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SOME WELL-BEHAVED COMPOSITION FUNCTIONS INVOLVING NONCONCAVE ARGUMENT FUNCTIONS

by

Raul V. Fabella

ABSTRACT

Quasi-concavity is the criterion for well-behavedness in our composition functions. Negishi (1963) has shown how to generate quasi-concave composition functions when the argument functions are concave.

We show here that one can generate quasi-concave composition functions even when the argument functions are nonconcave. The trick is to find interesting and "compatible" subfamilies in the quasi-concave family that, in composition, compensate for nonconcavity of the argument functions and as a result keep the composition function within the quasi-concave family.

Some Well-Behaved Composition Functions Involving Nonconcave Argument Functions

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Quasi-concavity is the criterion for well-behavedness in our composition functions. This choice is natural in many senses. First a quasi-concave function f displays a convex level set

$$A = \{x: f(x) \geq C, x \in D\} \qquad C > 0, \ D = \text{domain}$$

Arviel (1976) gives a proof (he may not be the first to show this) that shows that only quasi-concave functions display convex level sets. This means that only quasi-concave functions allow for well-behaved indifference curves and well-behaved isoquants. The quasi-concave family is also of great importance in mathematical programming. Arrow and Enthoven (1961) extended the Kuhn-Tucker-Lagrange Sufficiency Theorem to quasi-concave constraint and objective functions. In the area of value theory, Uzawa (1964) has shown that class of quasi-concave functions forms a unique class of numerical representations of the Uzawa Preference Ordering Axioms. That this class of functions has endeared itself to economists is no surprise. These pages are just another tribute to that endearment.

The Framework

Let X be an m x n allocation matrix with the representative element being $X_{ij} > 0$, the amount of X_i going to activity $j = 1, 2, \ldots, m$; $i = 1 \ldots n$. We thus have $\sum_j X_{ij} = X_i$. Let

$F = (f_1, f_2, \ldots, f_m): R_+^{n\times m} \to R_+^m$ be a transformation map, each of the f_j being a real-valued map from R_+^m to R_+^1 where index $+$ stands for nonnegativity. The range of f_j is the j^{th} row of X. Let $U: R_+^m \to R_1^+$, the representative point in R_+^m being F. We are interested in the properties of the composition map

$$W = UF: R_+^{m\times n} \quad R_+^1$$

In the foregoing, when we mention F, U and W we understand their domains of definition as given above. Likewise, when we give a property to describe F, we mean this property to be true of every element f_j of F.

An interesting composition map result on quasi-concave functions and which will serve as our point of departure is the Negishi (1963) and Berge (1963) result:

> If U is nondecreasing and quasi-concave and if F is nondecreasing and concave, then the composition map $W = UF$ is nondecreasing and quasi-concave. If U is furthermore concave, $W = UF$ is concave.

This gives a way to generate quasi-concave functions from other quasi-concave functions. Note that F is limited to concave functions and this sometimes poses a problem in application when it is desired that F display properties not encompassed by concavity such as say scale economies. Thus we are interested in complimenting the result of Negishi and Berge by focusing on conditions that allow for the generation of quasi-concave composition functions when the F is nonconcave (by nonconcave means it can have elements not of the concave family but does not exclude concave functions). An example of a nonconcave function is

$$f(X) = kx_1^a x_2^b \qquad a + b > 1$$

The function allows for economies of scale or nondiminishing marginal utility for some interval (as when the eating of the first apple makes the second more delicious (Gorman, 1959)).

The (h, ϕ) - Concave Family of Functions

In 1976, Arviel introduced a new family of functions that gathered under its wings many subfamilies of interest in mathematical programming. He defined it thus:

Definition 1: A function f on R^m is (h, ϕ) - concave if for every X^1 and X^2 in R^m and for $0 \leq t \leq 1$,

(1) $$f(h^{-1}(th(X^1) + (1 - t)h(X^2)) \geq \phi^{-1}(t \;\; \phi f(X^1) + (1 - t) \;\; \phi f(X^2))$$

where h and ϕ are real-valued functions with inverses h^{-1} and ϕ^{-1} respectively.

The scope of this family is very wide. If we let h = identity and ϕ = identity (function, we get the definition of the concave subfamily; letting h = identity and ϕ = log, we get the well-known log-concave family. For specific purposes we need to specify h and ϕ.

Our own specification for h and Φ is as follows: let g be a nondecreasing strictly concave function with an inverse g^{-1}. We will say that a function f is (i, g) - concave if it is $(h = \text{identity}, \Phi = g)$ - concave function. Likewise an $(h = g, \Phi = \text{identity})$ - concave function we designate as (g, i) - concave function. It is easy to see the following:

Proposition 1: Let U be nondecreasing and (g, i) - concave. Let the transformation F be nondecreasing and (i, g) - concave. Then the composition function $W = UF$ is nondecreasing and concave.

Proof: Since F is (i, g) - concave, for every X^1 and X^2 in the domain R_+^{m+n} and for every $0 \leq t \leq 1$, we have

$$F(tX^1 + (1 - t)X^2) \geq g^{-1}(tgF(X^1) + (1 - t)g\,F(X^2))$$

Since U is nondecreasing

$$UF(tX^2 + (1 - t)X^2) \geq U(g^{-1}(tgF(X^1) + (1 - t)F(X^2))$$

Since U is (g, i) - concave, we have

$$U(g^{-1}(tgF(X^1) + (1 - t)gF(X^2)) \geq tUF(X^1) + (1 - t)F(X^2)$$

so that

$$W(tX^1 + (1 - t)X^2 = UF(tX^1 + (1 - t)X^2) \geq tW(X^1) + (1 - t)W(X^2)$$

QED

Thus a combination of a (g, i) - concave U and an (i, g) - concave F produces a concave $W = UF$ which is then well-behaved by our criterion. Note that we did not use the properties of g in the proof. In reality, for the above result, any g with inverse g^{-1} will do. We will use the properties of g to show that the families that we use are of interest in economics, i.e., that they themselves are well-behaved. We first deal with the (i, g) - concave family and show the following property of g.

Lemma 1: If g is concave and nondecreasing and its inverse g^{-1} exists, then g^{-1} is convex.

Remark: The existence of g^{-1} is assured if the domain and the range have the same dimension and the Jacobian determinant of g is nonvanishing.

Proof: Let X^1 and X^2 be in R^m. Let $0 \leq t \leq 1$. Since g is concave

$$g(tX^1 + (1 - t)X^2) \geq tg(X^1) + (1 - t)\ (X^2)$$

Since g^{-1} exists, there exist points y^1 and y^2 in R^n (the range of g) such that

$$X^1 = g^{-1}(y^1) \text{ and } X^2 = g^{-1}(y^2) \quad ,$$

Substituting these into the above inequality gives

$$g(tg^{-1}(y^1) + (1 - t)\, g^{-1}(y^2)) \quad tgg^{-1}(y^1) + (1 - t)\, gg^{-1}(y^2) = ty^1 + (1 - t)y^2$$

Since g is nondecreasing, so is g^{-1} and so applying it we have

$$g^{-1}(ty^1 + (1 - t)y^2) \leq tg^{-1}(y^1) + (1 - t)\, g^{-1}(y^2)$$

and g^{-1} is convex. If g is strictly concave, g^{-1} is strictly convex.

Q.E.D.

For examples, we have the log function with inverse e. The inverse of $X^{-\frac{1}{2}}$ which is convex. We now show that the (i, g) - concave family includes the concave family.

Proposition 2: Every concave function is also an (i, g) - concave function.

Proof: For every X^1 and X^2 in R^n_+ and $0 \leq t \leq 1$,

$$f(tX^1 + (1 - t)X^2) \geq tf(X^1) + (1 - t)\, f(X^2)$$

Since g is strictly concave, g^{-1} is strictly convex and

$$tf(X^1) + (1 - t)f(X^2) > g^{-1}(tgf(X^1) + (1 - t)gf(X^2))$$

and

$$f(tX^1 + (1 - t)X^2) > g^{-1}(tgf(X^1) + (1 - t)gf(X^2))$$

Q.E.D.

In fact, the concave family is not only a subset but a proper subset of the (i, g) - concave family. To show this we have

<u>Proposition 3</u>: There exists (i, g) - concave function which is not concave.

<u>Proof</u>: Let $g = \log$ which is strictly concave and nondecreasing and so is allowed. If g is log the definition 1 collapses to

(2) $$f(tX^1 + (1 - t) X^2) \geq f(X^1)^t \, f(X^2)^{1-t}$$

X^1 and X^2 in R^n_+ and $0 \leq t \leq 1$.

$$\log f(tX + (1 - t)X^2) \geq t(\log f(X^1) + (1 - t) \log f(X^2)$$

which shows that if f is $(i, \log)$ - concave, $\log f$ is concave in R^n_+. Now consider the function.

$$F(X) = x_1^2 \; x_2^2$$

$\log f(X) = 2 \log X_1 + 2 \log X_2$ which is concave in X so that $f(X)$ is $(i, \log)$ - concave. But f is obviously not concave in R^2_+.

Q.E.D.

Are (i, g) - concave functions well-behaved? Are they quasi-concave?

<u>Proposition 4</u>: Every (i, g) - concave function is also quasi-concave.

<u>Proof</u>: Let X^1 and X^2 be in R^n_+. Let $0 \leq t \leq 1$. Suppose $f(X^1) \geq$

$f(X^2)$. Since f is (i, g) - concave

$$f(tX^1 + (1-t)X^2) \geq g^{-1}(tgf(X^1) + (1-t)f(X^2)$$

Since g is nondecreasing and $f(X^1) \geq f(X^2)$

$$tgf(X^1) + (1-t)g(f(X^2) \geq gf(X^2)$$

So that

$$g^{-1}(tgf(X^1) + (1-t)gf(X^2) \geq g^{-1}gf(X^2) - f(X^2)$$

and f is quasi-concave. Q.E.D.

To be interesting, the (i, g) - concave family must include members of interest to economists.

<u>Proposition 5</u>: The following functions are (i, g) - concave functions:

(a) $f(X) = \prod_{e=1}^{n} x_i^{Ci}$; $C_i \geq 0$; $x_i \geq 0$

(b) $f(X) = A(\sum_{i=1}^{n} a_i x_i^{-e})^{-r/e}$ $r \geq 1$, $a_i \geq 0$, $\sum_{i=1}^{n} a_i = 1$

$-1 \leq e \leq 0$.

Proof: (a) From the proof of Proof 3, we know that f is $(i, \log)$ - concave if log f is concave. Take the log of $f(X)$ in in (a) to get

$$\log f(X) = \sum_{i=1}^{n} C_i \log x_i$$

which is concave

(b) Again take the log to get

$$\log f(X) = \log A + -\frac{r}{e} \log \left(\sum_{i=1} a_i X_i^{-e}\right)$$

The parenthesized expression is a sum of concave functions and thus concave $(-\frac{r}{e}) > 0$. So $\log f(X)$ is concave. Q.E.D.

These two are the Cobb-Douglas (minus the constant returns to scale property) and the CES functions (again without constant returns to scale property) respectively.

We now turn our attention to (g, i) - concave functions.

Proposition 6: Every (g, i) - concave fu·ctions f is also concave but not vice-versa.

Proof: Let X^1 and X^2 be in R^n_+ and $0 \leq t \leq 1$. Since f is (g, i) - concave we have

$$f(g^{-1}(tg(X^1) + (1-t)\, g(X^2)) \geq tf(X^1) + (1-t)\, f(X^2)$$

Since g is concave, g^{-1} convex implying that

$$g^{-1}(tg(X^1) + (1-t)\, g(X^2)) \leq tX^1 + (1-t)\, X^2$$

Applying the nondecreasing property of f, we get

$$f(tX^1 + (1-t)X^2) > tf(X^1) + (1-t)\, f(X^2)$$

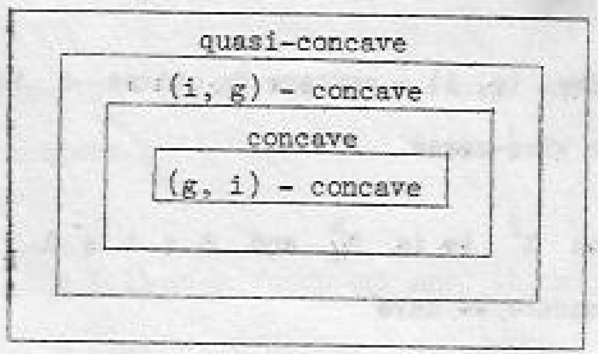
and f is concave. To prove the second part, consider the function $f(X) = x$ which is concave. If f is (g, i) - concave we have for every X^1 and X^2 in R^1

$$f(g^{-1}(tg(X^1) + (1 - t)\, gX^2) \geq tf(X^1) + (1 - t)\, f(X^2)$$

$$g^{-1}\,(tg(X^1) + (1 - t)\, g(X^2)) \geq tX^1 + (1 - t)\, X^2$$

Since f is an identity function. But this is a contradiction if g^{-1} is convex. Q.E.D.

So (g, i) - concave functions are also well behaved. An example of this type of function is $f(X) = \log x$. We have the following scheme



One of the most important subfamilies of the (i, g) - concave family is the (i, log) - concave family. If f is (i, log) - concave, l g f is the domain of f. In fact, it has another very interesting property:

<u>Proposition 7</u>: The (i, log) - concave class of functions is closed under multiplication

Proof: Let f^1 and f^2 be (i, log) - concave. Let X^1 and X^2 be in R^A_+ and $0 \leq t \leq 1$. We have from (2)

$$f^i(tX^1 + (1 - t)\, X^2) \geq f^i(X^1)^t\, f^2(X^2)^{1-t} \qquad l = 1, 2$$

so that

$$f^1(tX^1 + (1 - t)\, X^2)\, f^2(tX^1 + (1 - t)X^2) \geq (f^1(X^1)\, f^2(X^1))^t$$

$$(f^1(X^2)\, f^2(X^2))^{1-t}$$

The way we generated quasi-concave composition functions from other quasi-concave functions involves a kind of symmetric compensation. The nonconcavity of F is compensated for by the "very concave" nature of U. This combination of centripetal and centrifugal forces produces concavity. However, there is yet another way to generate quasi-concave functions when F is nonconcave. Our specification for g is unchanged. We show the following:

Proposition 8: Let U be a nondecreasing (g, g) - concave function. Let the transformation F be a nondecreasing (i, g) - concave function. Then the composition function $W = UF$ is nondecreasing and (i, g) - concave.

Proof: Let X^1 and X^2 be in R^n_+; $0 \leq t \leq 1$. Since F is (i, g) - concave.

$$F(tX^1 + (1 - t)\, X^2) \geq g^{-1}(tgF(X^1) + (1 - t)\, gF(X^2)$$

Since U is nondecreasing any (g, g) - concave, we have

$$W(tX^1 + (1 - t)\, X^2) = UF(tX^1 + (1 - t)\, X^2) \geq$$

$$U(g^{-1}(tgF(X^1) + (1-t)\, gF(X^2)) \geq$$

$$g^{-1}(tgUF(X^1) + (1-t)\, gUF(X^2) =$$

$$g^{-1}(tgW(X^1) + (1-t)\, gW(X^2))$$

which shows that W is (i, g) - concave. Q.E.D.

Now a (g, g) - concave function is (i, g) - concave but need not be concave nor (g, i) - concave. That it is (i, g) - concave follows because:

$$f(tX^1 + (1-t)X^2) > f(g^{-1}(tg(X^1) + (1-t)\, g(X^2)) \geq$$

$$g^{-1}(tgf(X^1) + (1-t)\, gf(X^2))$$

The second inequality defines (g, g) - concavity while the first follows from the convexity of g^{-1} and the nondecreasing property of f. Thus a (g, g) - concave function is also quasi-concave.

Note that the concave family is closed under addition but .ot under multiplication. The quasi-concave family on the other hand has no apparent closure property.

Applications:

(a) The Existence of the Bergson Social Welfare Function and Convex Social Indifference Curves.

We now interpret U to be an amalgamation function and F to be the set of individual utility functions. X is then the goods allocation matrix and i is the index for goods and j the index for individuals. Negishi defines the Bergson Social Welfare Function $B(X)$ as follows: Consider the programming problem.

$$\operatorname{Max}_{X} W(X) = UF(X)$$

$$\text{s.t.} \quad \sum_{j=1}^{M} X_{ji} = X_i \qquad i = 1, 2, \ldots n$$

Suppose X^* solves the programming problem. Then the Bergson Social Welfare Function $B(X) = W(X^*)$. We then have the following extension of the Negishi result:

If either:

(a) U is quasi-concave and nondecreasing and F is nondecreasing and concave

(b) U is (g, i) - concave and nondecreasing and F is nondecreasing and (i, g) - concave

(c) U is (g, g) - concave and nondecreasing and F is (i, g) - concave and nondecreasing then the Bergson Social Welfare Function exists with convex social indifference curves.

Negishi's result stems from the quasi-concavity and the nondecreasing character of $W(X)$. That is, if we allow some amount of nondiminishing marginal utility in the individual utility functions, we would still get convex social indifference curve. Note that $W(X^*)$ can be understood to be defined over the set of all goods since X^* can be defined to be a function of (X_i), $i = 1, \ldots n$.

(b) The Household Production Model

Let U be the household utility function over the set of commodities Z. Let $Z = F(X)$ where F is the m component transformation $W(X)$ subject to budget constraint Y. i.e.

$$\mathrm{Max}_X \; W(X) = U(F(X))$$

$$\text{S.t.} \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij} P_i = Y$$

If F is concave, nondecreasing and differentiable and U is quasi-concave, nondecreasing and differentiable, then the household problem can be solved via the Arrow-Enthoven extension of the Kuhn-Tucker Sufficiency result. We have shown that if F is (i, g) - concave and U is (g, i) - concave, a solution in the Arrow-Enthoven sense also exists. The same result is reached if F is (i, g) - concave and U is (g, g) - concave. The household indifference curve would also be convex.

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