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CONSISTENCY CONDITIONS FOR GROUP DECISION

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Abstract

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This paper argues that a number of consistency conditions for group choice cannot be considered as fundamental requirements since there are group decision rules which seem otherwise reasonable that do not satisfy them. In order to preserve Pareto optimality, which is essential, these rules represent possible alternatives by lexicographically ordered vectors that depend on the feasible set (because an alternative's Pareto property depends on the available alternatives). Revised versions of the consistency conditions are satisfied by the decision rules considered.

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1. Introduction

A number of propositions have been considered in the literature on social choice theory as possible desiderata for rational or consistent[1] group decisions. Independence of irrelevant alternatives, path independence, and the Chernoff-Sen condition called α are among the more prominent. The purpose of this paper is to show that these "consistency conditions" (see Sen (1977), sec. 4) cannot be considered as fundamental requirements since there are decision rules which seem otherwise reasonable that do not satisfy them. In order to preserve Pareto optimality, which is essential, these rules require that the alternatives be represented by lexicographically ordered vectors tha[illegible]pend on the feasible set. The consistency conditions are then violated because they ignore such dependence.

Section 2 states our assumptions and Section 3 describes a well defined decision rule as a particular example satisfying the assumptions. Section 4 on the consistency conditions shows that their present formulations fail but revised versions suggested by our assumptions turn out to be theorems. Section 5 makes concluding observations.

2. Assumptions

Let a group of individuals indexed by $k = 1, \ldots, N$ be put in a situation where they have to decide as a group on a course of action to be taken. The available alternatives constitute a nonnull set A in the set X of all possible alternatives, and a rule or procedure f is needed to determine a nonnull subset $f(A)$ as the group's choice in A. Conditions may be placed on f so that choices would conform to what are considered as necessary or at least reasonable requirements.

Each individual k is assumed to have a preference system U^k for evaluating possible alternatives $x, y, \ldots$ in X, which implies a subsystem U_A^k for evaluating alternatives in any A. From a knowledge of U^k one can infer whether or not xP^ky (k prefers x to y). Then $R^k = \{(x, y) | \sim yP^kx\}$, where $\sim$ stands for negation, is an ordering relation (i.e. one which is reflexive, complete and transitive) on X. If $(x, y) \in R^k$, we will write xR^ky.

Let $\{U^k\} = (U^1, \ldots, U^N)$ and $\{R^k\} = (R^1, \ldots, R^N)$. Using notation without superscript k for the group, the corresponding concepts for the group are U and R. The relationship between the group's ordering relation on A, call it R_A, and individual preference systems is usually put in the form $R_A = \Psi(\{R_A^k\})$, where R_A^k is k's ordering on A. This relationship is better written, however, as $R_A = \phi(\{U_A^k\})$ to allow for the possibility that $\{R_A^k\}$ may not suffice to determine R_A. One might have the following schema.

$$\begin{tikzcd} \{U^k\} \arrow[r] \arrow[d] \arrow[dr] & U \\ \{R_A^k\} & R_A \end{tikzcd} \tag{1}$$

It is possible (as will be seen in the next section) to have a group decision rule where $\{U^k\}$ determines the group's U which with $\{U_A^k\}$ yields R_A, and R_A is not determined solely by $\{R_A^k\}$. The formulation $R_A = \Psi(\{R_A^k\})$

assumes no difference between U^k and R^k.

As usual we will say that x is Pareto inferior (or, simply, inferior) to y if yR^kx for all k and yP^hx for some h, in which case we will also say that y dominates x (or yDx). If x is in A and not inferior to any y in A, then x is Pareto optimal in A. We want f to be such that only Pareto optimal points can be in $f(A)$.

<u>Assumption</u> 1. If x is Pareto inferior in A, then x is not in $f(A)$.

We will say that an ordering relation Q on X is nondictatorial if there is no k such that $Q_A = R^k_A$ for all A.

<u>Assumption</u> 2. There is a nondictatorial ordering relation Q on X that is determined by $\{U^k\}$.

An example is provided by what Sen (1977) calls the "majority closure method," i.e. the transitive closure of the majority decision relation, which would have xQy if and only if there exist $z_1, \ldots, z_n \in X$ such that $x = z_1$, $y = z_n$, and for $i = 1, \ldots, n - 1$, z_i gets at least as many votes in the group as z_{i+1} does in a pairwise comparison. The problem with this method as it stands, however, is that it allows an inferior alternative to belong to the choice set (Ferejohn and Grether (1977)). Since Assumption 1 is generally considered mandatory, we need a device that would permit an otherwise reasonable Q to serve as part of a group decision procedure.

<u>Assumption</u> 3. xR_Ay if and only if the first nonvanishing component of $r_A(x) - r_A(y)$ is nonnegative, where: $r_A(x) = (p_A(x), q(x))$; $p_A(x) = 1$ if

x is Pareto optimal in A, $p_A(x) = 0$ otherwise; and $q(x) \geq q(y)$ if and only if xQy.

The function q is either real-valued or vector-valued; in the latter case, $q(x) > q(y)$ means that the first nonvanishing component of $q(x) - q(y)$ is positive, i.e. the $q(x)$'s are ordered lexicographically. Finally,

<u>Assumption</u> 4. $f(A) = \{x \in A \mid \forall y \in A: xR_A y\}$.

Noting that not every x in A can be inferior if A is a closed set, which we assume, the lexicographic ordering of the alternatives by Assumption 3 assures a nonnull $f(A)$ that contains no dominated points. In effect the procedure f would have two stages f_1 and f_2 where f_1 first selects Pareto optimal points after which f_2 then picks out the choice in accordance with Q. Specifically, define

$$A_1 = f_1(A) = \{x \in A \mid \forall y \in A: \sim yDx\}$$

$$A_2 = f_2(A_1) = \{x \in A_1 \mid \forall y \in A_1: xQy\}.$$

Then $f(A) = f_2(f_1(A)) = A_2$. Let us say that x is Q-greatest in a set B if xQy for all y in B. We note the following relationships.

<u>Lemma</u> 1. If $A \subset B$, then if x is Q-greatest in B, x is Q-greatest in A.

<u>Lemma</u> 2. If $A \subseteq B$, then $A \cap B_1 \subset A_1$.

<u>Lemma</u> 3. $f_2(A_1) \cap B_1 \subset f_2(A_1 \cap B_1)$.

Lemma 1 is obvious. Since $A \cap B_1 = \{x \in A \cap B \mid \forall y \in B: \sim yDx\}$, and $(x \in A \cap B \;\&\; \forall y \in B: \sim yDx) \rightarrow (x \in A \;\&\; \forall y \in A: \sim yDx)$ if $A \subset B$, we get Lemma 2. From the fact that $A_1 \cap B_1 \subset A_1$, $\{x \in A_1 \cap B_1 | \forall y \in A_1: xQy\} \subseteq \{x \in A_1 \cap B_1 | \forall y \in A_1 \cap B_1: xQy\}$ gives Lemma 3.

Lemma 4. If $x \in A_1 \cap B_2$ and $A_2 \subset B_1$, then $x \in A_2$.

Suppose the hypothesis is true, so that $x \in A_1 \cap B_1$ (because $B_2 \subset B_1$) and $\forall y \in B_1: xQy$. Since $A_2 \subset B_1$, x is Q-greatest in A_2 by Lemma 1. Hence $x \in A_2$ given that $x \in A_1$.

Lemma 5. $f_2(A_2 \cup B_2) \subset f_2(A_1 \cup B_1)$.

Suppose $x \in f_2(A_2 \cup B_2)$. Then $x \in A_2 \cup B_2$ and $\forall y \in A_2 \cup B_2: xQy$. Since $A_2 \cup B_2 = \{y \in A_1 | \forall z \in A_1: yQz\} \cup \{y \in B_1 | \forall z \in B_1: yQz\}$, it follows that $x \in A_1 \cup B_1$ and $\forall z \in A_1 \cup B_1: xQz$.

Lemma 6. If $x \in A_1 \cup B_1$ and x is Q-greatest in $A_2 \cup B_2$, then $x \in A_2 \cup B_2$.

If the hypothesis is true, then xQy for all $y \in A_2 \cup B_2 = \{y \in A_1 | \forall z \in A_1: yQz\} \cup \{y \in B_1 | \forall z \in B_1: yQz\}$. So if $x \in A_1 \cup B_1$, we have $x \in A_2 \cup B_2$.

In the next section we will describe a particular rule satisfying the Assumptions (i.e. Assumptions 1 to 4) that relates the group choice to individual preference systems in a precise way. It is of some interest since the ordering by Q is itself lexicographic.

3. Lexicographic Preferences

The concept of lexicographic preferences assumes that there are multiple criteria of choice which are ranked in order of importance or priority (see Fishburn (1974) for a survey of this literature, especially Georgescu-Roegen (1954) and Chipman (1960); cf. Encarnación (1964)). Depending on the decision context, these criteria may correspond to different wants or needs, as in the case of the consumer whose need for food cannot be served by clothing or shelter, or more generally to objectives that permit no trade-offs among them until certain "satisfactory" levels of those objectives have been reached, as might be the case in the theory of the firm (Encarnación (1964a)).

In a group decision context we assume that the members of the group share a common set of objectives and rank them in the same order of importance, but they may differ as to the targets for those objectives. To each x corresponds a vector $u(x) = (u_1(x), u_2(x), \ldots)$ where u_i is a real-valued function such that $u_i(x) > u_i(y)$ if and only if everyone in the group prefers x to y on the basis of the ith criterion. It is assumed that there exist particular values u_i^{*k} such that if $u_i(x) \geq u_i^{*k}$, then k considers x satisfactory with respect to the ith criterion. Writing $v_i^k(x) = \min\,(u_i(x), u_i^{*k})$, we associate a vector $v^k(x) = (v_1^k(x), v_2^k(x), \ldots)$ to each x to define a relation L^{*k}.

<u>Definition.</u> $xL^{*k}y$ if the first nonvanishing $v_i^k(x) - v_i^k(y)$, $i = 1, 2, \ldots$, is positive.

We then put $P^k = L^{*k}$, thus determining R^k.

Because of different u_i^{*k} values for the same i, in general the members of the group will have different preferences over the alternatives. Fig. 1 may be used to illustrate possible preference orderings over three alternatives depending on the u_i^{*k} values, indicated in the diagram by a_i, b_i, c_i.

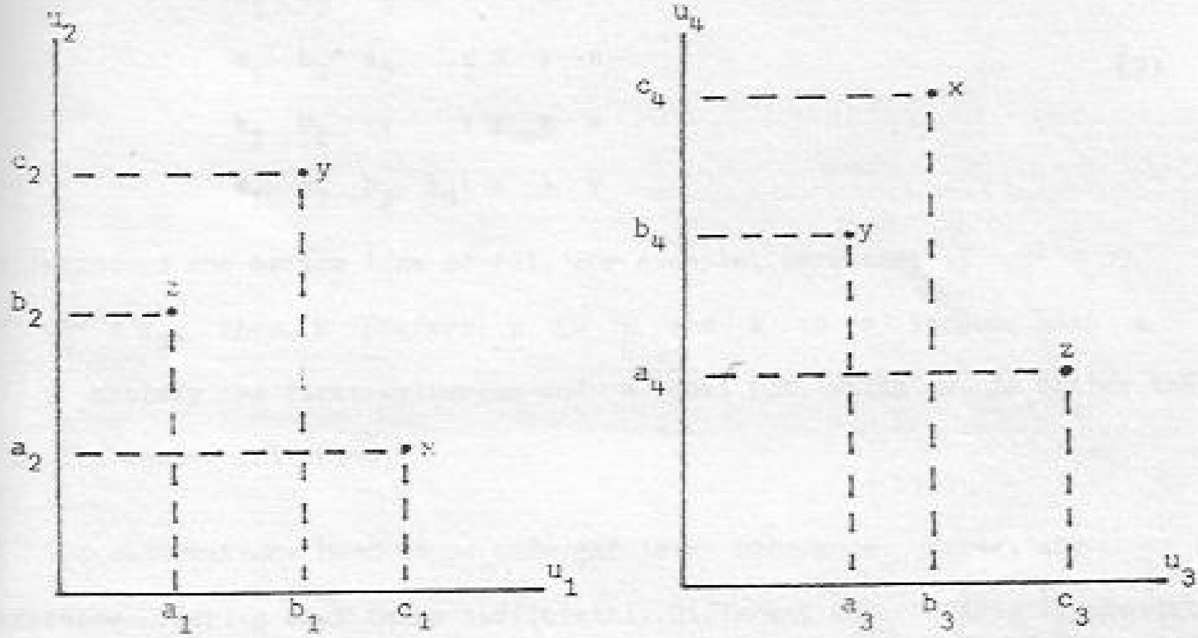

Figure 1

Some possibilities are shown in the following schema, where the symbols before a colon denote the u_i^{*k} values that suffice to give the ranking of x, y, and z after the colon, arranged in descending order of preference.

$$
\begin{array}{llll}
c_1 & & & : x\ y\ z \\
a_1 & c_2 & & : y\ z\ x \\
a_1 & a_2 & c_3 & : z\ x\ y \\
a_1 & b_2 & c_3 & : z\ y\ x \\
b_1 & c_2 & & : y\ x\ z \\
a_1 & a_2 & b_3 & c_4 : x\ z\ y
\end{array}
\tag{2}
$$

The second to the bottom line of (2), for example, says that if $u_1^{*k} = b_1$ and $u_2^{*k} = c_2$, then k prefers y to x and x to z because both x and y satisfy the first criterion and z does not, while y is better than x by the second criterion.

Two observations need to be made for later reference. First, any preference ordering over three sufficiently different alternatives is possible, as can be seen from (2). Second, a particular preference ordering may result from more than one set of u_i^{*k} $(i = 1, 2, \ldots)$ values. For example, the pattern x y z results also from $b_1\ a_2\ b_3$. Thus R^k does not suffice to recover U^k, which is k's preference system underlying the definition of L^{*k}.

Using the definition of L^{*k} without superscript k to get L^* for the group, we need only determine the group's u_i^* values as functions of the individual u_i^{*k} values. For this purpose we define u_i^* as the median of $u_i^{*1}, \ldots, u_i^{*N}$, or $u_i^* = \operatorname{med}_k \{u_i^{*k}\}$, $i = 1, 2, \ldots,$ assuming an odd number of persons in the group. The rationale is that with respect to any particular criterion u_i, Black's (1948) theorem on single-peaked preferences (over possible "candidates" for u_i^*) is applicable, since each k will want the group's u_i^* to be as close as possible to his own u_i^{*k}. Any higher value implies an unnecessarily high constraint on u_i while anything lower is less than satisfactory, so there would be no advantage for anyone to misrepresent his true u_i^{*k}. Accordingly, the members could vote for the value of u_i that is to serve as u_i^*, and only the median u_i^{*k} would win by simple majority rule over any other candidate.

We now put xQy equivalent to $- yL^*x$ so that $q(x) = (v_1(x), v_2(x), \ldots)$ in Assumption 3, and Assumption 4 then gives the choice set which we denote by $f^*(A)$ to distinguish it from other possible $f(A)$. This makes f^* a two-stage procedure, as in Section 2. Given a choice between x and y, if no one votes for y over x and someone votes for x over y, y is dropped from further consideration as a possible choice. Since pairwise comparisons can be made among all the elements of A, only Pareto optimal points will survive the first stage. The second stage then orders the remaining points lexicographically on the basis of the group's u_i^* values determined previously, giving $f^*(A)$.

The first stage is necessary because, as in the case of the majority closure method, the second stage by itself would not guarantee Pareto optimality. For example, let $u_i(x) > u_i(y)$ for $i = 1, 2, 3$ and $u_4(x) < u_4(y) \leq u_4^*$. Suppose that

$$G = m_1 \cup m_2 \cup m_3$$

$$= M_i \cup m_i \qquad (i = 1, 2, 3)$$

where M_i is a majority in the group G that finds x and y satisfactory and m_i a minority that considers y less than satisfactory as regards u_i.

Both alternatives therefore meet the group's u_i^* targets for $i = 1, 2, 3$ and yL^*x because of u_4. But individuals in m_i prefer x to y on account of u_i (if not because of a prior criterion) so everyone prefers x to y.

Referring back to schema (1) the f^* rule described above makes the group's U (i.e. the ordering implied by L^*) a function of $\{U^k\}$. The $p_A(x)$'s in Assumption 3 are determined by $\{U^k\}$ and the $q(x)$'s by U, giving R_A. The connection between $\{R^k\}$, which yields $\{R_A^k\}$, and R_A thus lies only in the first component of $r_A(x)$. We claim that the f^* rule is not unreasonable, relying only on repeated use of simple majority decision so that equal weights are given to all members' preferences, given the assumption that the members have the same priority ranking of the same set of objectives.[2/] This might seem to be a very special assumption, but it is a common experience for an appointed committee to receive terms of reference that specify the objectives of its assignment, though admittedly the priority ranking among them is not always made clear. At any rate the objective here is not a general model of group decision but simply a well defined rule that not only satisfies the Assumptions but also distinguishes between $\{U^k\}$ and $\{R^k\}$. It will serve as a specific counter-example to a number of consistency conditions.

Murakami (1961) showed in his formulation of Arrow's (1963) impossibility theorem that no group choice function g (reserving the use of f for a function satisfying the Assumptions) can satisfy all the following five conditions.[3/]

<u>Free triple</u>. There is at least one subset T of three alternatives in X over which each k can have any possible R_T^k.

Nondictatorship. There is no k such that for all x, y in T, $\{x\} = g(\{x, y\})$ if $xP^k y$.

Independence of irrelevant alternatives (IIR). $g(A)$ is invariant with respect to any changes in $\{R^k\}$ that do not change $\{R^k_A\}$.

Pareto principle. $\{x\} = g(\{x, y\})$ if $xP^k y$ for all k.

Collective rationality (CR). $g(A) = \{x \in A \mid \forall y \in A: xRy\}$, where R is reflexive, complete and transitive on X.

The possibility of any preference ordering over three alternatives has already been noted in schema (2), and the nondictatorship condition obviously holds. By construction, f^* satisfies the Pareto principle. That leaves IIR and CR which are discussed in the next section.

4. Consistency Conditions

Arrow's original rationale for the IIR condition was that $g(A)$ should be independent of alternatives outside A (Arrow (1963), p. 26). This is done by IIR, but it goes farther by also requiring that the group's ordering on A depend only on $\{R^k_A\}$.[4/] We observe that $f^*(A)$ is in fact independent of points not in A, but IIR can be violated because the same $\{R^k_A\}$ may result from different $\{U^k\}$--as already noted in connection with schema (2)--which is what determines R_A. The IIR condition as stated is thus really two conditions in one. The literal objective of Arrow's requirement can be accomplished by a straightforward reformulation.

Condition IIR'. $g(A)$ is independent of $\{R^k_{X-A}\}$, where $X - A$ is the set of x's not in A.

A review of the f^* rule shows that no information about $\{R^k_{X-A}\}$ is needed to determine $f^*(A)$, so IIR' is satisfied by $g = f^*$.

Comparing the CR condition with Assumption 4, CR makes the stronger requirement of a transitive R on X that is independent of A. Arrow's argument is that this would make group choice independent of the particular sequence in which the feasible alternatives are presented for choice: "The basic problem is ... the independence of the final choice from the path to it. Transitivity will insure this independence; from any [feasible set] there will be a chosen alternative" (Arrow (1963), p. 120). But then, transitivity on A and not necessarily on X would do for the purpose, as in the Assumptions. The CR condition demands more than is needed and is therefore an unnecessary requirement.

Plott (1973) has pointed out that if path independence ("the independence of the final choice from the path to it" in the literal sense) is the objective of transitivity, one may even dispense with the latter if path independence can be had without it. He has proposed a formalization of this property:

Path independence (PI). If $A = B \cup C$, then $g(A) = g(g(B) \cup g(C))$.

Clearly $f(A)$ is path independent in the literal sense since R_A is transitive, yet it can violate PI. Since PI implies Condition α (Chernoff (1954), Sen (1977)), a violation of α will also show failure of PI.

Condition α. If $A \subset B$, then $A \cap g(B) \subset g(A)$.

This condition has been considered as a "fundamental consistency requirement of choice" by many writers (see the references cited by Sen (1977), p. 67 and Kelly (1978), p. 26, n. 2). But suppose that $A = \{x, y\}$, x and y both

Pareto optimal in A, and $B = \{x, y, z\}$. One may have xQy and yQz (e.g. $xL^{*}y$ and $yL^{*}z$) while x is inferior to z and y is not. Then $A \cap f(B) = \{y\}$ but $f(A) = \{x\}$ so that α fails to hold for $g = f$.

Condition α seems a very natural requirement if the group's evaluation of alternatives is independent of the feasible set. Doing Venn diagrams for α shows its intuitive reasonableness if each point in A and B has a "value" to the group which does not depend on A or B, so that contour lines could be drawn to locate $g(A)$ and $g(B)$. Under the Assumptions, however, the value of an alternative is a vector whose first component depends on the feasible set, so that α can be violated. Similar remarks apply to the PI condition.

The idea behind the PI condition is that the choice in $A = B \cup C$ should come from the choices in B and C and should not depend on how A is disaggregated into B and C. But this means that if the condition is to be reasonable, the choices in B and C should at least qualify as possible candidates for choice in the larger set A, which requires under the Assumptions that they not be inferior in A. A revised version takes this requirement into account.

<u>Theorem</u> PI'. If $A = B \cup C$ & $f(B) \subset A_1$ & $f(C) \subset A_1$, then $f(A) = f(f(B) \cup f(C))$.

<u>Proof</u>. Suppose the hypothesis is true. First we show that $f(A) \subset f(f(B) \cup f(C))$. Clearly $A_1 \subset B_1 \cup C_1$. Hence if $x \in f(A)$, then $x \in A_1$, x is Q-greatest in A_1, and $x \in B_1 \cup C_1$. Also, $B_2 \cup C_2 \subset A_1$, so that x is Q-greatest in $B_2 \cup C_2$ by Lemma 1. This with $x \in B_1 \cup C_1$ and Lemma 6 gives $x \in B_2 \cup C_2$, and therefore $x \in f(B_2 \cup C_2)$.

To show $f(f(B) \cup f(C)) \subset f(A)$, suppose that $x \in f(f(B) \cup f(C))$. This means $x \in B_2 \cup C_2$ and x is Q-greatest in $B_2 \cup C_2$. Since $B_2 \cup C_2 \subset A_1$, we have $x \in A_1$ so that $x \in A_2$ if x is Q-greatest in A_1. This is so, by Lemma 1, since $A_1 \subset B_1 \cup C_1$ and x is Q-greatest in $B_1 \cup C_1$, by Lemma 5, from the fact that $x \in B_2 \cup C_2$ and is Q-greatest in $B_2 \cup C_2$. This completes the proof of PI', which we label as a theorem from the Assumptions.

The rationale for α is that an alternative chosen in B, if still available when the feasible set has been reduced to A, should be among those chosen in A because it is "best" in the larger set and should therefore be best also in the smaller one. This seems reasonable enough, but it implicitly assumes that the choices in A qualify as possible choices in B, which may not be the case. This is a recurring theme in violations of the consistency conditions, and when the assumption is made explicit, as in PI', the results are different.

<u>Theorem</u> α'. If $A \subset B$ & $f(A) \subset B_1$, then $A \cap f(B) \subset f(A)$.

<u>Proof</u>. Let the hypothesis be true. The conclusion is falsified if and only if there is an x such that $x \in A \cap B_2$ & $- x \in A_2$. Suppose such an x. Since $A \subset B$, Lemma 2 gives $A \cap B_1 \subset A_1$ so that $x \in A_1 \cap B_2$ since $x \in A \cap B_2$ and $B_1 \cap B_2 = B_2$. But $A_2 \subset B_1$ from the hypothesis, and therefore $x \in A_2$ by Lemma 4, contradicting $- x \in A_2$.

Four related conditions may be discussed together. Condition $\beta+$ was introduced by Bordes (1976), β by Sen (1969), ε by Blair (1974)--as reported by Sen (1977), p. 69--and δ by Sen (1971).

<u>Property</u> $\beta+$. If $A \subset B$ & $A \cap g(B) \neq \emptyset$, then $g(A) \subset A \cap g(B)$.

Condition β. If $x \in g(A)$ & $y \in g(A)$ & $A \subset B$, then $x \in g(B)$ if and only if $y \in g(B)$.

Condition ε. If $A \subset B$, then $\sim (g(B) \subset g(A)$ & $\sim g(A) \subset g(B))$.

Condition δ. If $x \in g(A)$ & $y \in g(A)$ & $A \subset B$, then $(\{x\} \neq g(B)$ & $\{y\} \neq g(B))$.

Since β+ implies β, β implies ε, and ε implies δ (Sen (1977)), the following violation of δ also shows failure of the others. Suppose x is inferior in B and y is not. Then $\{x\} \neq g(B)$ but $\{y\} = g(B)$ is possible.

These conditions put requirements on alternatives chosen in A when the feasible set is enlarged to B. As with PI and α, they fail to hold because of the possibility that an alternative chosen in a set may be dominated in a larger set. Suppose this possibility is restricted.

Theorem β+'. If $A \subset B$ & $f(A) \subset B_1$ & $A \cap f(B) \neq \emptyset$, then $f(A) \subset A \cap f(B)$.

Proof. Suppose there exists $y \in A \cap f(B)$, and suppose $x \in A \subset B$ and $x \in f(A)$. Then x is Q-greatest in A and therefore xQy since $y \in A$, so that $x \in f(B)$ since $y \in f(B)$ and x is undominated in B given the proviso that $f(A) \subset B_1$. Hence $x \in A \cap f(B)$.

If β, ε and δ are similarly revised by adding $f(A) \subset B_1$ to their hypotheses, the resulting propositions--call them β', ε' and δ' respectively--become theorems. For suppose the hypothesis of β' is true. Then x and y are Q-greatest in A_1 and both belong to B_1. Therefore if x is Q-greatest also in B_1, so must y, and vice versa. Propositions

ε' and δ' follow directly from β', which makes all points in $f(A)$ belong to $f(B)$ or none at all, so that $f(B)$ cannot be a proper subset of $f(A)$.

Sen's (1971) Condition γ, which is equivalent to the following statement, is quite different from the others as it follows from the Assumptions.

<u>Theorem</u> γ. If $x \in f(A)$ & $x \in f(B)$, then $x \in f(A \cup B)$.

<u>Proof</u>. Under the hypothesis, x is in A_1 and in B_1 and Q-greatest in A_1 and in B_1; therefore $x \in A_1 \cup B_1$ and x is Q-greatest in $A_1 \cup B_1$, giving the conclusion.

The reason for the difference is the fact that the hypothesis of γ does not allow x to be inferior in any of the sets considered. Plott's (1973) Axiom E, also called the Generalized Condorcet (GC) property by Blair, et al. (1976), is a weaker version of γ. It is therefore also true and shown in a similar way.

<u>Theorem</u> E. If $x \in A$ & $\forall y \in A$: $x \in f(\{x, y\})$, then $x \in f(A)$.

There are other consistency conditions--Axioms 1 and 2 of Plott (1973) which are variations of the PI condition, α- of Sen (1977) and B3 of Batra and Pattanaik (1972) which are weaker versions of α, and δ^* of Richelson (1978) which is a weaker version of δ--that are failed by f, but suitable reformulations are consequences of the Assumptions. In each case, the needed amendment (indicated by square brackets below) is simply to make the alternatives qualify as possible choices in some appropriate set. We merely catalogue those consequences, omitting the proofs which are similar to those above.

Theorem A1'. [If $f(A - B) \subset A_1$ & $f(A \cap B) \subset A_1$, then] $f(A) = f(f(A - B) \cup f(A \cap B))$.

Theorem A2'. If $A \cap f(B) \neq \emptyset$ [& $A \cap f(B) \subseteq A_1$], then $f(A) = f((A - B) \cup (A \cap f(B)))$.

Theorem α-'. If $A \neq \emptyset$ [& $\forall x, y \in A$: $f(\{x, y\}) \subset A_1$], then $\exists z \in f(A)$: $(\forall y \in A: z \in f(\{y, z\}))$.

Theorem B3'. If $\{x, y\} \subset A$ [& $\{x, y\} \subset A_1$] & $f(\{x, y\}) = \{x\}$, then $-(y \in f(A)$ & $-x \in f(A))$.

Theorem δ*'. If $f(\{x, y\}) = \{x, y\} \subset B$ [& $f(\{x, y\}) \subset B_1$], then $(\{x\} \neq f(B)$ & $\{y\} \neq f(B))$.

5. Concluding Observations

We have seen that a number of conditions for group decision can be failed by an internally consistent and not unreasonable procedure, and therefore they cannot be essential. When these conditions are reformulated so that the alternatives being considered at least qualify as possible choices, the revised versions turn out to be consequences of the Assumptions of this paper. The key observation is that if choice is required to satisfy Pareto optimality, the alternatives are better represented analytically by lexicographically ordered vectors. This permits a class of group orderings which do not automatically satisfy the requirement, e.g. the majority closure method and L^* ordering, to serve as the second stage in a two-stage procedure.

A related observation concerns the value of an alternative, which cannot be invariant with respect to changes in the feasible set if its Pareto property (i.e. whether it is inferior or not) is relevant. This is a feature of group choice which is absent from individual decisionmaking, so that conditions on choice that might be compelling for an individual are not necessarily so for the group. In particular, while it seems quite reasonable to say that one can infer an individual's choices over larger sets from his choices over two-element sets, this need not be the case for the group.

NOTES

1. Bordes (1976) makes a distinction between consistency and rationality in group choice: "Consistency is concerned with what happens to choices when the set of available alternatives expands or contracts. Rationality is concerned with how the choices are related to a binary relation on the set of all alternatives" (p. 451). As will be seen, however, there may be no such binary relation in group choice, in which case one would have consistency conditions only.

2. Cf. Savage (1954), pp. 172-173, in whose formulation of a group decision problem assumed the <u>same</u> utility function for the members of the group but allowed them to have "different ... judgments as to questions of fact."

3. It is implicitly assumed that the group has a finite number of members (Fishburn (1970)).

4. Arrow restricted the dependence of R_A to $\{R_A^k\}$ on the ground that $\{R_A^k\}$ contains all the admissible information needed to determine R_A, given only standard individual utility functions of the "ordinal" type. As schema (1) indicates and the example of Section 3 shows, one need not be bound by this restriction.

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