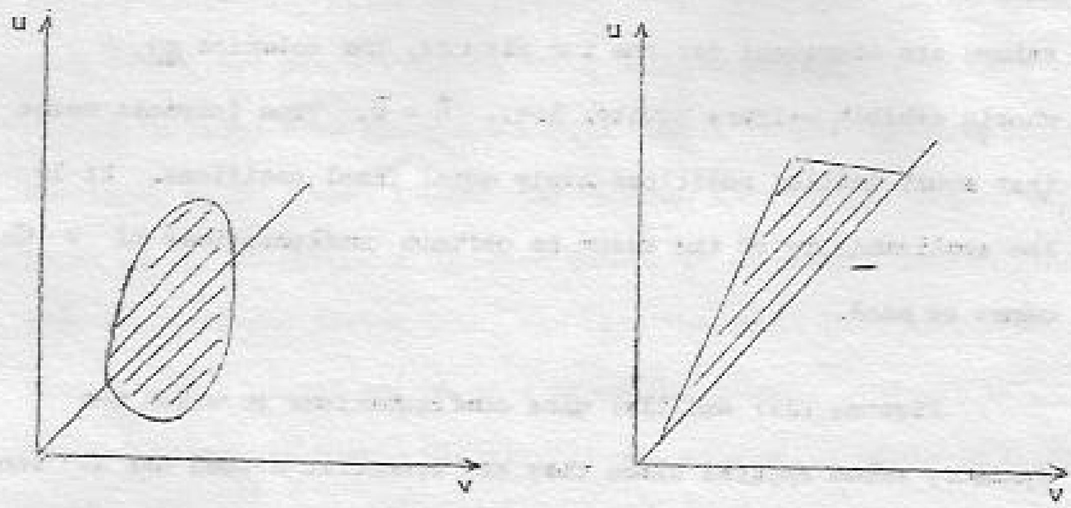


Fig. 2



This is no problem were there some stricture for S to be always symmetric. In fact, that can only happen by the remotest of chances. The intention is good but the implementation can very well be empty. Being a condition within a condition, it can easily be rendered void. "Fairness" we believe deserves a better deal than that. There is another reason, a minor technical one, why Nash's Symmetry axiom should be modified. It renders the proof of the Nash theorem more complicated than it should really be. Thus, here, we propose to replace Nash's "axiom of symmetry" with an "axiom of minimized inequality." We show that the axiom of symmetry is a proper subset of the "axiom of minimized inequality." To do this, we need to develop the concept and the properties of subsymmetric sets.

2. Symmetric and Subsymmetric Sets

In the following, we will develop the machinery which we will need to proceed in the next section. This will involve properties of symmetric and subsymmetric sets.

<u>Definition 1</u>: A set S is symmetric if $(u, v) \in S \leftrightarrow (v, u) \in S$.

<u>Definition 2</u>: $P(S)$ is the Pareto-efficient subset of set S, i.e., if $(u', v') \in P(S)$, $\tilde{\exists}\ (u, v) \in S$, $u > u'$ and $v \geq v'$ or $u \geq u'$ and $v > v'$.

<u>Definition 3</u>: A set S is subsymmetric if we can construct a symmetric and convex set SS such that $S \subseteq SS$ and $P(S) \subseteq P(SS)$. We call SS the shadow symmetric superset of S.

<u>Definition 4</u>: A point $(u^\circ, v^\circ) \in S$ is called the "regression" of the point $(u, v) \in S$ if $u^\circ = v^\circ = (u + v)/2$. The set of all regressions is called the regression set.

In two-space, the regression of any point is clearly on the 45° line. We prove the following:

<u>Lemma 1</u>: If S is symmetric and convex, then $(u, v) \in S \leftrightarrow (u^\circ, v^\circ) \in S$.

<u>Proof</u>: By symmetry, $(u, v) \in S \leftrightarrow (v, u) \in S$. If $u = v$, the claim follows. Let $u > v$. Find α, $0 \leq \alpha \leq 1$, such that $\alpha u + (1-\alpha)v = (u+v)/2$. Apply α to (u, v) and (v, u) so that

$$\alpha(u, v) + (1-\alpha)(v, u) = (\alpha u + (1-\alpha)v,\ \alpha v + (1-\alpha)u) \in S$$

by convexity. Let $\beta = \frac{1}{2}$. Apply β to the last point and its interchange in S to get

$$\left(\frac{\alpha}{2}u + \frac{(1-\alpha)v}{2} + \frac{\alpha v}{2} + \frac{(1-\alpha)u}{2},\ \frac{\alpha v}{2} + \frac{(1-\alpha)}{2}u + \frac{\alpha}{2}u + \frac{1-\alpha}{2}v\right) =$$

$$\left(\frac{u+v}{2}, \frac{u+v}{2}\right) \in S \qquad \text{Q.E.D.}$$

The regression set of a symmetric and convex set S in 2-space is a portion of the 45° line.

<u>Definition 5</u>: Let θ be the angle between the vertical axis and any line tangent (where well-defined) to S on $P(S)$. Let θ° be the angle between the vertical axis and a line orthogonal to the 45° line.

<u>Lemma 2</u>: A line connecting (u, v) and (v, u), i.e., $\lambda(u, v) + (1-\lambda)(v, u)$, $\forall\lambda$, $1 > \lambda > 0$, is an orthogonal projection on the 45° line and contains the regression of (u, v).

<u>Proof</u>: Let $u > v$. Consider point (v, v). Subtract this from (u, v) and (v, u) to get $(u - v, 0)$ and $(0, u - v)$. A line connecting the latter two points is orthogonal to the 45° line. By geometry, a line dropped from the intersection of 45° line with the connecting line and orthogonal to the horizontal axis bisects the distance $(0, u - v)$. Likewise, a line from the intersection orthogonal to the vertical axis bisects $(u - v, 0)$. Thus the intersection has coordinates $\left(\frac{u-v}{2}, \frac{u-v}{2}\right)$, the regression of (u, v). Q.E.D.

<u>Lemma 3</u>: If S is symmetric and convex then $\theta \geq \theta°$ above the 45° line and $\theta \leq \theta°$ below the 45° line.

<u>Proof</u>: Suppose $\theta < \theta°$ above the 45° line. Then there exists points (u, v) and (u', v') in $P(S)$ such that $(u, v) + (1 - \lambda)\ (v, u) > (u', v')$ for some λ, $1 > \lambda > 0$. Clearly, then, S cannot be convex since $(v, u) \in S$ and $(v', u') \in S$ and for some $\lambda, \lambda°$, $\lambda°(u, v) + (1 - \lambda°)\ (v, u) \not\in S$. Q.E.D.

<u>Definition 6</u>: $\theta \geq \theta°$ above 45° line and $\theta \leq \theta°$ below 45° line together form the slope rule.

<u>Lemma 4</u>: Along an orthogonal projection on the 45° line, $g(u, v) = (u - u^*)\ (v - v^*)$ with $u^* = v^*$, is monotonic increasing as (u, v) approaches the 45° line.

Proof: Take (u, v) with $u > v$. Consider the point

$(u-\varepsilon, v+\varepsilon)$, $\varepsilon > o$. Clearly, $(u-\varepsilon, v+\varepsilon)$ is on a

line orthogonal to the 45° line. Now $g(u-\varepsilon, v+\varepsilon) =$

$$uv + \varepsilon(u-v) - uu^* - vv^* + u^*v^* - \varepsilon^2 + u^* - \varepsilon v^* = uv + \varepsilon(u-v)$$

$$+ u^*v^* - uu^* - vv^* - \varepsilon^2$$

$$\lim_{\varepsilon \to o} g(u-\varepsilon, v+\varepsilon) = uv + \varepsilon(u-v) + u^*v^* - uv^* - vv^* > g(u, v) = uv + u^*v^* - uu^* - vv^*$$

This is true for all points on an orthogonal projection on

the 45° line. Q.E.D.

Lemma 5: A symmetric and convex set S is also subsymmetric but

not vice-versa.

Proof: Let S be symmetric and convex. Let S be itself the

shadow symmetric superset. Clearly, $S \subseteq S$ and $P(S) \subseteq P(S)$.

The converse is obvious. Q.E.D.

3. Fairness Extended

We propose the following set of axioms:

N_1 (Individual Rationality): $(\bar{u}, \bar{v}) > (u^*, v^*)$

N_2 (Feasibility): $(\bar{u}, \bar{v}) \varepsilon S$

N_3 (Pareto): If $(u, v) \varepsilon S$ and $(u, v) > (\bar{u}, \bar{v})$

then $u = \bar{u}$ and $v = \bar{v}$

N_4 (IIA à la Nash): If $(\bar{u}, \bar{v}) \in T \subset S$ and

$$(\bar{u}, \bar{v}) = r\,(S, U^*), \text{ then } (\bar{u}, \bar{v}) = r\,(T, U^*)$$

N_5 (ILT): If $r\,(S, U^*) = (\bar{u}, \bar{v})$, then

$$r\,(T, \alpha_1 u^* + \beta_1, \alpha_2 v^* + \beta_2) = (\alpha_1 \bar{U} + \beta_1, \alpha_2 \bar{v} + \beta_2)$$

$$\alpha_i > 0,\ \beta_i \in E$$

N_6 (Minimized Inequality): Let $u^* = v^*$ and let

$r(S, U^*) = (\bar{u}, \bar{v})$. If S is subsymmetric, then

$$E(\bar{u}, \bar{v}) < E(u, v) \qquad \forall (u, v) \in P(S)$$

Let us first discuss N_6 as a "fairness" condition. Specifically we focus on how it relates to Nash's Symmetry Axiom. First of all the "Minimized Inequality Axiom" applies to both symmetric and nonsymmetric sets and is thus more general. We show that the MIA includes the Symmetry axiom.

<u>Proposition 1</u>: If point $(\bar{u}, \bar{v})$ satisfies the Symmetry axiom, it satisfies MIA but not vice-versa.

<u>Proof</u>: Let S be symmetric and let $u^* = v^*$. By the Symmetry axiom, $\bar{u} = \bar{v}$ or that the solution should be on the 45° line. But this always satisfies MIA since by Lemma 5, S is subsymmetric and $E(\bar{u}, \bar{v}) = 0$, i.e., the Euclidean distance of an orthogonal line from the point to the 45° line is minimized, and $E(u, v) \geq 0$, $\forall (u, v) \in S$. Thus a solution that satisfies the Symmetry axiom satisfies MIA. If S is

subsymmetric and nonsymmetric, the Symmetry axiom does not apply but MIA does. Q.E.D.

Remark 1: Thus MIA is more general than the Symmetry axiom. The problem with strengthening an axiom in an axiom system is that you run the risk of derailing some existence result down the line. We shall see that this does not happen here and that an arbitration rule exists that satisfies the system.

Remark 2: MIA allows inequality to exist if S° is not symmetric. What it does not allow is unnecessary inequality. It cannot supersede the Pareto axiom but once this is satisfied, the field is its own. Now there is still an open philosophical question here. Why should Pareto take precedence over equity? We do not address this question here.

Remark 3: The central problem connected with generalizing an axiom in a system with a singleton for a universe is existence. If the universe is nonempty after the extension, the uniqueness property, if shown for the original set of axioms, remains.

Before we prove the main result, we first prove the following.

Lemma 6: $(\bar{u}, \bar{v})$ satisfies MIA if $(\bar{u}, \bar{v})$ maximizes $g(u, v) = (u - u^*)(v - v^*)$.

Proof: By definition $S \subset SS$ and $P(S) \subset P(SS)$ where SS is symmetric and convex. SS contains its regression set by Lemma 1. Connect $(\bar{u}, \bar{v})$ with its corresponding regression on 45° line. This line is orthogonal to the 45° line since by Lemma 2, this line extended connects $(\bar{u}, \bar{v})$ and $(\bar{v}, \bar{u})$. Clearly, $l \subset SS$ since SS is convex. Four possibilities arise:

(a) $\exists (u, v) \in P(S)$, $(u, v) \neq (\bar{u}, \bar{v})$ and $(u, v) \in l$. Then $g(\bar{u}, \bar{v}) < g(u, v)$ by Lemma 4, and $(\bar{u}, \bar{v})$ is not maximal, a contradiction.

(b) $\exists (u, v) \in P(S)$ and $(u', v') \in l$ such that $(u, v) < (u', v')$ and $E(u, v) < E(\bar{u}, \bar{v})$. This means that for some point in $P(S)$, the slope rule is violated and SS is not convex, a contradiction.

(c) $\exists (u, v) \in P(S)$ such that $(u, v) > (u', v') \in l$. Then $g(u, v) > g(u', v') > g(\bar{u}, \bar{v})$ contradicting the maximality of $(\bar{u}, \bar{v})$.

(d) The only possibility left is that $E(\bar{u}, \bar{v}) < E(u, v)$ $\forall (u, v) \in P(S)$. Q.E.D.

The role of the shadow symmetric superset in this case is crucial. We now go to the principal result. Consider the Nash arbitration function:

$$g(u, v) = (u - v^*)(v - v^*)$$

We prove the following:

<u>Proposition 2:</u> (Existence)

Let $(\bar{u}, \bar{v}) \varepsilon S$ be such that $(\bar{u}, \bar{v}) > (u^*, v^*)$ and $\forall (u, v) \varepsilon S$

$$g(\bar{u}, \bar{v}) \geq g(u, v)$$

Then $(\bar{u}, \bar{v})$ satisfies $N_1 - N_6$.

<u>Proof:</u> g is clearly continuous and S is compact so g attains a maximum in S and $(\bar{u}, \bar{v})$ is well-defined. It satisfies N_1 and N_2 by construction. It satisfies N_3 because if $\exists (u, v) \varepsilon S$ such that $(u, v) > (\bar{u}, \bar{v})$, then $g(u, v) > g(\bar{u}, \bar{v})$ contradicting the maximality of $(\bar{u}, \bar{v})$. Let $T \subset S$ and let $(\bar{u}, \bar{v}) \varepsilon T$. Let $(u, v) \varepsilon T$ $g(u, v) \geq g(\bar{u}, \bar{v})$. Then $(\bar{u}, \bar{v})$ is not maximal over S. Thus N_4 is satisfied. Let S' be $\alpha S + \beta$ where α & β are vectors we defined. Then

$$g(\alpha_1 u + \beta_1 ,\ \alpha_2 v + \beta_2) = (\alpha_1 u + \beta_1 - \alpha_1 u^* - \beta_1)$$

$$(\alpha_2 v + \beta_2 - \alpha_2 v^* - \beta_2) = \alpha_1 \alpha_2\ g(u, v)$$

$$\leq \alpha_1 \alpha_2\ g(\bar{u}, \bar{v})$$

Then N_5 is satisfied. That it satisfies MIA or N_6 is given by Lemma 6. Q.E.D.

<u>Remark 4</u>: What we have shown is that the Nash arbitration rule satisfies a more general set of desirables than the set of Nash axioms. The stroke of genius of Nash is to figure out just what is minimal sufficient for his uniqueness result.

<u>Remark 5</u>: Technically, what we avoided by using MIA instead of the Symmetry axiom is a uniqueness lemma which Nash uses to show contradiction with respect to N_6.

<u>Proposition 3</u>: The solution $(\bar{u}, \bar{v})$ that satisfies $N_1 - N_6$ is unique.

<u>Proof</u>: Since MIA or N_6 is a strengthening of Nash's Symmetry axiom and since the axiom system with MIA replaced by the Symmetry axiom allows only a unique solution by the Nash bargaining theorem, $N_1 - N_6$ guarantees a unique solution. Q.E.D.

Summary

In the foregoing, we substituted the "Minimized Inequality axiom" for Nash's "Symmetry axiom" so that the "fairness" notion would apply to more general configurations of the feasible utility space. We developed the concept and generated some properties of subsymmetric sets. We then showed that the "Minimized Inequality axiom" is more general than the "Symmetry axiom." We showed further that the Nash arbitration solution also satisfies the new set of axioms. Finally, it is clear that the solution should be unique, if it exists, since it is unique for a weaker set of axioms. The generalization of these results to n players is straightforward.

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