University of the Philippines
SCHOOL OF ECONOMICS

Discussion Paper 8103 February 1981

Lexicographic Consumer Theory

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Abstract

This paper develops some implications of consumer choice when preferences are lexicographic, by formulating the consumer's choice as a nonlinear programming problem. Under certain conditions, it turns out that some familiar results of standard consumer theory (including the Slutsky equation) have analogues under lexicographic choice. It is suggested that quality variations--where some goods in the consumer's market basket are replaced by similar goods of different quality--find an easier explanation with vector-valued utility.

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1. Introduction

The purpose of this paper is to develop some implications of consumer theory based on lexicographic preferences that represents the desirability of a consumption bundle in terms of a vector-valued utility function (Georgescu-Reogen [5], Chipman [1], Encarnación [3], Day and Robinson [2]). A natural application of Kuhn-Tucker nonlinear programming [8] provides a model for the purpose.

Section 2 is a quick review of lexicographic preferences. (For a survey of this literature, see Fishburn [4].) Section 3 takes the consumer's choice as a nonlinear programming problem, and a special case that parallels standard theory is treated in Section 4 which gives a generalization of the Slutsky equation. Section 5 considers "quality variations" (Houthakker [6])--where some goods are replaced by similar goods of different quality in the consumer's market basket--which find an easier explanation with vector-valued utility.

2. Lexicographic Preferences

It is assumed that there are various criteria of choice which are ranked in order of importance or priority. To each point $x = (x_1, \ldots, x_n)$ in the consumption goods space corresponds a utility vector $u(x) = (u_1(x),$

$u_2(x), \ldots)$ where u_i $(i = 1, 2, \ldots)$ is a real-valued function such that $u_i(x) > u_i(y)$ if x is preferred to y on the basis of the ith criterion. Suppose that x is preferred to y if and only if the first nonvanishing difference $u_i(x) - u_i(y)$, $i = 1, 2, \ldots,$ is positive, i.e. the preference ordering of the x's is given by the lexicographic ordering of the $u(x)$'s. Call such a preference ordering an L-ordering. L-ordering is rather unrealistic (cf. Malinvaud [9, p. 20]), or else it is largely superfluous for purposes of explaining consumer behavior since the first component of the utility vector would then normally suffice to determine the choice (Houthakker [7, p. 711]). A simple modification however results in a richer structure of preferences that has greater explanatory power than the standard utility function.

Suppose there exist critical or satisfactory levels u_i^* such that if $u_i(x) \geqq u_i^*$, the consumer considers x acceptable in terms of the ith criterion. More precisely, writing $v_i(x) = \min\{u_i(x), u_i^*\}$, let the preference ordering be given by the lexicographic ordering of the vectors $v(x) = (v_1(x), v_2(x), \ldots)$ and call it an L*-ordering. Accordingly, a criterion other than the first can be the determinant of choice subject to appropriate constraints regarding higher priority criteria. Specifically, if A_0 is the budget constraint set and

$$A_i = \{x \in A_{i-1} | u_i(x) \geqq \max_y \{v_i(y) | y \in A_{i-1}\}\} \qquad (1)$$

$i = 1, 2, \ldots,$ then x^0 will be chosen only if $x^0 \in A_0 \cap A_1 \cap A_2 \cap \ldots$. One goes through the choice criteria sequentially, beginning with the most important and discarding from further consideration those alternatives that fail to satisfy the constraint $x \in A_i$ at each stage, until just one alternative is left. The u_i^* are thus like objectives which are to be attained in the order of their importance.

Essentially different wants and needs (e.g. for water, food, shelter, medical services, etc. which are nonsubstitutable since no amount of one can satisfy the need for another) are then to be represented by different components of the utility vector. But as with the standard utility function, substitution possibilities exist within each component of the utility vector. Also, nothing prevents tradeoffs among some choice criteria once critical levels are surpassed. For example, u_k may be a function of $u_1, \ldots, u_{k-1}$. L*-ordering is thus quite flexible as an analytical framework, and it is possible to think of the standard utility function as such a u_k, the implicit assumption being that prior constraints on choice are always exceeded.

3. The Model

Let $p > 0$ be the given price vector and $B > 0$ the budget, and write $\theta_i = \max_y\{v_i(y) | y \in A_{i-1}\}$. If $\theta_1 < u_1^*$, it would be fortuitous for A_1 to have more than one element since this would mean that A_1 has a "flat" section that happens to coincide with part of $\{x | p'x = B\}$. In general, it would be exceptional for A_i to have more than one element if $\theta_i < u_i^*$.

Let j be the smallest index in $\{k | A_k$ is a one-element set$\}$. Excluding the exceptional cases described above, the consumer would have the following problem:

$$\text{Maximize} \quad u_j(x) \tag{2}$$

$$\text{subject to} \quad u_i(x) - u_i^* \geqq 0 \qquad i = 1, \ldots, j-1 \tag{3}$$

$$B - p'x \geqq 0 \tag{4}$$

$$x \geqq 0. \tag{5}$$

In the event that the exceptional case holds for a particular i, then θ_i simply replaces u_i^* in (3). With this understanding, our discussion will focus on the problem given by (2)-(5).

We need two theorems from Kuhn and Tucker [8] on the problem of maximizing $g(x)$ subject to $F(x) \geqq 0$, where $F(x)$ is a vector whose components are differentiable functions of $x \geqq 0$: (i) If a certain regularity condition (the Kuhn-Tucker constraint qualification) is satisfied--it would be satisfied if the constraint set $\{x \mid F(x) \geqq 0, x \geqq 0\}$ is convex--the following conditions are necessary for a particular x to solve the problem:

$$\nabla g(x) + \nabla F(x)'\lambda \leqq 0 \tag{6'}$$

$$(\nabla g(x) + \nabla F(x)'\lambda)'x = 0 \tag{7'}$$

$$F(x) \geqq 0 \tag{8'}$$

$$F(x)'\lambda = 0 \tag{9'}$$

$$x \geqq 0 \tag{10'}$$

$$\lambda \geqq 0 \tag{11'}$$

where $\nabla g(x)$ is the gradient of g at x, $\nabla F(x)$ the Jacobian of F, and λ a vector of Lagrange multipliers. (ii) Further, if $g(x)$ and the elements of $F(x)$ are concave functions, the above conditions are also sufficient.

Assuming that for each i, $\{x | u_i(x) \geqq \theta\}$ is convex for any real θ, the conditions (3)-(5) define a convex set. We therefore have the following necessary conditions:

$$\frac{\partial u_j}{\partial x_r} + \sum_{i=1}^{j-1} \lambda_i \frac{\partial u_i}{\partial x_r} - \lambda_0 p_r \leqq 0 \qquad r = 1, \ldots, n \tag{6}$$

$$\left(\frac{\partial u_j}{\partial x_r} + \sum_{i=1}^{j-1} \lambda_i \frac{\partial u_i}{\partial x_r} - \lambda_0 p_r\right) x_r = 0 \qquad r = 1, \ldots, n \tag{7}$$

$$u_i(x) - u_i^* \geqq 0 \qquad i = 1, \ldots, j-1 \tag{8a}$$

$$B - p'x \geqq 0 \tag{8b}$$

$$(u_i(x) - u_i^*)\, \lambda_i = 0 \qquad i = 1, \ldots, j-1 \tag{9a}$$

$$(B - p'x)\, \lambda_0 = 0 \tag{9b}$$

$$x \geqq 0 \tag{10}$$

$$\lambda \geqq 0. \tag{11}$$

As in the case of the standard utility function, each u_i is unique up to a positive monotone transformation; hence we can take it that the u_i's are concave functions. Conditions (6)-(11) are therefore also sufficient.

Write

$$U_r = \frac{\partial u_j}{\partial x_r} + \sum_{i=1}^{j-1} \lambda_i \frac{\partial u_i}{\partial x_r}. \tag{12}$$

$U_r > 0$ for some r, or the consumer would be at a point of complete satiety. Ruling out the latter, $\lambda_0 > 0$ from (6), so that using (9b), $B - p'x = 0$ in (8b). From (7),

$$\lambda_0 = U_r/p_r \quad \text{for all } r \text{ such that } x_r > 0 \tag{13}$$

and if $U_r < \lambda_0 p_r$, $x_r = 0$. λ_0 could be interpreted as the "marginal utility" of income, except that it is now a weighted sum of marginal utility components. Also,

$$p_r/p_s = U_r/U_s \quad \text{for } x_r,\ x_s > 0 \tag{14}$$

which is similar to another familiar result in standard theory. If all of (8a) are satisfied as strict inequalities, then all $\lambda_i = 0$ in (9a) so that (14) reduces to $p_r/p_s = (\partial u_j/\partial x_r)/(\partial u_j/\partial x_s)$ for $x_r, x_s > 0$. Only in such a case do we have marginal rates of substitution (with respect to the utility component being maximized) equal to relative prices.

It is clear that demand functions from the model are homogeneous of degree zero. Looking at (6)-(11), multiplying B and p by the same $\mu > 0$ merely changes λ_0 to λ_0/μ without affecting the solution x. Also, the revealed preference property (Samuelson [10, p. 111]) is satisfied, since this is not contingent on real-valued utility functions but only on consistent preferences.

4. A Special Case

To determine how the solution x changes in response to changes in the parameters p and B, we can examine the relatively simple case where the maximand u_j remains the same and all the indices r and i such that $x_r > 0$ and $\lambda_i > 0$, respectively, also remain the same. In this case we can work with the equations in (6) and (8a) implied by the $x_r > 0$ in (7) and the $\lambda_i > 0$ in (9a). Taking their total differentials gives the following system (which accordingly includes only those relationships that derive from those equations):

$$\begin{bmatrix} W & \nabla u_1 & \cdots & \nabla u_{j-1} & -p \\ \nabla u_1' & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \nabla u_{j-1}' & 0 & \cdots & 0 & 0 \\ -p' & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\lambda_1 \\ \vdots \\ d\lambda_{j-1} \\ d\lambda_0 \end{bmatrix} = \begin{bmatrix} \lambda_0 dp \\ 0 \\ \vdots \\ 0 \\ E \end{bmatrix} \tag{15}$$

where $W = (w_{rs})$, $w_{rs} = \partial^2 u_j/\partial x_r \partial x_s + \sum_{i=1}^{j-1} \lambda_i \, \partial^2 u_i/\partial x_r \partial x_s$, $E = -dB + \sum_s x_s \, dp_s$, $dx = (dx_1, \ldots, dx_n)'$, and similarly for dp. Note that $w_{rs} = w_{sr}$, so the coefficient matrix is symmetric. Let D be the determinant of this matrix and D_{ts} the cofactor of its (t, s)-element. Then

$$\frac{\partial x_s}{\partial p_r} = \frac{\lambda_0 D_{rs} + x_r D_{n+j,s}}{D} \tag{16}$$

and

$$\frac{\partial x_s}{\partial B} = \frac{-D_{n+j,s}}{D} \tag{17}$$

which are similar to standard results.

A generalized version of the Slutsky equation can also be shown. Let $dp_r \neq 0$ for a particular r and $dp_t = 0$ for all $t \neq r$, and choose dB so that $du_j = \sum_s (\partial u_j/\partial x_s)\, dx_s = 0$. From (13) and (12),

$$\lambda_0 \sum_s p_s \, dx_s = \sum_s \sum_{i=1}^{j-1} \lambda_i \frac{\partial u_i}{\partial x_s} dx_s \tag{18}$$

after putting $du_j = 0$. But from each equation in (8a), $\sum_s (\partial u_i/\partial x_s)\, dx_s = 0$; multiplying by λ_i and summing over i, both sides of (18) vanish. Hence $E = 0$ in (15) and we have (cf. (16))

$$\left.\frac{\partial x_s}{\partial p_r}\right|_{u[j]=\text{const}} = \frac{\lambda_0 D_{rs}}{D} \tag{19}$$

where $u[j]$ is a vector consisting of u_j and those among $u_1, \ldots, u_{j-1}$ as have corresponding equations in (8a). Noting that $D_{rs} = D_{sr}$, the implication for the left-hand side of (19) is clear. Finally, putting (19) and (17) in (16),

$$\frac{\partial x_s}{\partial p_r} = \left.\frac{\partial x_s}{\partial p_r}\right|_{u[j]=\text{const}} - x_r \left.\frac{\partial x_s}{\partial B}\right|_{p=\text{const}} \tag{20}$$

which is analogous to the Slutsky equation of standard theory.

We have been assuming in this special case that the component of the utility vector being maximized remains the same, as also the list of goods in positive amounts and the list of binding constraints. In general, such restrictive conditions will fail to hold. While the consumer's choice problem is always determinate, we expect that sufficiently large changes in income will change the maximand and the list of goods appearing in the consumer's market basket. In particular, at a sufficiently higher level of income, higher quality goods would replace similar goods of lower quality.

5. Quality Variations

In the treatment of this topic by Theil [11] and Houthakker [6], a commodity is a set of related goods (measured in the same physical units) of different quality levels, the price of a good being higher if its quality is higher. Judgment of quality is that of the consumer, who can always ignore a higher-priced good unless its quality is higher. Theil and Houthakker consider the quality and quantity pertaining to each commodity as separate arguments in the utility function. However, their formulations do not indicate why quality levels might be expected to rise with income, which is the empirical observation to be explained. We will sketch a different approach suggested by our previous discussion.

In place of the notation x_r for the amount of good r, let $x_{c\alpha}$ be the amount of commodity c whose quality is α. We expect that $\partial u_i/\partial x_{c\alpha} = 0$ identically for some i, c, α; that is, some goods do not contribute towards some objectives. Also, if two goods belong to the same commodity class c, $\partial u_i/\partial x_{c\alpha} > 0$ and $\partial u_i/\partial x_{c\beta} > 0$ for some i; otherwise there would be no basis for classifying them together. We propose to say that good $c\beta$ has higher quality than $c\alpha$ if $\partial u_i/\partial x_{c\beta} \geq \partial u_i/\partial x_{c\alpha}$ for all i, with $>$ holding for at least one i. If $>$ holds for j, we would then say that $c\beta$ is superior to $c\alpha$ with respect to u_j.

If $c\beta$ has higher quality than $c\alpha$ and superior with respect to u_j, we would expect $c\beta$ to replace $c\alpha$ in the consumer's market choice at a sufficiently high income. To illustrate, let the solution to the problem (2)-(5) be represented by point e in Fig. 1, abstracting from other goods. The (absolute value of the) slope of the "representative" constraint u_i^*, $i < j$, is greater than that of the budget line ba, which itself is steeper than u_j-indifference curves. With more income, the new solution would have a higher u_j resulting from the displacement of some $x_{c\alpha}$ by $x_{c\beta}$, and a sufficiently large increase in income would lead to complete displacement of the lower-quality good. Similarly, a sufficient reduction in income would imply zero consumption of the higher-quality good. Accordingly we expect different income classes to consume different goods, the income-consumption path being indicated by arrows in Fig. 1, since any good becomes inferior at a sufficiently high income level if a higher-quality good is available.

Price implications can also be seen from Fig. 1. If both goods are being consumed, a higher (lower) price for either one implies a reduction (an increase) in the quantity of the higher-quality good and an increase (a decrease) in that of the lower-quality one. Results are not symmetric, however, if only one of them is being consumed. In this case, an increase in its price implies a decrease in the quantity purchased, but a price reduction need not lead to a quantity increase. What may happen instead is a decrease because of a higher-quality good entering the consumer's basket.

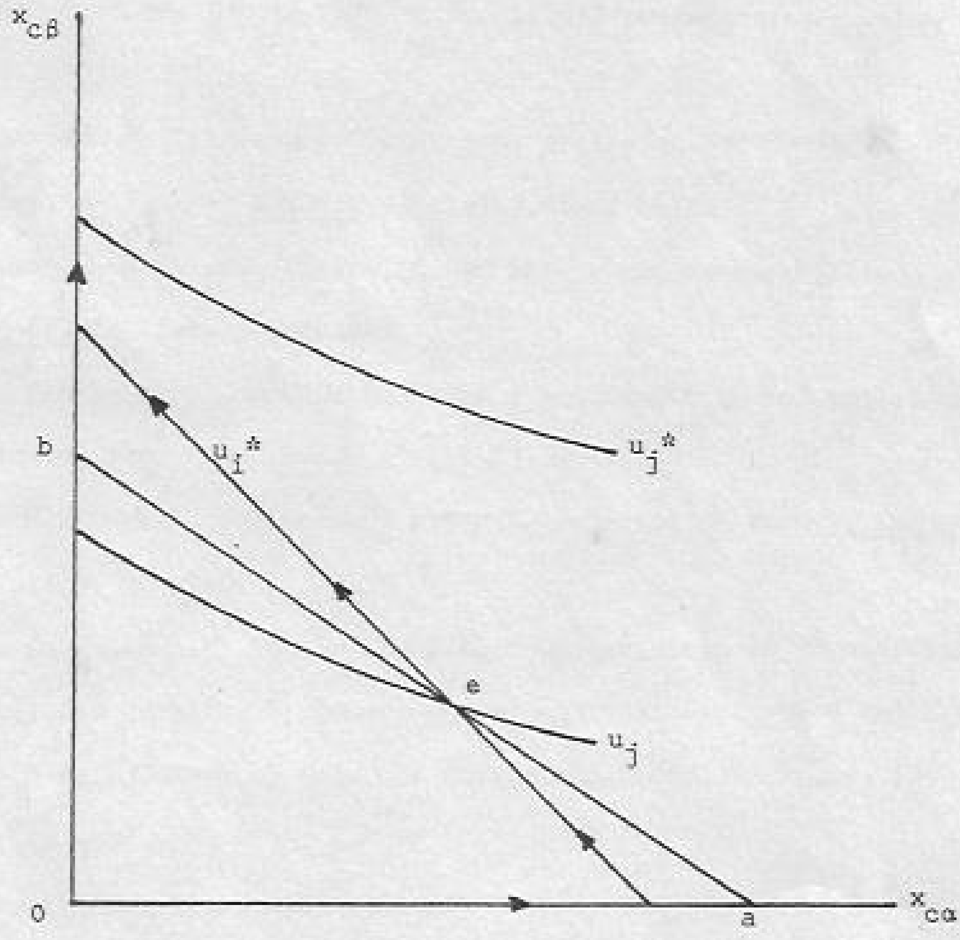

Fig. 1

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