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Portfolio Choice and Risk

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Abstract. Risk aversion and the riskiness of assets are interpreted in terms of a model of portfolio choice where the maximand is conditional on the probability of satisfying a minimum constraint on the future value of the portfolio. It is a consequence that the riskiness of the average asset in the portfolio increases with wealth, and when expected value is the maximand, low-risk low-return assets are inferior goods. The model also gives straightforward explanations of the Allais paradox and other puzzling patterns of choice under risk.

1. Introduction

According to Telser (1955), a person maximizes the expected value of his portfolio subject to the condition that the probability of a "disaster" does not exceed some critical level. According to the earlier Cramer-Roy criterion, the probability is minimized (Roy 1952). The latter gives a plausible explanation of behavior in some situations, but it is too limited in scope to be more than a special hypothesis. On the other hand, the Telser criterion is incomplete since it does not cover the case where the required condition is not satisfied.

In this paper we formulate a simple model of portfolio choice where the criterion function to be maximized is conditional: (i) the probability π of satisfying a minimum requirement on the future value of the portfolio--the minimum may be much better than what would normally be called a disastrous outcome--if π is less than some acceptable level π^* ; (ii) expected value if $\pi \geq \pi^*$. The concept of an acceptable probability is familiar from the standard statistical practice of taking some probability level as good enough for the purpose, say, of detecting batches of items containing more than a certain fraction of defectives. The classical Neyman-Pearson rule is similar: a specified probability of avoiding a Type I error is considered acceptable. Terms like "reasonable risk" and "acceptable risk" in common use carry the same idea.

In ordinary language, a low-risk low-return asset has large chances of yielding a relatively low yet reasonable return and small chances of either a much lower or higher one. At the other extreme, a high-risk high-return asset is likely to have a very low future value but it also has reasonable

chances of a relatively high one. (High-risk low-return assets would not sell, and any low-risk asset which is high-return to begin with cannot keep its status long--its price would be bid up.) It is a direct consequence of the portfolio model in Section 2 that a wealthier person will have riskier and therefore higher yielding assets.

Using an extension of the portfolio model, Section 3 gives strikingly simple resolutions of the Allais paradox and other puzzling choices under risk that have figured prominently in the literature. Section 4 makes some concluding remarks.

2. A Model of Portfolio Choice

Let $p > 0$ be the given price vector and $v \geq 0$ the random vector of future real values of n assets. (No asset is absolutely riskless with future values in real terms.) Suppose a person has an investment budget constraint $p'x \leq A$, $x \geq 0$. He wishes to maximize $E(v)'x$, where $E(v) = (E(v_1), \dots, E(v_n))'$ and $E(v_i)$ is the expected value of v_i , subject to the condition that the probability $P\{v'x \geq B^*\}$ is at least $\pi^* > 0$. $B^* > 0$ is a minimum or floor requirement on the future value of the portfolio which he wants satisfied with probability π^* or better, and the parameter π^* reflects his attitude towards risk: a higher π^* means greater risk aversion. Let

$$\pi(b) = P\{v \geq b\} = \int_{b_n}^{\infty} \dots \int_{b_1}^{\infty} f(v) dv_1 \dots dv_n$$

where $f(v)$ is the joint probability density of v . This would be a subjective estimate since that is what matters for a decision even though it would of course be influenced by past observation.

We assume that: (i) π is a quasi-concave function, which has the plausible implication of nondecreasing rates of substitution among the elements

of b at a constant level of $\pi(b)$; (ii) $\pi_i = \partial\pi/\partial b_i = 0$ only if b_i is so high that $\int_{b_i}^{\infty} f^i(v_i) dv_i = 0$, where $f^i(v_i)$ is the density of v_i , in which case $\pi(b) = 0$. Thus in what would be interest, $\pi_i < 0$.

The V problem

It is obvious that $P\{v'x \geq B^*\} \geq \pi^*$ if and only if for some b , $b'x \geq B^*$ and $\pi(b) \geq \pi^*$. Accordingly, suppose that we have the following V problem:

Maximize $V = E(v)'x$ subject to

$$g^1(z) = A - p'x \geq 0 \quad (1)$$

$$g^2(z) = b'x - B^* \geq 0 \quad (2)$$

$$g^3(z) = \pi(b) - \pi^* \geq 0 \quad (3)$$

$$z = (x, b) \geq 0. \quad (4)$$

If $g(z) = (g^1(z), g^2(z), g^3(z)) > 0$ for some $z \geq 0$ (which is generally the case when the constraint set defined by (1)-(4) is nonempty) the Kuhn-Tucker conditions (1)-(12), where we write $E_i = E(v_i)$, are necessary and sufficient for $x = x^0$ to solve the problem (see Appendix A).

$$E_i - \alpha p_i + \beta b_i \leq 0 \quad (5)$$

$$(E_i - \alpha p_i + \beta b_i)x_i = 0 \quad (6)$$

$$\beta x_i + \gamma \pi_i \leq 0 \quad (7)$$

$$(\beta x_i + \gamma \pi_i)b_i = 0 \quad (8)$$

$$(A - p'x)\alpha = 0 \quad (9)$$

$$(b'x - B^*)\beta = 0 \quad (10)$$

$$(\pi(b) - \pi^*)\gamma = 0 \quad (11)$$

$$(\alpha, \beta, \gamma) \geq 0 \quad (12)$$

$i = 1, \dots, n$. (If it is clear from the context, 0 superscripts to denote solution values of the variables will usually be omitted.)

Since $E_i > 0$ for some i , $\alpha > 0$ in (5) so (1) is a binding constraint from (9) and therefore $x_i > 0$ for some i . Hence from (7), $\beta > 0$ implies $\gamma > 0$. Condition (2) requires $b_i > 0$ and $x_i > 0$ for some i , so that in (8), $\gamma > 0$ implies $\beta > 0$ unless $\pi_i = 0$. But the latter implies $\pi(b) = 0$ which violates (3). Therefore, both (2) and (3) are binding or both are not. In the latter case the problem reduces to the uninteresting one of maximizing V subject only to the budget constraint. We shall therefore take $\beta, \gamma > 0$.

If $x_i = 0$, α holds in (7) and $b_i = 0$ in (8). Thus $b_i > 0$ implies $x_i > 0$. From (5) and (6),

$$x_i = 0 \text{ if } (E_i + \beta b_i)/p_i < \alpha \quad (13)$$

$$(E_i + \beta b_i)/p_i = \alpha \text{ if } x_i > 0. \quad (14)$$

An asset with a relatively low E_i/p_i could be bought provided it can "compensate" by having a sufficiently high b_i/p_i , and we would call it a low-risk low-return asset.

We propose therefore to measure the riskiness of an asset in terms of its b_i/p_i --it is more risky if the latter is lower. This view, which does not conflict with the idea that an asset is more risky if the variance of its future value is larger, agrees with the commonsense notion that an asset's riskiness is price dependent. Clearly, one would say that less risk attaches to buying an asset if its price were lower. From (13) and (14), an asset will not be bought if for the same expected value (per dollar's worth, in all that follows where appropriate) it is more risky than another, or if for the same risk its expected value is less. Note also that in (11), the components of b would have to be lower on average if π^* were higher. Greater risk

aversion thus implies an evaluation of assets that says they are more risky.

Variations in p , A , B^* and π^* generate the demand function $x = \psi(p, A, B^*, \pi^*)$. If the list of $b_i > 0$ and $x_i > 0$ remains the same, one has the Slutsky-type equation

$$\partial x_i / \partial p_j = \partial x_i / \partial p_j \Big|_{R=\text{const}} - x_j \partial x_i / \partial A \quad (15)$$

where $R = (B, \pi, V)$, $B = b'x$, and $\pi = \pi(b)$ (see Appendix B). One should expect, however, that if the arguments of the demand function vary sufficiently, the list of assets in the portfolio would change.

Suppose $b_i, b_j > 0$ and $E_i/p_i < E_j/p_j$ so that $b_i/p_i > b_j/p_j$ from (14). Let $dA > 0$. If an extra dollar were spent only on j , V would be larger by the amount $E_j/p_j < \alpha$ which cannot be optimal since $\partial V / \partial A = \alpha$. (As usual, the Lagrange multiplier α gives the sensitivity of the solution value $V = V^0$ of the objective function to the corresponding constraint A .) On the other hand, if x_j were increased by one unit and x_i reduced by $c = b_j/b_i$ to maintain the B^* constraint, there would be an additional outlay of $p_j - cp_i > 0$ but V would be larger by $E_j - cE_i > 0$. An extra dollar allocated this way raises V by the amount $(E_j - cE_i)/(p_j - cp_i) = \alpha$. This means that with higher A , not only would the average asset have a higher expected return as it should, there would be reductions in the quantities of lower yielding, less risky assets. The latter are thus inferior goods when one has a V objective, and at a sufficiently high A , they would be completely displaced from the portfolio. (One also expects new assets to enter the portfolio at some high A ; see Appendix C.)

Let $dB^* < 0$ instead, and put $dx_i = (p_j/p_i)dx_j$ to satisfy the A constraint. Since $db_i = db_j = 0$ in

$$dB = b_i dx_i + x_i db_i + b_j dx_j + x_j db_j \quad (16)$$

we have $(b_j/p_j - b_i/p_i)p_j dx_j < 0$ so that $dx_j > 0$ and $dx_i < 0$ as in the case of $dA > 0$.

Finally, let $d\pi^* < 0$ so that $db_i = -(\pi_j/\pi_i)db_j + a$ where $a > 0$.

Putting $dB = 0$ in (16), $(b_j/p_j - b_i/p_i)p_j dx_j + (x_j/\pi_j - x_i/\pi_i)\pi_j db_j = -ax_i$.

Using (8), $x_j/\pi_j = x_i/\pi_i$ and again, $dx_j > 0$ and $dx_i < 0$.

We have treated B^* as a parameter independent of A for the sake of convenience, but one can reasonably expect that $B^* = \phi(A)$ where $\phi'(A) \geq 0$. At the same time, one must require

$$\partial V/\partial A + \partial V/\partial B^* \cdot \phi'(A) > 0 \quad (17)$$

for otherwise, an increase in A would not raise V . We will assume that

$$\phi'(A)A/\phi(A) < 1 \quad (18)$$

which is plausible and gives (17). $(E(v))'x = \alpha p'x - \beta b'x > 0$ from (6) so

that $\alpha A > \beta \phi(A)$. Noting that $\alpha = \partial V/\partial A$ and $\beta = \partial V/\partial (-B^*)$, we get (17)

using (18).) While a higher A thus increases B^* as well, the net qualitative effect is that of A . Collecting the previous results, we therefore have

Proposition 1. If A is higher or π^* lower in the V problem, not only is the expected value per dollar of the portfolio higher, less risky and lower yielding assets are displaced by riskier ones.

The π problem

If the V problem is not solvable, the π^* constraint has to be relaxed since A is a datum and $B^* = \phi(A)$, and one can only maximize $\pi = \pi(b)$ subject to (1), (2) and (4). For a solution the corresponding Kuhn-Tucker conditions are necessary and sufficient, of which we need state only the following:

$$-\mu p_i + v b_i \leq 0 \quad (5')$$

$$(-\mu p_i + v b_i) x_i = 0 \quad (6')$$

$i = 1, \dots, n$, where μ and v are the Lagrange multipliers associated with (1) and (2) respectively. It is straightforward to show that $\mu > 0$, $\mu > 0$ if and only if $v > 0$, and $x_i > 0$ if and only if $b_i > 0$. A Slutsky-type equation can also be derived which has $(B, \pi) = \text{const}$ in place of $(B, \pi, V) = \text{const}$ in (15).

From (5') and (6'), all assets included in the portfolio are equally risky, and riskier ones are excluded. If A is higher, the solution value $\pi = \pi^0$ should be higher, which requires

$$\partial \pi / \partial A + \partial \pi / \partial B^* \cdot \phi'(A) > 0. \quad (17')$$

But under (18) we also have (17'). (From (6'), $\mu A = v \phi(A)$. Since $\mu = \partial \pi / \partial A$ and $v = \partial \pi / \partial (-B^*)$, (17') follows.) Thus at a high enough A , $\pi \geq \pi^*$ and the π criterion is replaced by the V criterion. We therefore have

Proposition 2. In the π problem, all assets in the portfolio have the same risk, and the risk level increases with A . At some value of A , the π problem is replaced by the V problem but (Proposition 1) the riskiness of the average asset still increases with A .

Discussion

The idea behind portfolio diversification is to reduce the chances of getting a low outcome. This consideration is directly incorporated in the model described above by a condition imposed on the probability of meeting a floor requirement on the future value of the portfolio. At a low level of A , the probability is maximized. When the probability level π^* is attainable, the objective $\pi = \max$ is converted into a constraint $\pi \geq \pi^*$ for the maximization of expected value. At that stage, lower yielding and less risky assets--which are defined in terms of the V problem--are displaced by higher yielding and riskier ones. (There could not be such a displacement in the $\pi = \max$ stage since one has to have the lower yielding assets to begin with before they can be displaced.)

The features of the model and its implications seem consistent with general knowledge. We can think of a person's π^* level as an individual psychological parameter independent of A . Some persons, e.g. the classical entrepreneur, appear naturally less risk-averse than others. Proposition 1 implies that wealth and a low degree of risk aversion are substitutes for the generation of more wealth. Entrepreneurs with low π^* 's go into high-risk high-return ventures with relatively small amounts of capital and the successful ones make large gains. It is also a common saying that a wealthier person can afford to take more risks. Proposition 1 implies that he takes not only absolutely more (which might be expected) but also relatively more. Perhaps the most interesting implication is that the expected value of the portfolio increases more than in proportion to A . This means that the multiplicative power of wealth is greater with more wealth, which can be tested against the facts.

3. An Extension

In this section we extend the portfolio model to provide a framework for explaining some interesting patterns of choice that have been observed from various experiments.

In addition to the n assets of Section 2, consider an "asset" (or gamble) G_j ($j = 1, 2, \dots$) which has a price q_j whose future value is a discrete random variable z_j . In all that follows, $q_j = 0$ unless stated otherwise. Write $\theta_j(c_j) = P\{z_j \geq c_j \in C_j\}$ where C_j is the set of possible values of z_j , and assume that v and z_j are independent so that $\pi(b)\theta_j(c_j) = P\{v \geq b \text{ \& } z_j \geq c_j \in C_j\}$. With G_j available—one unit may be bought or none at all—the V problem has to be extended to the following: Maximize $V_j = E(v)'x + E(z_j)$ subject to

$$A - p'x - q_j \geq 0 \quad (1')$$

$$b'x + c_j - B^* \geq 0 \quad (2')$$

$$\pi(b)\theta_j(c_j) - \pi^* \geq 0 \quad (3')$$

$$(x, b) \geq 0. \quad (4')$$

If there is no solution to this V_j problem, one would maximize $\pi(b)\theta_j(c_j)$ subject to (1'), (2') and (4').

The question of buying insurance will serve to introduce some terminology to be used. Let G_1 be an insurance policy which would cover a possible loss $-L$ whose probability is r for a price $q_1 > 0$. Although $q_1 > rL$, many people—not all—buy insurance. The alternative is G_2 (no insurance), the outcomes of which are then $-L$ and 0 with probabilities r and $1 - r$ respectively.

Consider G_1 . Its only net outcome is $c_1 = 0$ since the loss is covered if it occurs, and $\theta_1 = 1$. G_1 in effect reduces A in the V problem to $A - q_1$, so we shall say that G_1 has a negative A effect. With G_2 , the possible outcomes are $-L$ and 0 . Since $\theta_2(-L) = 1$, the result with $c_2 = -L$ is equivalent to that of raising the floor requirement in the V problem to $B^* + L$, and we shall speak of a negative B^* effect in this case. If $c_2 = 0$, $\theta_2 = 1 - r$ which virtually increases the π^* constraint level in the V problem to $\pi^*/(1 - r)$, in which case we shall say there is a negative π^* effect. Thus with G_1 there is only a negative A effect, while with G_2 there is either a negative B^* effect or a negative π^* effect (not both, since $c_2^0 = -L$ or $c_2^0 = 0$). Depending on their relative magnitudes, one can have $V_1 > V_2$ and buy insurance.

In the celebrated St. Petersburg paradox (see Samuelson 1977 for a recent discussion), a fair coin is tossed repeatedly and the game terminates when a head shows up, giving a monetary prize of 2^N if it terminates with the N th toss ($N = 1, 2, \dots$). Since the expected value $\sum_{N=1}^{\infty} 2^{-N} 2^N$ is infinite, the argument is that a person would be willing to pay any amount (that he has available) for a ticket to play the game. However, no one who has thought about it seems willing to do so.

Let G_3 be a ticket to participate in the game and G_4 the "asset" meaning nonparticipation. The V_4 problem is just the original V problem, so $V_4 = V$. Therefore if G_3 is to be considered at all, the constraint set defined by (1')-(4') must be nonempty in the first place. Suppose $q_3 = A$. The negative A effect is so large that $x = 0$, and one then requires a c_3 such that $c_3 \geq B^*$ and $\theta_3 \geq \pi^*$. Now $\theta_3(c_3) = 2/c_3$ (e.g. the probability

of winning at least 64 is $1/32$). In terms of our framework, one rejects G_3 simply because one cannot meet the B^* and π^* constraints simultaneously. (If $A = 1000$ say, how likely is it that one would have $B^* \leq 2$ and $\pi^* \leq 1$, or $B^* \leq 4$ and $\pi^* \leq 1/2$, or ...?) But if $q_3 \leq A$ and q_3 is sufficiently low so that the constraint set is nonempty, one would play the game.

The famous Allais paradox involves four options which are related as in Table I. The probabilities of the possible monetary outcomes $w > y > 0$ are indicated in the top row, and $0 < r_1 < r_2 < r_3$, r_1 being relatively small (say .01). Appropriate values are assigned such that, writing $E^j = E(z_j)$, $E^5 < E^6$ and therefore $E^7 < E^8$. It turns out that many persons who choose G_5 over G_6 also choose G_8 over G_7 , violating the "strong independence axiom" (see the symposium on this subject in the October 1952 issue of *Econometrica* and cf. Encarnación 1969). According to this axiom, the choice between G_5 and G_6 should be independent of the outcomes in the r_3 column where they are the same, and similarly for G_7 and G_8 . But G_5 and G_7 are identical except for their r_3 outcomes, and the same is true about G_6 and G_8 . The axiom thus requires that if G_5 is chosen over G_6 , so must G_7 over G_8 .

Consider the comparison between G_5 and G_6 . G_5 gives a sure outcome of y , so $c_5 = y$ and $\theta_5 = 1$. With G_6 there are three possibilities: (i) $c_6 = 0$ and $\theta_6 = 1$; (ii) $c_6 = y$ and $\theta_6 = 1 - r_1$; (iii) $c_6 = w$ and $\theta_6 = r_2$. If (i) holds, G_5 is better because of its positive B^* effect. If (ii) holds, both options would have the same B^* effect but G_6 is inferior on account of its negative π^* effect. In case (iii), G_5 is better on π^* but G_6 is better on B^* . If the advantage of G_5 on π^* is

sufficiently greater than the advantage of G_6 on B^* so that $V_5 > V_6$, we shall say that π^* is the deciding criterion in choosing G_5 over G_6 .

Clearly, G_5 can be chosen.

Consider G_7 and G_8 . If $c_7 = c_8 = 0$, $\theta_7 = \theta_8 = 1$, whence $V_8 > V_7$ since $E^8 > E^7$, which explains the observed pattern of choices. If $c_7 = y$, $\theta_7 = 1 - r_3$, and if $c_8 = w$, $\theta_8 = r_2 < 1 - r_3$, G_8 would be inferior on π^* but better on B^* , so if B^* is the deciding criterion the results are also explained.

Of more recent origin is the observation from experiments that risk aversion is common in a choice between positive prospects but risk seeking seems to take place in the case of negative prospects (see Kahneman and Tversky 1979, p. 268, and the references cited there). The options in Table II involve monetary outcomes $w > y > 0$ and probabilities $s_1 \geq s_2 > s_3 \geq 0$. The outcomes with G_{11} and G_{12} are the negative of those with G_9 and G_{10} respectively, so $E^{11} \geq E^{12}$ if $E^9 \leq E^{10}$. Values are assigned so that E^9 and E^{10} are equal or about the same. The majority of respondents who choose the safer G_9 over G_{10} also choose the riskier G_{12} over G_{11} . (Note that if $s_2 = s_3$, G_9 would never be picked over G_{10} .)

Consider G_9 and G_{10} . If $c_9 = y$, $\theta_9 = s_1 + s_2$, and if $c_{10} = w$, $\theta_{10} = s_1 + s_3$, G_{10} would be better on B^* but G_9 on π^* , so if the latter is the deciding criterion, G_9 would be chosen.

Suppose $s_3 > 0$. If $c_{11} = c_{12} = 0$, $\theta_{11} = s_3$ and $\theta_{12} = s_2$, in which case G_{12} would beat G_{11} . Suppose $s_3 = 0$. Then $c_{11} = -y$ and $\theta_{11} = 1$. If $c_{12} = 0$, $\theta_{12} = s_2$, G_{11} would be better on π^* but G_{12} on B^* , so

if the latter is the deciding criterion, G_{12} would be chosen.

An interesting special case of Table II which has been investigated by Kahneman and Tversky (1979, p. 273) puts $s_1 = s_2$, $s_3 = 0$, $w = 2y$. In addition, advance bonuses are given so that $q_9 = q_{10} = -w$ and $q_{11} = q_{12} = -2w$. Thus, in terms of total outcomes including the bonuses, G_9 becomes identical to G_{11} and G_{10} to G_{12} . The same pattern of choices is observed--there is an apparent reversal of preferences--and can be explained in the same way. The A effects of the bonuses are the same for G_9 and G_{10} so they do not count in the choice between them, and similarly for G_{11} and G_{12} .

Finally, consider the results summarized by Grether and Plott (1979), where in experimental situation (i) a person is made to choose between a less risky G_{13} and a riskier G_{14} --their expected values are nearly the same--while in (ii) he is asked his minimum selling prices for them. Most subjects who choose G_{13} in (i) place a higher value on G_{14} in (ii). An explanation is immediate from the portfolio model. A person in (ii) is wealthier since he has both assets, so it is not surprising that he should put a higher valuation on the riskier of the two.

While various explanations of some of the choices concerning G_1 to G_{14} exist in the literature, none seems to cover all of them (see Kahneman and Tversky 1979, p. 284; Machina 1982, p. 308; Loomes and Sugden 1982, p. 819). The explanatory power of the portfolio model thus seems greater.

4. Concluding Remarks

Risk aversion in this paper is of course different from the usual Arrow-Pratt sense, and our interpretation of the riskiness of an asset is

also different from that of current analysis (see e.g. Nachman 1979). In our view, the concepts in this paper correspond more closely to ordinary language. In particular, the contrast between low-risk low-return assets and high-risk high-return assets has a precise formulation in the V problem of the portfolio model. Several consequences are interesting: the riskiness of an asset depends on its price; a more risk-averse person evaluates the same assets as more risky; a less risk-averse person (e.g. the classical entrepreneur) has a riskier and higher yielding portfolio; similarly, a wealthier person takes relatively more risks, so the expected value per dollar of his portfolio is higher, and therefore more wealth will generate increasingly greater wealth. All these appear to be commonplace observations, but they do not seem forthcoming from more standard analysis (see Cass and Stiglitz 1972, p. 350).

As a by-product, the portfolio model gives straightforward explanations of the Allais paradox and other puzzling patterns of choice in risk situations. Especially interesting is the "preference reversal phenomenon" which turns out to have the simplest explanation.

(1A)

(1B)

(1C)

Appendix A

Restricting $u = (x, b)$ throughout by $u \geq 0$, suppose: (i) π is a differentiable quasiconcave function; (ii) $g(u) > 0$ for some u ; (iii) for all u , $g(u) \geq 0$ implies $\nabla g^h(u) \neq 0$ ($h = 1, 2, 3$) where $\nabla g^h(u)$ is the vector of partial derivatives of $g^h(u)$; and (iv) $g(x, b) \geq 0$ for some (x, b) with $x_i > 0$ and $E_i > 0$ for some i . Applying Theorem 1 of Kuhn and Tucker (1951) and Theorems 1(b) and 2(b) of Arrow and Enthoven (1961), the conditions (1)-(12), where the superscripts in (x^0, b^0) are suppressed, are then necessary and sufficient for (x^0, b^0) to solve the V problem. Conditions (iii) and (iv) are clearly satisfied, and we have assumed (i). That (ii) holds in general is shown by

Proposition A: If the constraint set is nonempty and $b_i/p_i \neq b_j/p_j$ for some i, j with $b_i, b_j, x_i, x_j > 0$, then $g(u) > 0$ for some $u \geq 0$.

Proof: Suppose (x, b) is in the constraint set and $b_1/p_1 \neq b_2/p_2$ in particular. Putting $dx_i = db_i = 0$ for all $i \neq 1, 2$, it suffices to show the possibility of

$$p_1 dx_1 + p_2 dx_2 < 0$$

$$\pi_1 db_1 + \pi_2 db_2 > 0$$

$$b_1 dx_1 + x_1 db_1 + b_2 dx_2 + x_2 db_2 > 0$$

or

$$dx_2 = -(p_1/p_2)dx_1 - k_1, \quad k_1 > 0 \quad (A1)$$

$$db_2 = -(\pi_1/\pi_2)db_1 - k_2, \quad k_2 > 0 \quad (A2)$$

$$(b_1 - b_2 p_1/p_2)dx_1 + (x_1 - x_2 \pi_1/\pi_2)db_1 > k_1 b_2 + k_2 x_2. \quad (A3)$$

We can always choose dx_1 and db_1 so that the left side of (A3) is positive, and then choose k_1 and k_2 sufficiently small so that (A3) holds. It is thus possible to have (A1)-(A3).

$$\begin{array}{c} \text{(A1)} \\ \text{(A2)} \\ \text{(A3)} \end{array} \quad \begin{array}{c} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \\ dx_5 \\ dx_6 \\ dx_7 \\ dx_8 \\ dx_9 \\ dx_{10} \\ dx_{11} \\ dx_{12} \\ dx_{13} \\ dx_{14} \\ dx_{15} \\ dx_{16} \\ dx_{17} \\ dx_{18} \\ dx_{19} \\ dx_{20} \\ dx_{21} \\ dx_{22} \\ dx_{23} \\ dx_{24} \\ dx_{25} \\ dx_{26} \\ dx_{27} \\ dx_{28} \\ dx_{29} \\ dx_{30} \\ dx_{31} \\ dx_{32} \\ dx_{33} \\ dx_{34} \\ dx_{35} \\ dx_{36} \\ dx_{37} \\ dx_{38} \\ dx_{39} \\ dx_{40} \\ dx_{41} \\ dx_{42} \\ dx_{43} \\ dx_{44} \\ dx_{45} \\ dx_{46} \\ dx_{47} \\ dx_{48} \\ dx_{49} \\ dx_{50} \\ dx_{51} \\ dx_{52} \\ dx_{53} \\ dx_{54} \\ dx_{55} \\ dx_{56} \\ dx_{57} \\ dx_{58} \\ dx_{59} \\ dx_{60} \\ dx_{61} \\ dx_{62} \\ dx_{63} \\ dx_{64} \\ dx_{65} \\ dx_{66} \\ dx_{67} \\ dx_{68} \\ dx_{69} \\ dx_{70} \\ dx_{71} \\ dx_{72} \\ dx_{73} \\ dx_{74} \\ dx_{75} \\ dx_{76} \\ dx_{77} \\ dx_{78} \\ dx_{79} \\ dx_{80} \\ dx_{81} \\ dx_{82} \\ dx_{83} \\ dx_{84} \\ dx_{85} \\ dx_{86} \\ dx_{87} \\ dx_{88} \\ dx_{89} \\ dx_{90} \\ dx_{91} \\ dx_{92} \\ dx_{93} \\ dx_{94} \\ dx_{95} \\ dx_{96} \\ dx_{97} \\ dx_{98} \\ dx_{99} \\ dx_{100} \end{array}$$

the cofactor of the (k, h) -element. We have

$$\begin{array}{c} \text{(B1)} \\ \text{(B2)} \end{array} \quad \begin{array}{c} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \\ dx_5 \\ dx_6 \\ dx_7 \\ dx_8 \\ dx_9 \\ dx_{10} \\ dx_{11} \\ dx_{12} \\ dx_{13} \\ dx_{14} \\ dx_{15} \\ dx_{16} \\ dx_{17} \\ dx_{18} \\ dx_{19} \\ dx_{20} \\ dx_{21} \\ dx_{22} \\ dx_{23} \\ dx_{24} \\ dx_{25} \\ dx_{26} \\ dx_{27} \\ dx_{28} \\ dx_{29} \\ dx_{30} \\ dx_{31} \\ dx_{32} \\ dx_{33} \\ dx_{34} \\ dx_{35} \\ dx_{36} \\ dx_{37} \\ dx_{38} \\ dx_{39} \\ dx_{40} \\ dx_{41} \\ dx_{42} \\ dx_{43} \\ dx_{44} \\ dx_{45} \\ dx_{46} \\ dx_{47} \\ dx_{48} \\ dx_{49} \\ dx_{50} \\ dx_{51} \\ dx_{52} \\ dx_{53} \\ dx_{54} \\ dx_{55} \\ dx_{56} \\ dx_{57} \\ dx_{58} \\ dx_{59} \\ dx_{60} \\ dx_{61} \\ dx_{62} \\ dx_{63} \\ dx_{64} \\ dx_{65} \\ dx_{66} \\ dx_{67} \\ dx_{68} \\ dx_{69} \\ dx_{70} \\ dx_{71} \\ dx_{72} \\ dx_{73} \\ dx_{74} \\ dx_{75} \\ dx_{76} \\ dx_{77} \\ dx_{78} \\ dx_{79} \\ dx_{80} \\ dx_{81} \\ dx_{82} \\ dx_{83} \\ dx_{84} \\ dx_{85} \\ dx_{86} \\ dx_{87} \\ dx_{88} \\ dx_{89} \\ dx_{90} \\ dx_{91} \\ dx_{92} \\ dx_{93} \\ dx_{94} \\ dx_{95} \\ dx_{96} \\ dx_{97} \\ dx_{98} \\ dx_{99} \\ dx_{100} \end{array}$$

choose dx_1 so that $dx_1 = E(v)dx_1 - 0$. From the equations in (2),

$$\begin{array}{c} \text{(C1)} \\ \text{(C2)} \\ \text{(C3)} \end{array} \quad \begin{array}{c} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \\ dx_5 \\ dx_6 \\ dx_7 \\ dx_8 \\ dx_9 \\ dx_{10} \\ dx_{11} \\ dx_{12} \\ dx_{13} \\ dx_{14} \\ dx_{15} \\ dx_{16} \\ dx_{17} \\ dx_{18} \\ dx_{19} \\ dx_{20} \\ dx_{21} \\ dx_{22} \\ dx_{23} \\ dx_{24} \\ dx_{25} \\ dx_{26} \\ dx_{27} \\ dx_{28} \\ dx_{29} \\ dx_{30} \\ dx_{31} \\ dx_{32} \\ dx_{33} \\ dx_{34} \\ dx_{35} \\ dx_{36} \\ dx_{37} \\ dx_{38} \\ dx_{39} \\ dx_{40} \\ dx_{41} \\ dx_{42} \\ dx_{43} \\ dx_{44} \\ dx_{45} \\ dx_{46} \\ dx_{47} \\ dx_{48} \\ dx_{49} \\ dx_{50} \\ dx_{51} \\ dx_{52} \\ dx_{53} \\ dx_{54} \\ dx_{55} \\ dx_{56} \\ dx_{57} \\ dx_{58} \\ dx_{59} \\ dx_{60} \\ dx_{61} \\ dx_{62} \\ dx_{63} \\ dx_{64} \\ dx_{65} \\ dx_{66} \\ dx_{67} \\ dx_{68} \\ dx_{69} \\ dx_{70} \\ dx_{71} \\ dx_{72} \\ dx_{73} \\ dx_{74} \\ dx_{75} \\ dx_{76} \\ dx_{77} \\ dx_{78} \\ dx_{79} \\ dx_{80} \\ dx_{81} \\ dx_{82} \\ dx_{83} \\ dx_{84} \\ dx_{85} \\ dx_{86} \\ dx_{87} \\ dx_{88} \\ dx_{89} \\ dx_{90} \\ dx_{91} \\ dx_{92} \\ dx_{93} \\ dx_{94} \\ dx_{95} \\ dx_{96} \\ dx_{97} \\ dx_{98} \\ dx_{99} \\ dx_{100} \end{array}$$

Appendix B

Consider the equations implied by $x_i, b_i, \alpha, \beta, \gamma > 0$ in (6) and (8)-(11). Taking their total differentials gives the following system (which thus includes the relationships deriving from those equations).

$$\begin{bmatrix} 0 & \beta I & -p & b & 0 \\ \beta I & \gamma(\pi_{ij}) & 0 & 0 & (\pi_i) \\ -p' & 0 & 0 & 0 & 0 \\ b' & 0 & 0 & 0 & 0 \\ 0 & (\pi_i)' & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx \\ db \\ d\alpha \\ d\beta \\ d\gamma \end{bmatrix} = \begin{bmatrix} \alpha dp \\ -d\beta x \\ -dA + x'dp \\ dB^* - x'db \\ d\pi^* \end{bmatrix} \quad (B1)$$

where $\pi_{ij} = \partial \pi_i / \partial b_j$. Since the matrix (π_{ij}) is symmetric, so is the coefficient matrix. Let D be the determinant of the coefficient matrix and D_{kh} the cofactor of its (k, h) -element. We have

$$\partial x_i / \partial p_j = (\alpha D_{ji} + x_j D_{2n+1,i}) / D \quad (B2)$$

$$\partial x_i / \partial A = -D_{2n+1,i} / D. \quad (B3)$$

Let $dp_j \neq 0$ for a particular j and $dp_i = 0$ for all $i \neq j$, and choose dA so that $dV = E(v)'dx = 0$. From the equations in (5), $\alpha p'dx = E(v)'dx + \beta b'dx$ and therefore $\alpha p'dx = \beta b'dx$. Putting $dB^* = 0$ and $db = 0$ in (B1), $b'dx = 0$ so that $p'dx = 0$. Hence $-dA + x'dp = 0$ in (B1), and

$$\partial x_i / \partial p_j \Big|_{R=\text{const}} = \alpha D_{ji} / D. \quad (B4)$$

(B2)-(B4) then give (15). Since $D_{ij} = D_{ji}$, the matrix of substitution terms is symmetric; negative semidefiniteness also follows along the lines of Samuelson (1947, pp. 109-113).

Appendix C

While $b_i > 0$ implies $x_i > 0$ as noted earlier, the reverse does not hold. If $x_i > 0$, (14) gives $b_i > 0$ if and only if $E_i/p_i < \alpha$. Consider a situation where, in the solution (x^0, b^0) to the V problem, $x_i^0 > 0$ and $b_i^0 = 0$ for a particular i and $x_j^0 > 0$ if and only if $b_j^0 > 0$ for all $j \neq i$. Another problem is obtained by changing A to $A - dA$ where $dA = p_i dx_i > 0$ and $x_i^0 - dx_i = 0$. As can be quickly verified, conditions (1)-(12) are satisfied by $(x^0 - dx, b^0)$ where $dx_j = 0$ for all $j \neq i$, and the same α, β, γ . (In particular, (1) is still binding and because $b_i^0 = 0$, so is (2).) Thus with a budget of $A - dA$, asset i is not bought. But with a larger budget, it enters the portfolio.

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Table I

	r_1	r_2	r_3
$G_5:$	y	y	y
$G_6:$	0	w	y
$G_7:$	y	y	0
$G_8:$	0	w	0

Table II

	s_1	s_2	s_3
$G_9:$	y	y	0
$G_{10}:$	w	0	w
$G_{11}:$	-y	-y	0
$G_{12}:$	-w	0	-w