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Assuming that the parties to a conflict have lexicographic preferences, an arbitration model is formulated whose proposed solution satisfies five reasonable conditions. Four of these conditions are analogous to the four conditions that characterize the Nash solution to the bargaining problem. The fifth condition, called "nonfavoritism," is new. It turns out that if any possible solution is required to satisfy the five conditions, it can only be the proposed solution.

1. Introduction

This paper is concerned with the problem of finding a "fair" solution to a conflict between two individuals or parties. The conflict arises because if either person were to have his way, he would choose a point in the feasible set different from what the other person would choose. If the parties to the conflict feel that some agreement would be better than none, an arbiter is then needed to decide for both of them in a manner that does not deliberately favor either one over the other.

In the context of game theory, Luce and Raiffa "define an arbitration scheme to be a function, i.e. rule, which associates to each conflict, i.e. two-person non-strictly competitive game, a unique payoff to the players" [7, p. 121]. The Nash solution [8], for example, maximizes (after a suitable transformation) the product of the two players' utilities which are of the von Neumann-Morgenstern "cardinal utility" type. What is especially interesting about the Nash solution is that given the cardinal utility assumptions, it satisfies four apparently reasonable conditions and it is the only solution that does so (cf.[7, pp. 126-127]).

In this paper we will instead treat the problem under the assumption that the two individuals' preferences are representable by lexicographic utility functions (see Fishburn [5] for a survey of this literature, the major references being Georgescu-Roegen [6] and Chipman [2]). It will turn out that our proposed solution (though a set and not in general a unique point) satisfies four properties similar to the Nash conditions and also a fifth condition, called "nonfavoritism," that we introduce; moreover, it is the only solution that does so.

After a quick review of lexicographic utility functions in Section 2, the model will be described in Section 3. Section 4 gives some properties of the solution, and Section 5 extends the results to the case of more than two persons. Section 6 is a concluding remark.

2. Lexicographic Utility

We assume that alternatives are evaluated in terms of various criteria of choice $u_1, u_2, \ldots,$ so that to each alternative x corresponds a vector $u(x) = \{u_1(x), u_2(x), \ldots\}$. Each u_i is a real valued function (like the standard utility function, admitting of arbitrary positive monotonic transformations) that indicates the desirability of x on the basis of the ith criterion. We postulate particular values u_i^* such that if $u_i(x) \geq u_i^*$, the alternative x is considered satisfactory in terms of the ith criterion. (If strict inequality holds, we will say that x is supra-satisfactory.) For example, in the theory of the firm, one might speak of a satisfactory rate of profit, the profit rate being a criterion of choice [3].

Writing $v_i(x) = \min \{u_i(x), u_i^*\}$, we associate with each x the vector $v(x) = \{v_1(x), v_2(x), \ldots\}$ and say that x is preferred to y if and only if the first nonvanishing difference $v_i(x) - v_i(y)$ $(i = 1, 2, \ldots)$ is positive. That is, the preference ordering over the x's is given by the lexicographic ordering of the corresponding $v(x)$'s. The u_i^* thus play the role of targets or objectives to be attained, if they are attainable, in accordance with their relative priorities. We will refer to such a preference ordering as an L^*-ordering.

It is to be noted that $v(x)$ is possibly infinite-dimensional. Under L^*-ordering, the common statement that wants are unlimited has to do with the dimensionality of $v(x)$ and not with the unboundedness of a real-valued utility function.

3. The Model

Consider two persons indexed by $k = 1, 2$ who agree to arbitration over their conflict on the ground that they would do better through an arbitrated solution than if they were to go their separate ways. They have L^*-orderings over the possible outcomes but their objectives and priorities are not identical. That is, Person 1 looks at the alternatives in terms of his $u_1^{1*}, u_2^{1*}, \ldots,$

while 2 looks at them in terms of his own $u_1^{2*}, u_2^{2*}, \ldots,$ and u_i^1 need not be the same function as u_i^2 (using superscripts on the notation in Section 2 to label the persons). We assume that while each party may have a general idea of the other party's concerns, he does not know exactly the other's priorities and targets. The arbiter needs to know, however, each party's L*-ordering, and information on this is communicated to him as the need arises. This implies that he can tell, given any two alternatives x and y, whether $u_i^k(x) > u_i^k(y)$ holds or not. As we shall argue in due course, there would be no reason for either party to misrepresent his preferences to the arbiter.

Let S_0 be the subset of the feasible alternatives which, from the viewpoint of both parties, are at least as good as what each can get himself with no agreement. An alternative in S_0 will be represented as a point in $2n$-dimensional space, where $n = \min(n^1, n^2)$ and n^k is the number of objectives that k has. Since S_0 contains the no-agreement point, S_0 is nonnull. We assume that S_0 is closed and convex, by appropriate transformations of the u_i^k $(i = 1, 2, \ldots;$ $k \in K = \{1, 2\})$ if necessary.

Consider

$$A_1 = \{x \in S_0 \mid \forall k: u_1^k(x) \geqq u_1^{k*}\}$$

which satisfies both parties' first objectives. If A_1 is nonnull, the arbiter can then narrow the choice further by considering next how the alternatives in A_1 rank with respect to the second objectives. If A_1 is null, we would have one of the situations typified in Fig. 1, which depicts possible relevant boundaries of S_0 in (u_1^1, u_1^2)-space.

Let

$$B_1^k = \{x \in S_0 | \ u_1^k(x) \geqq \max_y \ \{v_1^k(y) | \ y \in S_0\}\}.$$

Given the choice, k would want a point in B_1^k. The problem here is that 1 and 2 cannot both be accomodated. In Fig. 1(a) for example, 1 wants a while 2 wants b. Confining himself to the two parties' first objectives only, there is no way for the arbiter to decide the issue without favoring one over the other.

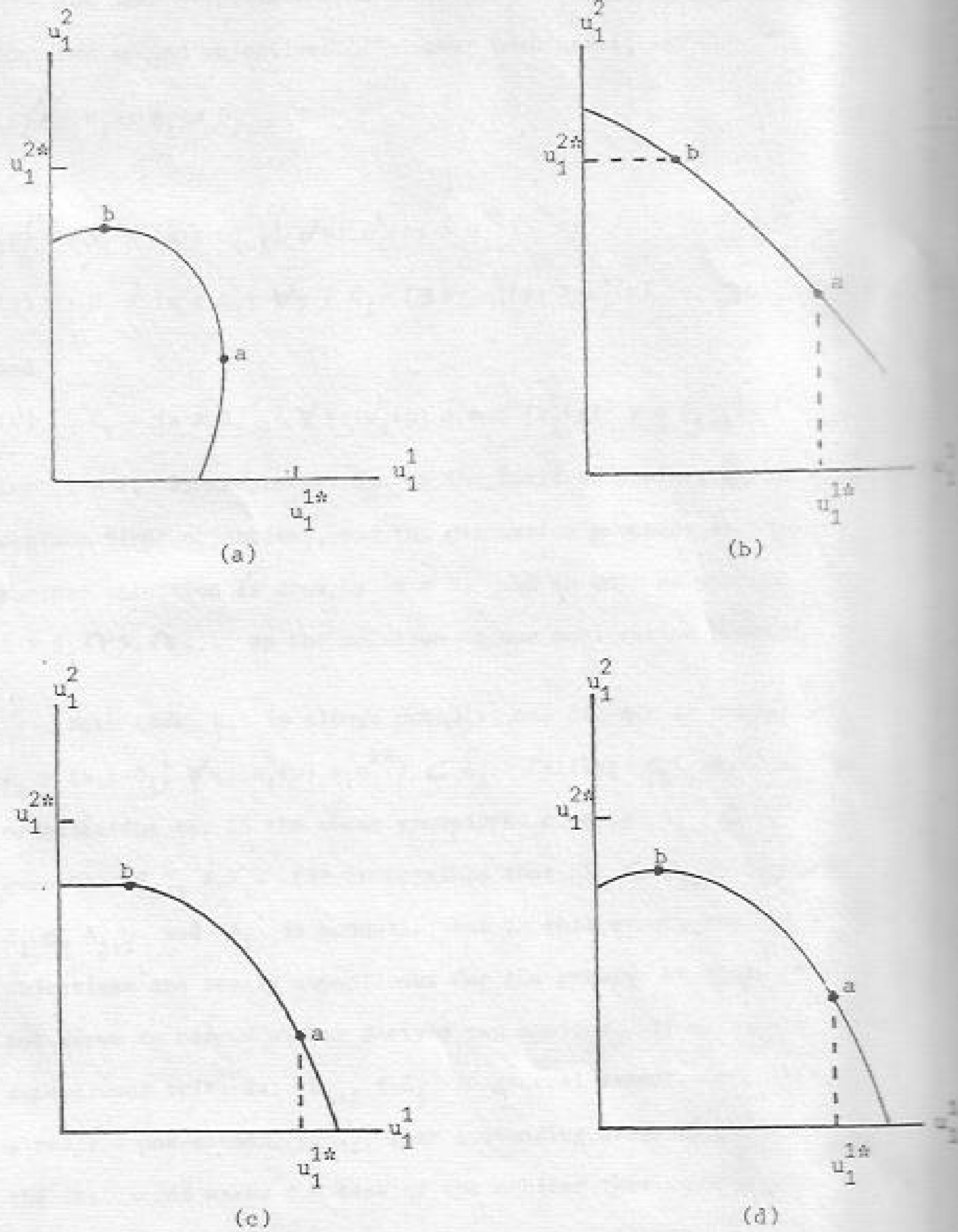
In order to resolve this issue, we propose to proceed as follows. Let

$$C_1 = \{x \in S_0 | \ \forall k: u_1^k(x) \leqq \max_y \ \{v_1^k(y) | \ y \in S_0\}$$

and define what might be called the undominated set in C_1 by

$$D_1 = \{x \in C_1 | \ \forall y \in C_1: (\exists k: u_1^k(y) > u_1^k(x) \rightarrow \ \exists h: u_1^h(x) > u_1^h(y))\}$$

whose endpoints are a and b in Fig. 1. Let us say that x is v_i-inferior to y if $v_i^k(x) \leqq v_i^k(y)$ for all k and $v_i^h(x) < v_i^h(y)$ for some h. It may then be observed that if A_1 is null, every point in $S_0 - D_1$ (the set of points in S_0 that do not belong to D_1) is either supra-satisfactory to someone or else v_1-inferior to some point in D_1. We say that the choice must lie in D_1 since, so far as first objectives go, there is no warrant for 1 to get more than $u_1^1(a)$ nor for 2 to get more than $u_1^2(b)$. In Fig. 1(b) for example, 1 would be satisfied with a; there can be no justification then for him to exceed $u_1^1(a)$, which would give 2 an even smaller u_1^2 that is already less than u_1^{2*}.



u_1^2
u_1^{2*}
b
a
u_1^1
u_1^{1*}
(a)
u_1^2
u_1^{2*}
b
a
u_1^1
u_1^{1*}
(b)
u_1^2
u_1^{2*}
b
a
u_1^1
u_1^{1*}
(c)
u_1^2
u_1^{2*}
b
a
u_1^1
u_1^{1*}
(d)

Figure 1

The next step, as in the case where A_1 is nonnull, is to consider second objectives. To cover both cases, let

$$S_i = A_i \cup D_i \qquad (i = 1, 2, \ldots) \tag{1}$$

where

$$A_i = \{x \in S_{i-1} \mid \forall k: u_i^k(x) \geq u_i^{k*}\} \tag{2}$$

$$D_i = \{x \in C_i \mid \forall y \in C_i: (\exists k: u_i^k(y) > u_i^k(x) \rightarrow \exists h: u_i^h(x) > u_i^h(y))\} \tag{3}$$

and

$$C_i = \{x \in S_{i-1} \mid \forall k: u_i^k(x) \leq \max_y \{v_i^k(y) \mid y \in S_{i-1}\}\}. \tag{4}$$

For $i = 2$, S_1 replaces S_0 as the field of choice, second objectives replace first objectives, and the discussion proceeds as in $i = 1$. Further selection is done by $i = 3$, and so on. We propose $S = S_1 \cap S_2 \cap \ldots$ as the solution to our arbitration problem.

Note that D_i is always nonnull, and if A_i is nonnull, $D_i = \{x \in C_i \mid \forall k: u_i^k(x) = u_i^{k*}\} \subset A_i$. Calling S_i the ith stage or selection set in the above procedure, clearly $S_{i+1} \subset S_i$ but in general, $S_{i+1} \neq S_i$. (It is possible that $S_i \subset S_{i+1}$ as when $A_i \subset A_{i+1}$ and A_i is nonnull. But in this event, the $(i + 1)$th objectives are really superfluous for the problem at hand, as they do not serve to narrow either party's own choices. If we rule out such superfluous criteria, $S_{i+1} \neq S_i$ in general except where S_i is already a one-element set.) Each succeeding stage thus narrows down the choice and makes the task of the arbiter that much easier. (Since there is no assurance that S will be a one-element set, the procedure could be supplemented by a random device to select a point in S, which would be a considerably smaller set than S_0. Differences among the elements of S being the nth order of magnitude, so to speak, the result of a random device on S should occasion little difference.)

There would seem to be no reason for either party to misrepresent his preferences to the arbiter, for this would only work against his own interests (except possibly when he has complete knowledge of the other party's preferences and the other party does not know his). Obviously he would not understate his u_1^{k*}, say, since he wants to reach it. On the other hand, if he were to overstate his u_1^{k*}, he would be restricting later stages of selection unnecessarily and thereby possibly eliminating a more preferred alternative.

What may appear objectionable about the procedure is that we are letting lower priority criteria decide, in effect, the choice with respect to higher priority ones. This seems, however, to be precisely what one does when he does not know how the alternatives rank in terms of the more important criterion. For example, in deciding where to spend a holiday (subject to a budget constraint), having a pleasant one is presumably more important than where it is spent. It may happen however that one has no knowledge of the ranking of the possibilities in terms of the more important objective--he also wants to go to a place which he has not visited before--so he may narrow the field of choice by considering only those near the sea if he has a minor preference for the sea as against the mountains. The situation of the arbiter seems similar, for he has no way of selecting a point in D_i (with A_i null) on the basis of the ith objectives only without favoring one party over the other, so he turns to the next pair of objectives in order to narrow the selection.

4. Properties of the Solution

Denoting the solution of S_0 by $g(S_0)$, not necessarily our proposed solution S, consider the following properties that one might require of $g(S_0)$.

CONDITION 1 (invariance with respect to utility transformations): The solution $g(S_0)$ is unchanged by arbitrary positive monotonic transformations of the u_i^k $(i = 1, 2, \ldots;\ k \in K)$.

CONDITION 2 (Pareto optimality): No point of $g(S_0)$ is Pareto inferior to any point in S_0. (As usual, x is Pareto inferior to y if someone prefers y to x and no one prefers x to y.)

CONDITION 3 (Property α): If $S_0 \subset S_0'$, then $S_0 \cap g(S_0') \subset g(S_0)$.

CONDITION 4 (symmetry): Suppose that $u^1 = (0, 0, \ldots)$ and $u^2 = (0, 0, \ldots)$ at the no-agreement point and $u_i^{1*} = u_i^{2*}$ $(i = 1, 2, \ldots)$. Writing $\sigma_i(S_0) = \{u_i^1(x), u_i^2(x) | x \in S_0\}$ and $\gamma_i(S_0) = \{u_i^1(x), u_i^2(x) | x \in g(S_0)\}$, suppose further that $(p, q) \in \sigma_i(S_0) \to (q, p) \in \sigma_i(S_0)$ for all i, p, q. Then $(p, q) \in \gamma_i(S_0) \to (q, p) \in \gamma_i(S_0)$.

These conditions correspond to the four properties that characterize the Nash solution (see Luce and Raiffa [7, pp. 126-127]). Condition 2 is actually the same. We have followed Sen [10] in using the term "Property α" for Condition 3, as Luce and Raiffa's use of Arrow's term "independence of irrelevant alternatives" [1, p. 27] tends to be misleading (cf. [4] and [9] in regard to the latter). Conditions 1 and 4 are called for in an L^α-ordering framework.

We wish to show that if $g(S_0) = S$, Conditions 1 to 4 are satisfied. As any arbitrary positive monotonic transformation of u_i^k carries along u_i^{k*}, Condition 1 is clearly satisfied. Condition 2 follows from the fact that x is Pareto inferior to y only if x is v_i-inferior to y for some i. Since S_i contains no v_i-inferior points, S contains no Pareto inferior points.

Condition 3 is false only if, given $S_0 \subset S_0'$, there is a z such that (i) $z \in S_0 \cap g(S_0')$ but (ii) $z \in S_0 - g(S_0)$. With $g(S_0) = S$ and $g(S_0') = S'$, statement (ii) implies that there is an S_i $(i = 1, 2, \ldots)$ when z gets excluded from the selected set because $-(z \in A_i)$ with A_i nonnull and $-(z \in D_i)$ with A_i null. Writing $S_i' = A_i' \cup D_i'$, we want to show the following two propositions to contradict (i).

(iii) $A_i \neq \emptyset$ & $z \in S_0$ & $-(z \in A_i) \rightarrow -(z \in S_0 \cap A_i')$.

(iv) $A_i = \emptyset$ & $z \in S_0$ & $-(z \in D_i) \rightarrow -(z \in S_0 \cap D_i')$.

If A_i is nonnull, (iii) evidently holds. If A_i is null while $z \in S_0$ and $-(z \in D_i)$, it must be that z is supra-satisfactory to someone or else v_i-inferior to some element of D_i. But then, if A_i' is also null, D_i' similarly excludes supra-satisfactory points, and if z is v_i-inferior to an element of D_i, it must also be v_i-inferior to some element of D_i' since $S_0 \subset S_0'$. In the case where A_i' is nonnull, $D_i' \subset A_i'$ and therefore z cannot be in D_i'. Hence in either case, $-(z \in S_0 \cap D_i')$ so that (iv) is established. This suffices to show Condition 3.

In view of the symmetry of $\sigma_i(S_0)$ with respect to the line $u_i^1 = u_i^2$ $(i = 1, 2, \ldots)$ under the hypothesis of Condition 4, it is clear that this condition holds under the procedure that determines S. Conditions 1 to 4 are thus satisfied if $g(S_0) = S$.

Another property may be added.

CONDITION 5 (nonfavoritism): For all $x \varepsilon g(S_0)$,
$\exists k: u_i^k(x) < u_i^{k*} \rightarrow \forall h: \sim(u_i^h(x) > u_i^{h*}),\ i = 1, 2, \ldots$.

This may be considered a minimum requirement for fairness: we do not permit any party to exceed his ith target if another is short of his own. It is equivalent to the statement that if $x \varepsilon g(S_0)$, then for any given i, either (i) $u_i^k(x) \geq u_i^{k*}$ for all k, or else (ii) $u_i^k(x) < u_i^{k*}$ for some k and $u_i^h(x) \leq u_i^{h*}$ for all h. Accordingly, since (i) is the case for A_i nonnull and (ii) for A_i null, Condition 5 holds if $g(S_0) = S$. We have therefore established the following lemma, and several theorems can be derived.

LEMMA: _If_ $g(S_0) = S$, _then_ $g(S_0)$ _satisfies Conditions_ 1 _to_ 5.

THEOREM 1: _If_ $g(S_0)$ _satisfies Conditions_ 2 _and_ 5, _then_ $g(S_0) = S$.

PROOF: Suppose it is false that $S \subset g(S_0)$. Then there is an x such that $x \varepsilon S - g(S_0)$. If $x \varepsilon S_0$ is Pareto inferior, $x \varepsilon S_0 - g(S_0)$ by Condition 2. But since S contains no Pareto inferior points, $S \cap S_0 - g(S_0) = S - g(S_0)$ is null. Hence $S \subset g(S_0)$.

If $x \varepsilon g(S_0)$, then $x \varepsilon A_1$ if A_1 is nonnull since x cannot be Pareto inferior by Condition 2. If A_1 is null, x cannot be supra-satisfactory to any party--by Condition 5--and x cannot be Pareto inferior, so x must be in D_1. Therefore $x \varepsilon S_1$.

Repetition of the argument gives $x \in S_2$, etc., so that $g(S_0) \subset S$. This completes the proof.

THEOREM 2: If $g(S_0)$ satisfies Conditions 2 and 5, it also satisfies Conditions 1, 3 and 4.

PROOF: This is direct from Theorem 1 and the lemma.

THEOREM 3: $g(S_0) = S$ if and only if $g(S_0)$ satisfies Conditions 1 to 5.

PROOF: From Theorem 1, obviously $g(S_0) = S$ if $g(S_0)$ satisfies Conditions 1 to 5. Combining this with the lemma yields the theorem.

5. Generalization to N Persons

A review of Section 3 will show that the procedure for determining S is not dependent in any way on the fact that there are only two parties involved in the conflict. We can replace $K = \{1, 2\}$ by $K = \{1, \ldots, N\}$, consider each alternative as a point in Nn-dimensional space where $n = \min(n^1, \ldots, n^N)$, and discussion of the arbitration problem for N persons would be the same almost word for word. Expressions (1)-(4) would be completely unchanged.

A review of Section 4 will also show that it applies to N persons after modifying Condition 4, which would now read as follows.

CONDITION 4' (N-person symmetry): Suppose that $u^k = (0, 0, \ldots)$ for all k at the no-agreement point and $u_i^{k*} = u_i^{h*}$ $(i = 1, 2, \ldots)$ for all h and k. Writing $c_i(S_0) = \{u_i^1(x), \ldots, u_i^N(x) \mid x \in S_0\}$

and $\gamma_i(S_0) = \{u_i^1(x), \ldots, u_i^N(x) \mid x \in g(S_0)\}$, suppose further that $(r^1, \ldots, r^N) \in \sigma_i(S_0) \rightarrow (r^N, \ldots, r^1) \in \sigma_i(S_0)$ holds for all $i, r^1, \ldots, r^N$ and all possible permutations $(r^N, \ldots r^1)$ of the components of $(r^1, \ldots, r^N)$. Then $(r^1, \ldots, r^N) \in \gamma_i(S_0) \rightarrow (r^N, \ldots, r^1) \in \gamma_i(S_0)$ also holds for all possible permutations.

With this revision, the symmetry of $\sigma_i(S_0)$ is with respect to the hyperline $u_i^1 = \ldots = u_i^N$. Section 4 is then word for word applicable to N persons with Condition 4' in place of Condition 4, and we have the corresponding theorems for the N-person case.

6. Concluding Remark

Condition 1 (invariance with respect to utility transformations), 2 (Pareto optimality) and 4 or 4' (symmetry) are clear requirements of any solution to the arbitration problem. Condition 5 (nonfavoritism) seems called for by a simple concept of fairness under L*-ordering: one should not have more than enough, by his own standards, if another is still wanting. Condition 3 (Property α) has been considered by many authors as "a fundamental consistency requirement of choice" (see the references cited by Sen [10, p. 67]). It is interesting then that if $g(S_0)$ is required to satisfy Conditions 1, 2, 3, 5 and 4' (where $N \geqq 2$), this can only be the solution S.

The case $N > 2$ falls subject of course to Arrow's theorem on the impossibility of a social choice function that satisfies his conditions [1]. Suffice it to say that the model of this paper fails "collective rationality" (which requires a social preference ordering of all possible alternatives) and "independence of irrelevant alternatives."

both of which however seem the least important among the Arrow conditions.

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