THE LINEAR COMPLEMENTARITY PROBLEM: SOLVING LINEAR AND CONVEX QUADRATIC PROGRAMMING PROBLEMS BY LEMKE'S ALGORITHM

Rolando A. Danao*

1. THE LINEAR COMPLEMENTARITY PROBLEM

1.1 Statement of the Problem

Let R^n be the n-dimensional real Euclidean space and let R^{nxn} be the set of nxn matrices with real entries. Given $M \in R^{nxn}$ and $q \in R^n$, the linear complementarity problem, denoted by (q,M), is the following problem:

$$\text{Find } w, \; z \in R^n$$

$$\text{Subject to } \quad w = Mz + q \tag{1}$$

$$w \geq 0, \quad z \geq 0 \tag{2}$$

$$w^t z = 0 \tag{3}$$

where w^t denotes the transpose of w. The linear complementarity problem (q,M) is said to be of order n.

Since $w \geq 0$ and $z \geq 0$, the constraint $w^t z = \sum_{j=1}^{n} w_j z_j = 0$ implies that $w_j z_j = 0$ for each $j = 1, 2, \ldots, n$. The variables w_j and z_j are said to be complementary. Any pair (w;z) that satisfies (1), (2), and (3) is called a complementary solution and the set of all complementary solutions of (q,M) is denoted by C(q,M). Note that if $q \geq 0$, the linear complementarity problem (q,M) always has a complementary solution given by (w;z) = (q;0).

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*Associate Professor, School of Economics, University of the Philippines.

The linear complementarity problem has been shown to be a unified form of problems in diverse fields of study such as mathematical programming, game theory, mechanics, and economics.

Cottle and Dantzig [1] and Lemke [6] showed that linear programming problems, convex quadratic programming problems, and finding Nash equilibrium points of bimatrix games can be formulated as linear complimentarity problems. Ingleton [5] describes Lagrange's equation of motion for the initial velocities under applied impulses of a dynamical system subject to smooth unilateral constraints and shows how this equation can be reduced to the form of the linear complementarity problem. Dantzig and Manne [2] analyzed the invariant capital stock problem from the viewpoint of Lemke's linear complementarity algorithm.

In the next two sections the linear programming problem and the convex quadratic programming problem will be formulated as linear complementarity problems. These formulations are due to Cottle and Dantzig [1].

1.2 The Linear Programming Problem

Consider the following pair of dual linear programming problems:

Primal: $\text{Min } c^t x$

$$\text{s.t. } Ax \geq b$$

$$x \geq 0$$

$$\text{Dual:} \quad \begin{array}{ll} \text{Max} & b^t y \\ \text{s.t.} & A^t y \leq c \\ & y \geq 0 \end{array} \tag{D}$$

The following theorem gives a necessary and sufficient condition for the primal and the dual problems to have optimal solutions:

1.1.1 <u>Theorem</u> (Weak Theorem of Complementary Slackness)

Given the pair of dual linear programming problems P and D, a necessary and sufficient condition for $\bar{x}$ and $\bar{y}$ to be optimal is that they satisfy the equations

$$y^t(Ax - b) = 0$$
$$(c - A^t y)^t x = 0.$$

<u>Proof</u>: See Simmonard [9]. □

If we let $u = -A^t y + c$ (4)

and $v = Ax - b,$ (5)

then $u \geq 0$ and $v \geq 0$ and the equations in Theorem 1 become

$$y^t v + x^t u = 0. \tag{6}$$

We can rewrite conditions (4), (5), and (6) as follows:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -A^t \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ -b \end{bmatrix} \tag{7}$$

$$x^t u + y^t v = 0, \tag{8}$$

where $x \geq 0$, $y \geq 0$, $u \geq 0$, and $v \geq 0$. (9)

The following identifications express conditions (7), (8), and and (9) as a linear complementarity problem:

$$w = \begin{bmatrix} u \\ v \end{bmatrix} \quad M = \begin{bmatrix} 0 & -A^t \\ A & 0 \end{bmatrix} \quad z = \begin{bmatrix} x \\ y \end{bmatrix} \quad q = \begin{bmatrix} c \\ -b \end{bmatrix}.$$

1.3 The Convex Quadratic Programming Problem

1.3.1 Definition. Let C be a convex subset of R^n. A function f defined on the set C is convex iff for each pair $x, y \in C$ and $\lambda \in [0,1]$, $f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda)f(y)$.

The following is the usual formulation of the convex quadratic programming problem:

$$\begin{aligned} \text{Min} \quad & c^t x + \tfrac{1}{2} x^t Q x \\ \text{Subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

where $c \in R^n$, Q is a symmetric nxn matrix, A is an mxn matrix, $x \in R^n$, $b \in R^m$, and the objective function $c^t x + \frac{1}{2}x^t Qx$ is a convex function.

Remark: The requirement that Q be symmetric is not restrictive. For if Q were not symmetric, replace Q by $\overline{Q} = \frac{1}{2}(Q + Q^t)$, which is symmetric. One can easily show that $x^t Qx = x^t \overline{Q}x$.

In order to formulate the convex quadratic programming problem as a linear complementarity problem, we need a well-known theorem in mathematical programming namely, the Kuhn-Tucker Theorem. The statement of the theorem is taken from Zangwill [10].

1.3.2 Theorem (Kuhn-Tucker): Consider the nonlinear programming problem

$$\text{Max} \quad f(x)$$

$$\text{Subject to} \quad g_i(x) \geq 0, \; i = 1, \ldots, m$$

where all functions are differentiable. Let $\overline{x}$ be an optimal solution and assume the constraint qualification[1/] holds.

[1/] The constraint qualification is a restriction on the constraint functions. It precludes certain irregularities (e.g., a cusp) on the boundary of the feasible region which would invalidate the Kuhn-Tucker conditions should the optimal solution occur there. For a discussion of the constraint qualification, see Zangwill[10] or Mangasarian [7].

Then there exist multipliers $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$ such that

$$(1) \quad \lambda_i g_i(\bar{x}) = 0, \; i = 1, 2, \ldots, m \tag{10}$$

$$(2) \quad \nabla f(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0. \tag{11}$$

<u>Remarks</u>: 1. If the functions $f(x)$, $g_i(x)$ $(i = 1, 2, \ldots, m)$ are concave[2/], then the Kuhn-Tucker conditions (10) and (11) are also sufficient. (See Zangwill [10]).

2. One can show that if the constraints are linear, then the constraint qualification holds. (See Zangwill [10]).

The Kuhn-Tucker conditions for the convex quadratic programming problem is obtained by first rewriting the problem into the following form:

$$\text{Max} \quad -c^t x - \tfrac{1}{2} x^t Q x$$

$$\text{Subject to} \quad g_1(x) = A_{1\cdot} x - b_1 \geq 0$$

$$\vdots$$

$$g_m(x) = A_{m\cdot} x - b_m \geq 0$$

$$g_{m+1}(x) = x_1 \geq 0$$

$$\vdots$$

$$g_{m+n}(x) = x_n \geq 0$$

[2/] A function h defined on a convex set C is <u>concave</u> iff -h is convex.

where $A_{i.}$ is the ith row of A. Note that the objective function $-c^t x - \frac{1}{2}x^t Qx$ is concave and the constraint functions, being linear, are concave. Hence, a necessary and sufficient condition for $\bar{x}$ to be optimal is that there exist multipliers $\bar{y}_i \geq 0$ $(i = 1, 2, \ldots, m)$ and $\bar{u}_i \geq 0$ $(i=1, 2, \ldots, n)$ such that

$$\bar{y}_i \, (A_{i.}\bar{x} - b_i) = 0, \quad i = 1, 2, \ldots, m \tag{12}$$

$$\bar{u}_i \, \bar{x}_i = 0, \qquad i = 1, 2, \ldots, n \tag{13}$$

$$-c - Q\bar{x} + \sum_{i=1}^{m} \bar{y}_i A_{i.}^t + \sum_{i=1}^{n} \bar{u}_i I_{i.}^t = 0 \tag{14}$$

where $I_{i.}$ is the ith row of the identity matrix I. The conditions (12), (13), and (14) can be written as

$$\bar{y}^t(A\bar{x} - b) = 0$$

$$\bar{u}^t\bar{x} = 0$$

$$-c - Q\bar{x} + A^t\bar{y} + \bar{u} = 0.$$

Therefore, a necessary and sufficient condition for $\bar{x}$ to be an optimal solution of the convex quadratic programming problem is that there exist $\bar{y} \geq 0$, $\bar{u} \geq 0$ and $\bar{v} = A\bar{x} - b$ that solve the system of equations

$$u = Qx - A^t y + c \tag{15}$$

$$v = Ax - b \tag{16}$$

$$u^t x + y^t v = 0 \tag{17}$$

By setting $w = \begin{bmatrix} u \\ v \end{bmatrix}$, $z = \begin{bmatrix} x \\ y \end{bmatrix}$, $M = \begin{bmatrix} Q & -A^t \\ A & 0 \end{bmatrix}$,

and $q = \begin{bmatrix} c \\ -b \end{bmatrix}$, the system (15), (16), (17) has the form

of a linear complementarity problem.

2. THE GEOMETRY OF THE LINEAR COMPLEMENTARITY PROBLEM

The linear complementarity problem admits of a geometric interpretation that is intuitively appealing (Murty [8]). First, we rewrite the linear complementarity problem (q,M) into the following form:

$$\text{Find } w, z \in R^n$$

$$\text{Subject to } \left[I \;\vdots\; -M \right] \left[\begin{matrix} w \\ z \end{matrix} \right] = q \tag{18}$$

$$w \geq 0, \; z \geq 0 \tag{19}$$

$$w^t z = 0. \tag{20}$$

2.1 Complementary Cones

Notation. The jth column of a matrix A is denoted by $A_{\cdot j}$.

2.1.1 Definition. Given an mxn matrix A, the cone generated by the columns of A, denoted by Pos[A], is the set of all non-negative linear combinations of the columns of A, i.e.,

$$\text{Pos}[A] = \{u \mid u = \sum_{j=1}^{n} x_j A_{\cdot j}, \; x_j \geq 0\}.$$

In matrix notation,

$$\mathrm{Pos}[A] = \{u \mid u = Ax,\ x \geq 0\}.$$

Each column $A_{\cdot j}$ is called a <u>generator</u> of the cone Pos[A].

2.1.2 <u>Definitions</u>. Let $M \in R^{n \times n}$ and let I be the nxn identity matrix. Consider the matrix $\left[\, I \,\vdots\, -M \,\right]$. For each $j = 1, 2, \ldots, n$, the vectors $I_{\cdot j}$ and $-M_{\cdot j}$ are called <u>complementary vectors</u>. An nxn matrix whose jth column is either $I_{\cdot j}$ or $-M_{\cdot j}$ is called a <u>complementary matrix</u>. The cone generated by a complementary matrix is called a <u>complementary cone</u>.

<u>Example</u>

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$$

The complementary matrices are

$$I = \left[I_{\cdot 1} \,\vdots\, I_{\cdot 2} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \left[I_{\cdot 1} \,\vdots\, -M_{\cdot 2} \right] = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}$$

$$B = \left[-M_{\cdot 1} \,\vdots\, I_{\cdot 2} \right] = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$-M = \left[-M_{\cdot 1} \,\vdots\, -M_{\cdot 2} \right] = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}$$

The complementary cones are shown in Figure 1.

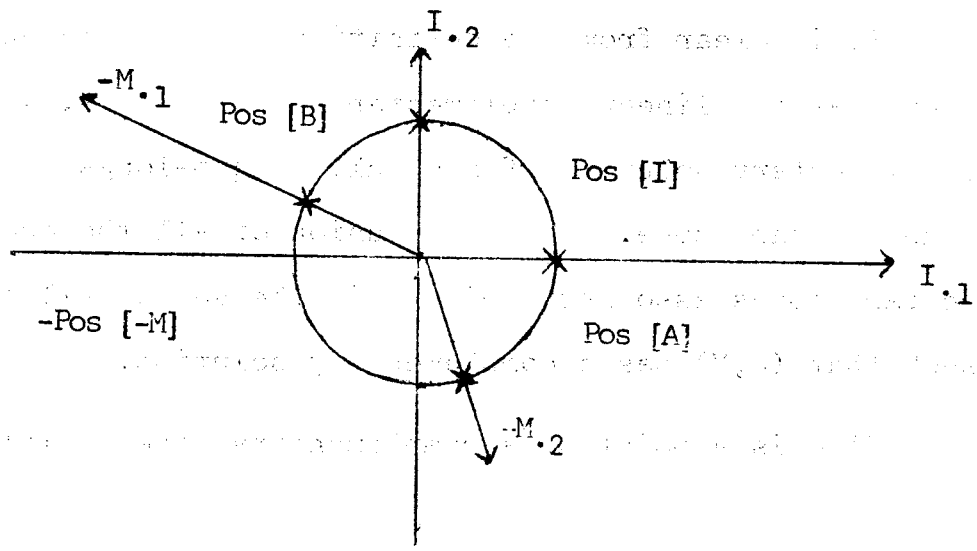


Figure 1

Note: The vectors $I_{.1}$, $I_{.2}$, $-M_{.1}$, and $-M_{.2}$ are not drawn to scale. A curved arrow joining two vectors indicate the cone generated by these vectors.

Remarks: 1. If M is an nxn matrix, then there are 2^n complementary cones associated with M. The set of complementary cones associated with M is denoted by K(M).

2. Whenever a complementary cone is denoted by Pos[A], it is assumed that A is a complementary matrix. The complementary cone Pos[I] is simply the non-negative orthant of R^n.

2.2. Complementary Solutions Induced by Complementary Cones

It is clear from the definition of a complementary cone that the linear complementarity problem (q,M) has a complementary solution if and only if q belongs to some complementary cone. Thus, the union of all the complementary cones associated with M is the set of all $q \in R^n$ such that (q,M) has a complementary solution.

If q is a point of a complementary cone Pos[A], then

$$q = \sum_{j=1}^{n} x_j A_{.j} \quad \text{where } x_j \geq 0 \ (j = 1, 2, \ldots, n).$$

A complementary solution $(w;z)$ of (q,M) can be obtained by setting the variable associated with $A_{\cdot j}$ equal to x_j for $j = 1, 2, \ldots, n$ and all other variables in $(w;z)$ equal to zero, i.e.,

$$w_j = \begin{cases} x_j & \text{if } A_{\cdot j} = I_{\cdot j} \\ 0 & \text{if } A_{\cdot j} = -M_{\cdot j} \end{cases}, \quad j = 1, 2, \ldots, n$$

$$z_j = \begin{cases} 0 & \text{if } A_{\cdot j} = I_{\cdot j} \\ x_j & \text{if } A_{\cdot j} = -M_{\cdot j} \end{cases}, \quad j = 1, 2, \ldots, n$$

In this case, $(w;z)$ is referred to as a <u>complementary</u> solution induced by Pos[A].

<u>Example</u>

$$M = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix} \qquad q = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

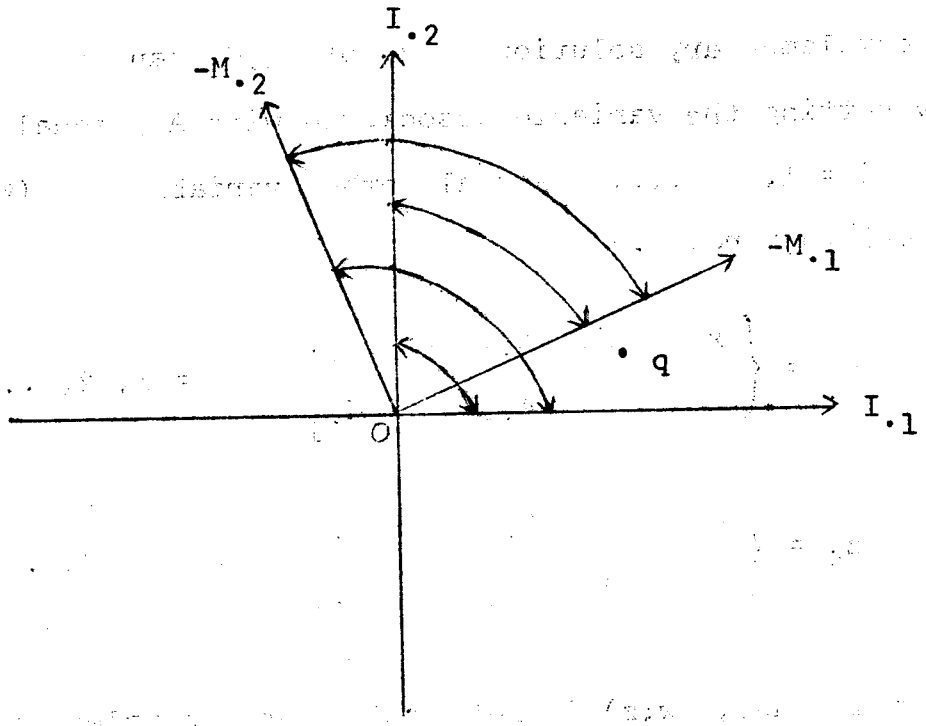


Figure 2

Note that q is a point of Pos $[I_{.1} \vdots I_{.2}]$ and Pos $[I_{.1} \vdots -M_{.2}]$. The complementary solution of (q,M) induced by Pos $[I_{.1} \vdots I_{.2}]$ is given by

$$w_1 = 3 \qquad z_1 = 0$$

$$w_2 = 1 \qquad z_2 = 0.$$

The complementary solution of (q,M) induced by Pos $[I_{.1} \vdots -M_{.2}]$ is easily obtained by solving the following equation:

$$w_1 I_{.1} + z_2(-M_{.2}) = q,$$

i.e.

$$w_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

We get

$$w_1 = \frac{10}{3}, \quad z_2 = \frac{1}{3}.$$

The complementary solution induced by Pos $[I_{.1} \vdots -M_{.2}]$ is, therefore, given by

$$w_1 = \frac{10}{3} \qquad z_1 = 0$$

$$w_2 = 0 \qquad z_2 = \frac{1}{3}$$

2.3 Principal Submatrices and Complementary Cones

2.3.1 Notation. Let $M \in R^{nxn}$. Let J and $\bar{J}$ be subsets of $\{1, 2, \ldots, n\}$. $M_{J\bar{J}}$ denotes the submatrix of M obtained by deleting the rows of M corresponding to indices not contained in J and the columns of M corresponding to the indices not contained in $\bar{J}$.

Example:

$$M = \begin{bmatrix} 1 & 0 & -2 & 4 & 3 \\ 0 & 6 & 5 & -3 & 1 \\ 8 & 9 & 0 & -4 & 6 \\ 7 & 2 & 5 & 1 & 0 \\ 1 & 0 & 3 & 1 & 1 \end{bmatrix}.$$

$$J = \{1,4\}$$

$$\bar{J} = \{2,4,5\}$$

$$M_{J\bar{J}} = \begin{bmatrix} 0 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

$$M_{\bar{J}J} = \begin{bmatrix} 0 & -3 \\ 7 & 1 \\ 7 & 1 \end{bmatrix}$$

$$M_{JJ} = \begin{bmatrix} 1 & 4 \\ 7 & 1 \end{bmatrix}$$

$$M_{\bar{J}\bar{J}} = \begin{bmatrix} 6 & -3 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

2.3.2 <u>Definitions</u>. Let $M \in R^{nxn}$ and let $J \subseteq \{1, 2, \ldots, n\}$ consisting of r elements. The submatrix M_{JJ} of M is called a <u>principal submatrix</u> of order r. When $r < n$, M_{JJ} is called a <u>proper principal submatrix</u>. When $J = \{1, 2, \ldots, n\}$, then $M_{JJ} = M$. When $J = \phi$, then $M_{\phi\phi}$ is called the <u>empty principal submatrix.</u>

<u>Remark</u>: If $M \in R^{nxn}$, then M has 2^n principal submatrices.

The determinant of a principal submatrix is called a <u>principal minor</u>. We adopt the convention that $\det M_{\phi\phi} = 1$.

2.3.3 <u>Definition</u>. $\overline{M}$ is a <u>principal rearrangement</u> of a square matrix M if there exists a permutation matrix P such that

$$\overline{M} = P^T M P.$$

<u>Remarks</u>: 1. Given $M \in R^{nxn}$, and a principal submatrix M_{JJ}, there exists a principal rearrangement $\overline{M}$ of M such that

$$\overline{M} = \begin{bmatrix} M_{JJ} & M_{J\overline{J}} \\ M_{\overline{J}J} & M_{\overline{J}\overline{J}} \end{bmatrix}$$

where $\overline{J} = \{1, 2, \ldots, n\} - J$

2. If $\overline{M}$ is a principal rearrangement of M, then $\det \overline{M} = \det M$.

2.3.4 <u>Definition</u>. Let $M \in R^{nxn}$, $N = \{1, 2, \ldots, n\}$. Let $J \subseteq N$ and $\overline{J} = N - J$.

Define

$$A_{\cdot j} = \begin{cases} -M_{\cdot j}, & j \in J \\ I_{\cdot j}, & j \in \overline{J} \end{cases}$$

The matrix $A(J) = [A_{\cdot 1} \vdots A_{\cdot 2} \vdots \cdots \vdots A_{\cdot n}]$ is the complementary matrix associated with the set of indices J.

Define the correspondence

$$\mathrm{Pos}[A(J)] \longleftrightarrow M_{JJ}.$$

This is a 1-1 correspondence between the set of complementary cones associated with M and the set of principal submatrices of M.

2.3.5 <u>Theorem</u>. Let M_{JJ} be a principal submatrix of M and let Pos[A(J)] be its corresponding complementary cone. Then

$$\det A(J) = (-1)^r \det M_{JJ}$$

where r is the order of M_{JJ}.

<u>Proof:</u> If $J = \phi$, $A(J) = I$

$$M_{JJ} = M_{\phi\phi}$$

$$\det I = 1 = (-1)^0 \det M_{\phi\phi}.$$

If $J \neq \phi$, then by a principal rearrangement, the matrix A(J) can be transformed into

$$A(J)^* = \begin{bmatrix} -M_{JJ} & 0 \\ -M_{\bar{J}J} & I_{\bar{J}J} \end{bmatrix}.$$

$$\det A(J) = \det A(J)^* = (-1)^r \det M_{JJ}. \quad \square$$

3. LEMKE'S ALGORITHM

Lemke's Algorithm is an iterative method of processing[1/] the linear complementarity problem (q,M). Although the algorithm terminates in a finite number of steps, it may fail to process the linear complementarity problem for some classes of matrices M. However, Eaves [3] has shown that Lemke's Algorithm will process the linear complementarity problem for a large class of matrices.

3.1. Preliminaries; Definitions and Notation

Consider the linear complementarity problem

$$
\begin{aligned}
(q,M): \qquad & \text{Find } w,\ z \in R^n \\
& \text{Subject to } w = Mz + q \\
& \qquad w \geq 0,\ z \geq 0 \\
& \qquad w^t z = 0.
\end{aligned}
$$

[1/] Processing the linear complementarity problem means finding a complementary solution or showing that none exists.

If $q \geq 0$, then $w = q$, $z = 0$ is a complementary solution of (q,M). Assume $q \ngeq 0$. Introduce an artificial variable z_o and define the artifical problem

$$
\begin{array}{lll}
(q,M)^*: & \text{Find} & w,\ z \in R^n,\quad z_o \in R \\
 & \text{Subject to} & w = Mz + q + z_o e_n \\
 & & w \geq 0,\ z \geq 0,\ z_o \geq 0 \\
 & & w^t z = 0
\end{array}
$$

where e_n is the n-vector $[1, 1, \ldots, 1]^t$.

<u>Remark</u>: $w = \bar{w}$, $z = \bar{z}$, $z_o = 0$ constitute a solution of $(q,M)^*$ iff $w = \bar{w}$, $z = \bar{z}$ constitute a complementary solution of (q,M).

The artificial problem can be rewritten as follows:

$$
\begin{array}{lll}
(q,M)^*: & \text{Find} & w,\ z \in R^n,\ z_o \in R \\
 & \text{Subject to} & [e_n \,\vdots\, M \,\vdots\, -I] \begin{bmatrix} z_o \\ \hline z \\ \hline w \end{bmatrix} = -q \qquad (21)
\end{array}
$$

$$
\begin{bmatrix} z_o \\ \hline z \\ \hline w \end{bmatrix} \geq 0 \qquad (22)
$$

$$
w^t z = 0 \qquad (23)
$$

3.1.1 Definition. A vector $[z_o \mid z \mid w]^t$ that satisfies (21) and (22) is called a feasible solution of $(q,M)^*$.

Remark: Note that the matrix $[e_n \mid M \mid -I]$ is of rank n; hence, a basic feasible solution of $(q,M)^*$ has n basic variables.

3.1.2. Definition. A basic feasible solution of $(q,M)^*$ is called complementary iff z_o is nonbasic and exactly one variable in each pair $\{w_i, z_i\}$, $i = 1, 2, \ldots, n$, is basic.

3.1.3. Definition. A basic feasible solution of $(q,M)^*$ is called almost-complementary iff z_o is basic and except for one index, say $i=k$, exactly one variable in each pair $\{w_i, z_i\}$, $i = 1, 2, \ldots, n$, $i \neq k$, is basic. The pair $\{w_k, z_k\}$ is called a nonbasic pair.

Remarks: (1) Both complementary and almost-complementary basic feasible solutions satisfy the constraints of $(q,M)^*$.

(2) A complementary basic feasible solution of $(q,M)^*$ is a complementary solution of (q,M).

3.1.4. Theorem. Let $q \in R^n$ such that $q \not\geq 0$. Define the index k by

$$q_k = \min_{1 \leq i \leq n} q_i .$$

Then

$$z_o = -q_k$$

$$z = 0$$

$$w_i = q_i - q_k \qquad (i=1, 2, \ldots, n)$$

constitute an almost complementary basic feasible solution with $\{w_k, z_k\}$ as the nonbasic pair.

<u>Proof</u>: Note that

(i) $$z_o e_n + Mz - w = -q_k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + M \cdot 0 - \begin{bmatrix} q_1 - q_k \\ q_2 - q_k \\ \vdots \\ q_n - q_k \end{bmatrix}$$

$$= -q.$$

(ii) $$z_o = -q_k \geq 0$$

$$z = 0$$

$$w_i = q_i - q_k \geq 0, \; i = 1, \ldots, n.$$

We show that the variables z_o, w_i $(i \neq k)$, are basic. This is equivalent to showing that the column vectors associated

with these variables are linearly independent. The variables with their corresponding vectors are given below:

$$\begin{array}{ccccc} z_o & w_1 & \cdots & w_k & \cdots & w_n \\ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{kth coordinate} & \cdots & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \end{array}$$

The linear independence of the vectors associated with z_o, w_j, $j \neq k$, is now easy to show. $\square$

3.1.5. <u>Definition</u>. A basic feasible solution is said to be <u>nondegenerate</u> if all the basic variables are positive. The artificial problem $(q, M)^*$ is said to be <u>nondegenerate</u> if all the basic feasible solutions are nondegenerate.

3.2 Lemke's Algorithm

Assumptions: (1). $q \not\geq 0$

(2) $(q, M)^*$ is nondegenerate.

<u>Notation</u>: At the rth iteration, let the column associated with z_o be denoted by $Z^r_{\cdot o}$, the column associated with $z_j (j=1, 2, \ldots, n)$ by $Z^r_{\cdot j}$ the column associated with $w_j (j=1, 2 \ldots, n)$ by $W^r_{\cdot j}$, and the right hand side (RHS) by q^r.

Step 0.1. <u>Initialization</u>. Set up the initial tableau:

z_0	z_1	z_2	...	z_n	w_1	w_2	...	w_n	RHS
1	M_{11}	M_{12}	...	M_{1n}	-1	0	...	0	$-q$
1	M_{21}	M_{22}	...	M_{2n}	0	-1	...	0	$-q_2$
⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮
1	M_{n1}	M_{n2}	...	M_{nn}	0	0	...	-1	$-q_n$

0.2. Determine k by

$$q_k = \min_{1 \le i \le n} \{q_i\}.$$

0.3. Pivot on Z_{ko}.

0.4. Multiply row i, $i \neq k$, by -1.

<u>Remark</u>: This initial step sets the tableau in feasible canonical form with respect to the first almost-complementary basic sequence $\langle z_o, w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n \rangle$.

0.5. Set $r=1$, $j=k$, $Z^r_{\cdot j} = A^r_{\cdot j}$. Let z_j be the incoming variable.

 Ten. c.2

Step 1. Determine the index t according to the equation

$$\frac{q_t^r}{A_{tj}^r} = \min \left\{ \frac{q_i^r}{A_{ij}^r} \;\middle|\; A_{ij}^r > 0,\ 1 \le i \le n \right\}$$

If $A_{ij}^r \le 0$ for all $i = 1, 2, \ldots, n$, stop. The incoming variable can be increased indefinitely without driving any of the basic variables towards zero. Otherwise, go to step 2.

Step 2. Pivot on A_{tj}^r.

2.0. If z_o drops from the basis, stop. The new basic feasible solution is complementary.

2.1. If z_j, $j \neq 0$ drops from the basis, set $W_{\cdot j}^{r+1} = A_{\cdot j}^{r+1}$, let w_j be the incoming variable and go to step 1.

2.2. If w_j drops from the basis, set $Z_{\cdot j}^{r+1} = A_{\cdot j}^{r+1}$, let z_j be the incoming variable and go to step 1.

<u>Remarks</u>: (1) The ratio test in Step 1 is similar to the ratio test in the Simplex Algorithm in linear programming. It ensures that the basic solution at each iteration remains feasible.

(2) The stopping rule in Step 1 is also similar to the unboundedness stopping rule in the Simplex Algorithm. In this case, Lemke's Algorithm terminates in a ray. It will be shown that this stopping rule in Lemke's Algorithm fails to give any definite conclusion on the existence of a complementary solution for an arbitrary matrix M. However, for some classes of matrices (Eaves [3]), this stoppin rule indicates that the linear complementarity problem (q,M has no complementary solution.

3.3 Finiteness of Lemke's Algorithm

This section will show that Lemke's Algorithm terminates in a finite number of iterations.

3.3.1. Definition. If P_{r+1} is a basic feasible solution obtained from a basic feasible solution P_r by pivoting in Step 2 of Lemke's Algorithm, then P_r and P_{r+1} are said to be neighbors. The point P_r is called the predecessor of P_{r+1} and P_{r+1} is called a successor of P_r.

3.3.2. Definition. The sequence of almost-complementary basic feasible solutions generated by Lemke's Algorithm is called an almost-complementary path.

3.3.3. Definition. Let P be an almost-complementary basic feasible solution with $\{w_k, z_k\}$ as the nonbasic pair. If w_k or z_k can be increased indefinitely without driving any basic variable to zero, the ray emanating from P generated by increasing w_k or z_k indefinitely is called an almost-complementary ray.

3.3.4. Lemma. An almost-complementary basic feasible solution in an almost-complementary path can have at most two neighbors.

Proof: This follows from the fact that every almost-complementary basic feasible solution has exactly one nonbasic pair, say $\{w_k, z_k\}$, and pivoting requires holding all nonbasic variables at value zero and increasing w_k or z_k. $\square$

3.3.5. Lemma. The initial almost-complementary basic feasible solution cannot have a predecessor.

Proof: Let P_o be the initial almost-complementary basic feasible solution with $\{w_k, z_k\}$ as the nonbasic pair. Note that the successor of P_o is obtained by pivoting z_k into the basis. Hence, if P_o has a predecessor, then it can be obtained by pivoting w_k into the basis. We claim

that we cannot pivot w_k into the basis, i.e., w_k can be increased indefinitely without driving a basic variable to zero

The relevant portion of the initial tableau follows:

	z_o	...	w_k	...	Right Hand Side
	1		0		$-q_1$
	⋮		⋮		⋮
pivot element ←	$\boxed{1}$	...	1	...	$-q_k$
	⋮		⋮		⋮
	1		0		$-q_n$

After pivoting z_o into the basis (with pivot element as shown above), the initial tableau in feasible canonical form with respect to the first almost-complementary basic sequence is:

z_o	...	w_k	...	Right Hand Side
0		-1		$-q_1 + q_k$
⋮		⋮		⋮
1	...	-1	...	$-q_k$
⋮		⋮		⋮
0		-1		$-q_n + q_k$

Since the column associated with w_k is negative, w_k can be increased indefinitely without driving any basic variable to zero. $\square$

3.3.6. <u>Theorem</u>. Along an almost-complementary path, an almost complementary basic feasible solution cannot recur.

<u>Proof</u>: Let P_o, P_1, ... be an almost complementary path and let P_j be the first almost-complementary basic feasible solution that recurs. Let P_k be P_j's predecessor when it recurs. First, we note that $P_j \neq P_o$ since P_o does not have a predecessor.

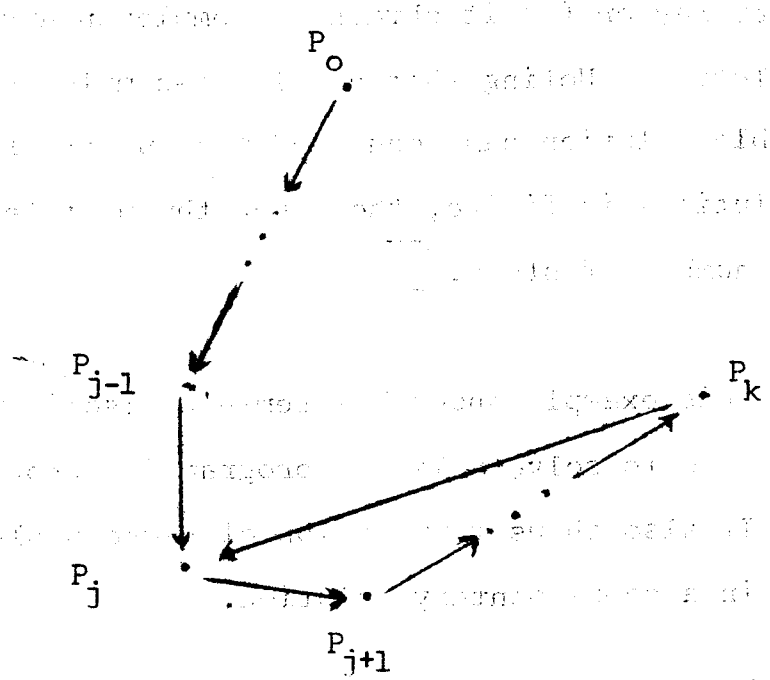


Figure 3

By the nondegeneracy assumption, the extreme points of the feasible region of $(q,M)^*$ are in one-to-one correspondence with the basic feasible solutions. Hence, P_{j-1}, P_{j+1}, and P_k are distinct basic feasible solutions that are neighbors of P_j. This is impossible since P_j can have at most two neighbors. $\square$

3.3.7. Theorem. Lemke's Algorithm terminates in a finite number of steps.

Proof: Starting from an almost-complementary basic feasible solution, Lemke's Algorithm generates an almost-complementary path and stops when (a) it generates an almost complementary ray or (b) it obtains a complementary basic feasible solution. Noting that no almost-complementary basic feasible solution can recur and the number of basic feasible solutions is finite, the algorithm must terminate in a finite number of steps. □

Example 1. This example shows how Lemke's Algorithm is used to solve a linear programming problem. It also shows termination of Lemke's Algorithm in a complementary solution.

Primal Problem

$$\begin{array}{lrcrcl} \text{Min} & 144x_1 & + & 108x_2 & & \\ \text{s.t.} & 4x_1 & + & x_2 & \geq & 4 \\ & 3x_1 & + & 3x_2 & \geq & 6 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

Dual Problem

$$\begin{array}{lrcrcl} \text{Max} & 4y_1 & + & 6y_2 & & \\ \text{s.t.} & 4y_1 & + & 3y_2 & \leq & 144 \\ & y_1 & + & 3y_2 & \leq & 108 \\ & y_1, & & y_2 & \geq & 0 \end{array}$$

Let $v = [v_1, v_2]^t$ be the vector of surplus variables for the primal problem and let $u = [u_1, u_2]^t$ be the vector of slack variables for the dual problem. The associated linear complementarity problem is that of finding $w, z \in R^4$ such that $w = Mz + q$, $w \geq 0$, $z \geq 0$ and $w^t z = 0$ where

$$w = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} \quad z = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 0 & -4 & -3 \\ 0 & 0 & -1 & -3 \\ 4 & 1 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \quad q = \begin{bmatrix} 144 \\ 108 \\ -4 \\ -6 \end{bmatrix}$$

Initial Tableau

z_0	z_1	z_2	z_3	z_4	w_1	w_2	w_3	w_4	RHS
1	0	0	-4	-3	-1	0	0	0	-144
1	0	0	-1	-3	0	-1	0	0	-108
1	4	1	0	0	0	0	-1	0	4
$\boxed{1}$	3	3	0	0	0	0	0	-1	6

$$\min_{i \leq i \leq 4} q_i = q_4 = -6$$

Initial nonbasic pair = $\{z_4, w_4\}$

Initial almost complementary basic sequence: $\langle z_0, w_1, w_2, w_3 \rangle$

[illegible]

Basic Variables	z_0	z_1	z_2	z_3	z_4	w_1	w_2	w_3	w_4	RHS
w_1	0	0	0	$\boxed{3}$	0	1	-1	0	0	36
z_4	0	$\frac{3}{2}$	0	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	37
z_2	0	$-\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1
z_0	1	$\frac{9}{2}$	0	0	0	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	3

Introduce z_3 (the complement of w_3) into the basis; w_1 drops from the basis.

Basic Variables	z_0	z_1	z_2	z_3	z_4	w_1	w_2	w_3	w_4	RHS
z_1	0	0	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	12
z_4	0	$\frac{3}{2}$	0	0	+1	-1	$\frac{4}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	33
z_2	0	$-\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1
z_0	1	$\boxed{\frac{9}{2}}$	0	0	0	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	3

Introduce z_1 (the complement of w_1) into the basis; z_0 drops from the basis.

Basic Variables	z_0	z_1	z_2	z_3	z_4	w_1	w_2	w_3	w_4	RHS
z_3	0	0	0	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	12
z_4	$-\frac{1}{3}$	0	0	0	1	-1	$\frac{4}{3}$	0	$\frac{1}{3}$	32
z_2	$\frac{1}{9}$	0	1	0	0	0	0	$\frac{5}{6}$	$-\frac{4}{9}$	$\frac{4}{3}$
z_1	$\frac{2}{9}$	1	0	0	0	0	0	$-\frac{1}{3}$	$\frac{1}{9}$	$\frac{2}{3}$

Since z_o dropped from the basis, the algorithm stops and the following complementary solution is obtained:

$$w_1 = 0 \qquad z_1 = \frac{2}{3}$$
$$w_2 = 0 \qquad z_2 = \frac{4}{3}$$
$$w_3 = 0 \qquad z_3 = 12$$
$$w_4 = 0 \qquad z_4 = 32$$

In terms of the given linear programming problems,

$$u_1 = w_1 = 0 \qquad x_1 = z_1 = \frac{2}{3}$$
$$u_2 = w_2 = 0 \qquad x_2 = z_2 = \frac{4}{3}$$
$$v_1 = w_3 = 0 \qquad y_1 = z_3 = 12$$
$$v_2 = w_4 = 0 \qquad y_2 = z_4 = 32$$

The optimal solution of the primal problem is

$$x_1 = \frac{2}{3}$$
$$x_2 = \frac{4}{3}$$

The optimal solution of the dual problem is

$$y_1 = 12$$
$$y_2 = 32.$$

<u>Example 2.</u> This example shows Lemke's Algorithm terminating in an almost-complementary ray and the linear complementarity problem has no complementary solution.

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

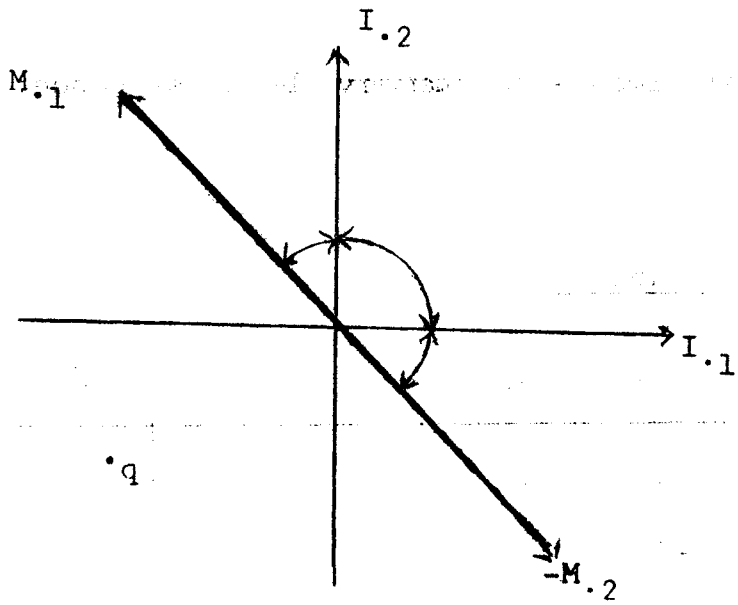


Figure 4.

Note: The nondegenerate complementary cones are indicated by the curved arrows. The degenerate complementary cone $\mathrm{Pos}[-M_{\cdot 1} \vdots -M_{\cdot 2}]$ is indicated by the bold line.

Since q does not belong to any complementary cone, the linear complementarity problem (q,M) has no complementary solution.

$$\min_{1 \le i \le 2} q_i = q_2 = -2$$

Initial nonbasic pair: $\{w_2, z_2\}$

Initial almost-complementary basic sequence: $\langle z_0, w_1 \rangle$

Initial Tableau

z_0	z_1	z_2	w_1	w_2	RHS
1	1	-1	-1	0	1
1	-1	1	0	-1	2

Initial Tableau (Feasible Canonical Form)

Basic Variables	z_0	z_1	z_2	w_1	w_2	RHS
w_1	0	-2	$\boxed{2}$	1	-1	1
z_0	1	-1	1	0	-1	2

Introduce z_2 (the complement of w_2) into the basis; w_1 drops from the basis.

Basic Variables	z_0	z_1	z_2	w_1	w_2	RHS
z_2	0	-1	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
z_0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

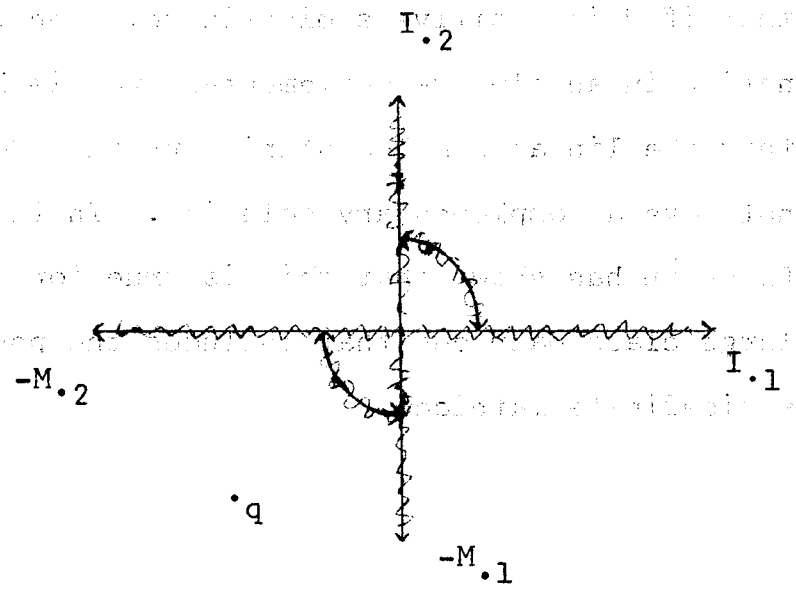
Introduce z_1 (the complement of w_1) into the basis; since the column under z_1 is nonpositive, the algorithm terminates in an almost-complementary ray.

<u>Remark</u>: The matrix M in the preceding example is a positive semidefinite matrix. It has been shown (Eaves [3]) that if M is positive semidefinite, then termination in an almost-complementary ray implies that the linear complementarity problem does not have a complementary solution. In fact, Eaves [3] has shown that this is true for a large class matrices that includes the positive semidefinite matrices.

<u>Example 3</u>. This example shows that Lemke's Algorithm can terminate in an almost-complementary ray even if the linear complementarity problem has a complementary solution. Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Figure 5



The point q is contained in the complementary cone $\mathrm{Pos}[-M_{\cdot 1} \mid -M_{\cdot 2}]$. Hence, (q,M) has a complementary solution.

Initial Tableau

z_0	z_1	z_2	w_1	w_2	RHS
1	0	1	-1	0	1
$\boxed{1}$	1	0	0	-1	2

$$\min_{1 \le i \le 2} q_i = q_2 = -2$$

Initial almost-complementary basic sequence: $\langle z_0, w_1 \rangle$

Initial nonbasic pair: $\{w_2, z_2\}$

Initial Tableau (Feasible Canonical Form)

Basic Variables	z_0	z_1	z_2 ↓	w_1	w_2	RHS
w_1	0	1	-1	1	-1	1
z_0	1	1	0	0	-1	2

Introduce z_2 (the complement of w_2) into the basis. Since the column under z_2 is nonpositive, stop. The variable z_2 can be increased indefinitely without driving any of the basic variables z_0 or w_1 to zero. Hence, Lemke's algorithm terminates in an almost-complementary ray.

<u>Example 4</u>. A convex quadratic programming problem solved by Lemke's Algorithm.

$$\text{Min} \quad f(x_1, x_2) = x_1 + x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2$$

$$\text{s.t.} \quad x_1 + 2x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

This problem can be written as follows:

$$\text{Min} \quad [1 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2}[x_1 x_2] \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{s.t.} \quad [1 \ 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0.$$

In the notation of Section 1.3,

$$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$b = [\ 2\ ] \qquad A = [\ 1 \quad 2\ ]$$

The objective function is convex since the matrix Q is positive definite. The associated linear complementarity problem (q,M) has

$$q = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and } M = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$w = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \end{bmatrix}, \quad \begin{aligned} u &= Qx - A^t y_1 + c \\ v_1 &= Ax - b \end{aligned}$$

$$z = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \end{bmatrix} \qquad y_1 = \text{Kuhn-Tucker multiplier}$$

<u>Initial Tableau</u>

z_0	z_1	z_2	z_3	w_1	w_2	w_3	RHS
1	4	2	-1	-1	0	0	-1
1	2	4	-2	0	-1	0	-1
$\boxed{1}$	1	2	0	0	0	-1	2

$$\min_{1 \le i \le 3} q_i = q_3 = -2$$

Initial nonbasic pair: $\{w_3, z_3\}$

Initial almost-complementary basic sequence: $\langle z_0, w_1, w_2 \rangle$.

Initial Tableau (Feasible Canonical Form)

Basic Variables	z_0	z_1	z_2	z_3	w_1	w_2	w_3	RHS
w_1	0	-3	0	1	1	0	-1	3
← w_2	0	-1	-2	$\boxed{2}$	0	1	-1	3
z_0	1	1	2	0	0	0	-1	2

Introduce z_3 (the complement of w_3) into the basis; w_2 drops.

Basic Variables	z_0	z_1	z_2	z_3	w_1	w_2	w_3	RHS
w_1	0	$-\frac{5}{2}$	1	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
z_3	0	$-\frac{1}{2}$	-1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
← z_0	1	1	$\boxed{2}$	0	0	0	-1	2

Introduce z_2 (the complement of w_2) into the basis; z_0 drops;

Basic Variables	z_0	z_1	z_2	z_3	w_1	w_2	w_3	RHS
w_1								$\frac{1}{2}$
z_3								$\frac{5}{2}$
z_2								1

Since z_o dropped from the basis, the algorithm stops at a complementary solution given by:

$$w_1 = \frac{1}{2} \qquad z_1 = 0$$

$$w_2 = 0 \qquad z_2 = 1$$

$$w_3 = 0 \qquad z_3 = \frac{5}{2}$$

In terms of the given convex quadratic programming problem, the optimal solution is given by

$$x_1 = z_1 = 0$$

$$x_2 = z_2 = 1$$

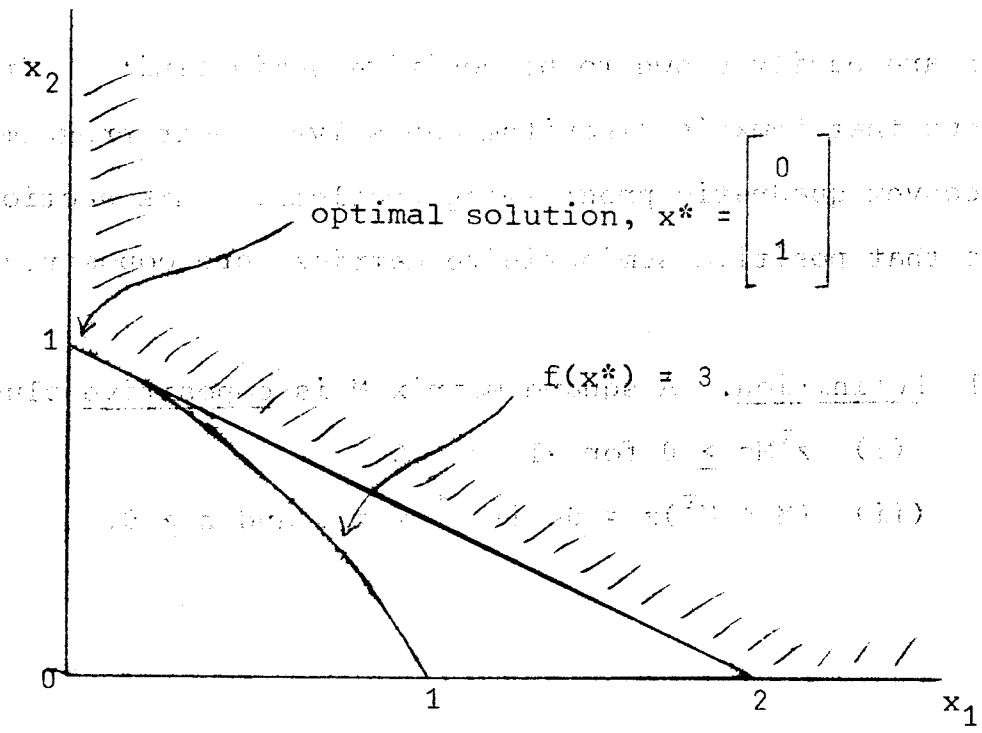


Figure 6

3.4 Termination of Lemke's Algorithm in a Ray

Examples 2 and 3 show that termination in a ray does not lead to a definite conclusion on the existence of a complementary solution. However, for some classes of matrices M, termination in a ray implies that the linear complementarity problem has no complementary solution. Lemke proved this result for the class of copositive plus matrices which include the positive semidefinite matrices. We note that the matrices in the linear complementarity problems associated with the linear programming problem and the convex quadratic programming problem are of the form

$$\begin{bmatrix} 0 & -A^t \\ A & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q & -A^t \\ A & 0 \end{bmatrix}$$

which are easily shown to be positive semidefinite. It follows that Lemke's Algorithm can solve linear programming and convex quadratic programming problems. This section shows that positive semidefinite matrices are copositive plu

3.4.1 Definition. A square matrix M is copositive plus iff

(i) $z^tMz \geq 0$ for all $z \geq 0$

(ii) $(M + M^t)z = 0$ if $z^tMz = 0$ and $z \geq 0$.

3.4.2 <u>Lemma</u>. If M is a real symmetric matrix, then there exists a nonsingular matrix U such that U^tMU is a diagonal matrix whose diagonal entries are characteristic roots of M.

Proof: See Höhn [4].

3.4.3 <u>Lemma</u>. If M is a real symmetric positive semidefinite matrix and $z^tMz = 0$, then $Mz = 0$.

Proof: Let M be an nxn real symmetric positive semidefinite matrix. Let U be the nonsingular matrix in Lemma 3.4.2 and let

$$U^tMU = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \Lambda$$

where $\lambda_i \geq 0$ $(i=1, 2, \ldots n)$ are the characteristic roots of M. Define

$$\bar{z} = U^{-1}z.$$

Then $$z = U\bar{z}$$

By assumption, $z^tMz = 0$, i.e.,

$$\bar{z}^tU^tMU\bar{z} = 0$$

$$\bar{z}^t\Lambda z = 0$$

$$\sum_{i=1}^{n} \lambda_i \bar{z}_i^2 = 0.$$

Since $\lambda_i \bar{z}_i^2 \geq 0$ for $i = 1, 2, \ldots, n$, then

$$\lambda_i \bar{z}_i^2 = 0, \quad i = 1, 2 \ldots, n;$$

hence, $\lambda_i \bar{z}_i = 0$, $i = 1, 2 \ldots, n$.

Now, $U^t M U \bar{z} = \Lambda \bar{z} = \begin{bmatrix} \lambda_1 \bar{z}_1 \\ \lambda_2 z_2 \\ \vdots \\ \lambda_n \bar{z}_n \end{bmatrix} = 0$

Since U^t is nonsingular, then

$$MU\bar{z} = 0$$

or $$Mz = 0. \quad \square$$

3.4.4 <u>Theorem</u>. Let M be a real positive semidefinite matrix. If $z \geq 0$ and $z^t M z = 0$, then $(M + M^t)z = 0$.

<u>Proof</u>: If M is positive semidefinite then $M + M^t$ is symmetric positive semidefinite and

$$\frac{1}{2} z^t (M + M^t) z = z^t M z.$$

If $z^t M z = 0$, then $z^t (M + M^t) z = 0$.

By Lemma 3.4.3

$$(M + M^t)z = 0. \quad \square$$

3.4.5 <u>Corollary</u>. A positive semidefinite matrix is copositive plus.

<u>Proof</u>: Follows from the definition of a positive semidefinite matrix and Theorem 3.4.4. $\square$

REFERENCES

[1] Cottle, R.W. and Dantzig, G.B., "Complementary Pivot Theory of Mathematical Programming", Mathematics of the Decision Sciences, Part 1, edited by G.B. Dantzig and A.F. Veinott, American Mathematical Society, Providence, Rhode Island (1968), 115-135.

[2] Dantzig, G.B. & Manne, A.S., "A Complementarity Algorithm for an Optimal Capital Path with Invariant Proportions", Journal of Economic Theory, 9 (1974), pp. 312-232.

[3] Eaves, B.C., "The Linear Complementarity Problem", Management Science, Vol. 17 (1971) pp. 612-634.

[4] Hohn, F.E., Elementary Matrix Algebra, The MacMillan Company, New York, 1965.

[5] Ingleton, A.W., "A Problem in Linear Inequalities", Proc. London Math. Soc., 16 (1966), pp. 519-536.

[6] Lemke, C.E., "Bimatrix Equilibrium Points and Mathematical Programming", Management Science, Vol. 11 (1965).

[7] Mangasarian, O.L., Non-linear Programming, Mc-Graw-Hill, Inc., N.Y. 1969.

[8] Murty, K.G., "On the Number of Solutions to the Complementarity Problem and Spanning Properties of Complementary Cones", Linear Algebra and its Applications, 5 (1972), 65-108.

[9] Simmonard, M., Linear Programming, Prentice-Hall, Inc. Englewood Cliffs, N.J., 1966.

[10] Zangwill, W.I., Nonlinear Programming: A Unified Approach, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1969.