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### I Reply to Borch

1 Borch here essentially repeats his old point that the E-S indifference map is incapable of giving an unconditionally consistent representation of the preference ordering of uncertain returns. This is an incontrovertible truth which I have fully recognized in my original article.<sup>1/</sup> He, however, refuses to grant my point that, nevertheless, for investors who normally undertake only small risk relatively to his total wealth, the E-S analysis remains a highly useful approximation.

Let me take this opportunity to clarify this point by a numerical demonstration using Borch's own examples. He gives an example of three gambles A, B and C. A and B are assumed to be equivalent (on the same indifference curve). B and C have the same mean and the same variance (2, 4). Yet C is so constructed that it is axiomatically preferable to A and hence should be preferable to B also, even though B and C have identical means and variances. This is regarded as a proof that a consistent preference ordering of all probability distributions cannot be represented by a utility function of mean and variance, i.e.,  $U(E, V)$ .

A casual observation would reveal why gamble C should be preferred to gamble B even though they have identical means and variances. While C is symmetric, B has a negative skewness as measured by a negative

third central moment of  $\bar{m}_3 = -13.6$ . There are, however, very common circumstances under which an investor would hardly be concerned with the skewness at all. These circumstances involve the initial wealth of the investor and the risk he has already undertaken. The trouble with Borch is that in giving these cute classroom examples he never specifies what these initial circumstances are. He seems to assume that the one or two dollars mean values of the gambles constitute the only wealth of the player (investor) concerned in this world. This neglect cause him to miss my point entirely. Once these informations are filled in, the difference between our points of view would be immediately clarified.

Let us assume for argument's sake that these gambles are to be played by a person who has initially a riskless wealth of \$10,000. Let us assume that his utility function is of the negative exponential type, i.e.,  $B(1 - e^{-\alpha y})$ , where  $y$  represents the net wealth. Furthermore, let us assume that he is reasonably ambitious (or greedy) about wealth that a tenfold multiplication of his present wealth would not send him right into a state of utter bliss (full contentment), but would bring him no closer than one per cent away from that blissful state. Thus  $\alpha$  must be taken as a small number of the order of magnitude of, say, roughly  $\frac{1}{2y_0}$ , where  $y_0$  is the level of wealth that he is accustomed to.<sup>2/</sup>

Thus game B would increase his expected utility from the initial level of  $B(1 - e^{-\frac{1}{2}})$  to

$$B = Be^{-\frac{10,002}{20,000}} \left( 1 + \frac{4}{8} \times \left(\frac{1}{10}\right)^8 - \frac{(-13.6)}{48} \times \left(\frac{1}{10}\right)^{12} + \dots \right)$$

whereas game C would raise it to

$$B = Be^{-\frac{10,002}{20,000}} \left( 1 + \frac{4}{8} \times \left(\frac{1}{10}\right)^8 - 0 + \dots \right)$$

Since the marginal utility of one dollar of sure wealth in this case is roughly  $\alpha Be^{-\alpha y} = \frac{1}{20,000} Be^{-\frac{10,002}{20,000}}$ , therefore, the difference in monetary values of the two gambles to this particular person is only  $\$ \frac{13.6}{24} \times \left(\frac{1}{10}\right)^8 = 5.7 \times \left(\frac{1}{10}\right)^7$  cents, which I hope Borch would agree is not a sum worth quarelling about even before the devaluation of the dollar.

Borch claims that he can easily escalate his example to demonstrate that, of two gambles with identical mean-variance-skewness triples, one can be so constructed as to be clearly superior to the other. I have no doubt at all about his ability to do that. But it is just as easy to escalate our numerical illustration by extending the above expansion of the expected utility function to include the fourth and higher central moments. It will be easy too to demonstrate that the differences in fourth or higher moments would be of rapidly diminishing monetary values to the person concerned as compared with the already negligible value of skewness.

2 Borch next uses an example of gamma distributions to demonstrate that two (E, V) vectors supposedly on the same indifference curve cannot be of equal expected utility, if the two gamma distributions are constructed in the way he indicated. The difficulty of handling gamma distributions by means of E-V analysis or E-S indifference curves is again due to the degrees of skewness of these distributions, which are



closely linked with their means and variances. The problem is essentially the same as that of the previous example.

Borch, moreover, intends to use this example to demonstrate that my proposition that positive skewness is an attractive property cannot be generally valid. This demonstration, however, is based upon an obvious misunderstanding. He argues that the skewness of a gamma distribution decrease with an increase of the parameter  $n$  in his formula for gamma density. This is true if "skewness" is measured by the pure number measure  $\mu_3 = \bar{m}_3/S^3$ . Measured by the third moments about the means, however, it necessarily increases with  $n$  given  $\alpha$ , for  $\bar{m}_3 = 2(n+1)/\alpha^3$ .  $\mu_3$  decreases with  $n$  only because  $S^3$  increases faster with  $n$ , as  $S = \sqrt{(n+1)/\alpha}$ . Therefore,  $\mu_3 = \bar{m}_3/S^3 = 2/\sqrt{n+1}$ , which decreases with  $n$ .

In the case of the two gamma densities constructed according to his specification, viz., that the two densities share the same  $\alpha$  but one has a greater  $n$  than the other, the density with a great  $n$  must have greater mean and variance and also a greater third central moment,  $\bar{m}_3$ . Since the basic Taylor expansion of the expected utility function is

$$E[U(y)] = U(\bar{y}) + U'(\bar{y}) \frac{S^2}{2} + U''(\bar{y}) \frac{\bar{m}_3}{3!} + \dots,$$

therefore, with  $U''$  understood to be positive,  $\bar{m}_3$  must be a desirable property. Thus the greater  $\bar{m}_3$  of the gamma density with greater mean and variance would help to explain why it is preferable to the other, just as in the previous case of two Bernoulli distributions.

My proposition that skewness is desirable is certainly generally

valid, so long as  $U''$  is positive. Even if skewness is to be measured by  $\mu_3 = \frac{\bar{m}_3}{S^3}$ , this proposition still holds, provided other moments, especially  $S$ , remain the same.

3 Borch then reminds us that we should not build the EV analysis on the basis of the fact that, when all stochastic variables has distributions belonging to the same two parameter family of distributions, there will be a one to one correspondence between the two parameters and the mean and the variance, and, hence, a preference ordering over this set of stochastic variables will be a preference ordering over the set of  $(E, V)$  vectors. For the return of a portfolio of two or more assets with stochastic returns belonging even to the same family of distributions will in general belong to a different type of distribution unless the distributions of the component assets are what is known as "stable" distributions. The only member this class of distributions, that has a finite variance, is the normal distribution. Even normal distributions, when they are truncated and modified by insurance, stop loss arrangements, progressive taxation, etc., would no longer be "stable" in this sense, nor normal for that matter.

This is a very pertinent reminder in view of Bierwag's comment we are going to discuss next. But it is fully recognized in my paper. In fact this is why I emphasize that the EV analysis, if it is to be useful for portfolio analysis must, to some extent be "distribution free".<sup>3/</sup>

4 Of course, to be completely distribution free is inherently impossible. Since my analysis is based upon the Taylor expansion of the expected utility function, its validity depends upon the convergence of the series.

I have already pointed out that for Paretian distributions with infinite moments, such as the Cauchy distribution, this analysis is of course inapplicable. Borch has now ferreted out another special family of distributions, viz.,

$$f(x) = \frac{1}{24} [1 - \alpha \sin 4\sqrt{x}] \exp(-4\sqrt{x}), \quad (0 < \alpha < 1)$$

all the member of which have identical moment sequences, viz.,

$$m_n = \frac{(4n + 3)!}{6}$$

which is independent of the only parameter  $\alpha$ . A preference ordering based on mean, variance, skewness, etc., is then out of question. In fact, the Taylor expansion of the expected utility function involving this density would not converge with the successive moments rapidly increasing in a factorial function as above.

However, I would not regard this as a fatal defect of the ES analysis at all in its limited field of application as I suggested. For the ES analysis is meant only to be a practical method of portfolio analysis for investors who regularly take rather small risk relatively to their total wealth. It is not meant to be a universally valid mathematical theorem on preference ordering of all stochastic variables, that may be regarded as conclusively disproved by a single counter-example which mathematicians can dream up in their pipe smoke. Unless Borch can point out that there are actually investments in the financial market with returns belong this type of density, we need not even be bothered with it.

## II Reply to Bierwag

1 Bierwag take issue with me chiefly on my proposition that the

indifference curves in the (E, S) half-plane, in so far as such analytical device is justified, must have slopes less than 45°. He contends that this proposition is quite false. To support his contention, he attempt to derive the slope of indifference curves by using an artificial example of an investor investing all his wealth in two assets, one riskless and the other risky. He lets  $x_0$  and  $x_1$  stand for the actual number of dollars invested in the riskless and the risky assets, respectively, without imposing any wealth constraint or sign constraint on them.

In order to save space, I shall merely point out the queer implications and obvious contradictions in his results without making a thorough going research for the sources of these difficulties. I think it is incumbent rather upon himself to make this big effort.

If the derivation of his equations are correct, then equations (3) and (4) combined would give the slope of indifference curves as

$$\frac{dE}{dS} = \frac{\mu\sigma}{\sigma^2 + \mu/x_1}.$$

Where  $\mu$  and  $\sigma$  are the mean and the variance of the random return of the risky asset here assumed to be gamma distributed.

To confront this set of indifference curves with the opportunity locus of

$$E = \frac{\mu - 1}{\sigma} S + W, \text{ with } \frac{dE}{dS} = \frac{\mu - 1}{\sigma}$$

where  $W = x_0 + x_1$  is the total wealth of the investor.  $x_1$  can be readily solved as

$$x_1 = \frac{\mu(\mu - 1)}{\sigma^2}.$$



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This is a most astonishing result, as it implies that the investor concerned would always invest the same amount of dollars on the risky asset (depending only upon the parameters of the distribution of the random return of that asset) no matter how rich or how poor he is<sup>4/</sup>.

*Bierwag*

If / is aware of this, he certainly shows no qualm for this strange implications. He boldly proclaims that it proves that the E-S indifference curves can have slopes greater than  $45^\circ$ , since if we let  $x_1$  approach infinity,  $\frac{dE}{dS}$  would approach  $\mu/\sigma$ , which is greater or smaller than one according as  $\mu \gtrless \sigma$ .

The basic question is whether the E-S indifference curves, which is supposed to represent the subject preference ordering over combinations of E and S, should be derived from a knowledge of the investor's subjective utility function (or scale of preference) alone, or should they incorporate the investment opportunities on the market yet without any wealth constraint, as is implied in Bierwag's procedure. If the latter, then how are we to derive the properties of the indifference curves of an investor who is free to invest his wealth on any of the hundreds of stocks listed in the various exchange and on any of the hundreds of government or corporate bonds and bills plus a host of other assets, such as real estates, time deposits, cash, and what not. Furthermore, if we recall the section 3 of our discussion of Borch's comment, we should know that if there are just two risky assets, the random returns of which are not distributed according to the same family of "stable" distributions (in this connection, it should be pointed out that gamma distributions, which Bierwag used in his example, are not

"stable" in this sense), then we can no longer say that "expected utility can be consistently expressed as a function of only the mean and the standard deviation of the portfolio".<sup>5/</sup> That is to say, if there are two or more risky assets (unless they all have normally distributed returns), the theretical basic for his analysis would disappear.

2 Moreover, Bierwag should have noticed the glaring contradiction between his conclusion of the first section and his conclusion in the second section. It is a good thing that Bierwag has now admitted that if  $\text{Prob} [\lambda < 0] = 0$ , then the asset concerned cannot have a negative marginal expected utility.<sup>6/</sup> In other words, "contamination" of the portfolio on account of having too much of that asset is impossible.

In the case of his two asset model of the section 1, both the riskless asset, cash, and the risky one, assumed to have a gamma distributed return, cannot have a negative gross return, i.e.,  $\text{Prob} [\lambda < 0] = 0$ , and, hence, cannot be a source of "contamination" of the portfolio. If so, it is clearly unreasonable to find that a ray from the zero origin of the E-S plane to cut any E-S indifference curve twice. For this would imply that two vectors of E and S, one of which is a scalar multiple of the one, which can be represented by two portfolios, one of which is the same scalar multiple of the other, in the two assets in question, would yield the same level of expected utility. Given the convexity of the indifference curves, this would imply that starting with the bigger of the two utility equivalent portfolios, some destruction of both assets, cash and the asset with gamma distributed return, would increase the expected utility of the investor. This is of course absurd.

Bierwag has now accepted all this and tried to express the same fact with the proposition that the indifference curves involving such assets must have an elasticity of less than unity.<sup>7/</sup> That is, a ray from the zero origin can only cut the indifference curves from below, and, hence, cannot cut them twice.

However, the indifference curves he derived in Section 1 have a slope of zero when  $x_1 = 0$  and a slope of  $\mu/\sigma$  when  $x_1 = \infty$ , i.e.,

$$\frac{dE}{dS} = \frac{\mu}{\sigma} + \frac{1}{\sigma} \left( \frac{-\mu^2}{x_1 \sigma^2 + \mu} \right) = \begin{cases} 0 & \text{when } x_1 = 0 \\ \mu/\sigma & \text{when } x_1 = \infty \end{cases}$$

Thus any ray with a tangent between 0 and  $\frac{\mu}{\sigma}$  would cut all the indifference curves it encounters at two points except the one it is tangent to. How can he reconcile this fact with his proposition that these indifference curves must have an elasticity less than unity?

I have no ready explanation for this contradiction. One tentative conjecture would be that the difficulty probably arises chiefly because Bierwag tries forcibly to apply the E-S analysis to obviously skewed distributions, such as the gamma distributions, not realizing that its skewness is bound to lead to unreasonable conclusions, once the ratio of S to E becomes fairly large and the influence of skewness and other higher moments on the expected utility can no longer be safely neglected. It remains, however, for Bierwag himself to make the big effort to solve these puzzles.

## FOOTNOTES

1/ Tsiang (1972), pp. 354-355.

2/ See Tsiang, op. cit., p. 358.

If he is more ambitious (or greedy) and aspire to be a multi-millionaire, we must assign an even smaller value to the parameter  $\alpha$ . Thus it can be easily deduced that the more ambitious and greedy the investor is with respect to material wealth, the less aversion for risk or preference to skewness he would have.

3/ See also Tsiang (1973).

4/ For instance, if  $\mu = 2$ , meaning that the net rate of return is 100 per cent on the capital, and  $\sigma = 1$ ,<sup>making the risky asset</sup> by no means unattractive investment, this formula would say that only exactly two dollars would be invested on it, no matter whether the investor is a multi-millionaire or a pauper.

5/ Bierwag's comment, p. .

6/ Op. cit., p. .

7/ Op. cit., p. .

#### REFERENCES

- G. O. Bierwag, "The Rationale of the Mean-Standard Deviation Analysis: Comment," American Economic Review, this issue.
- K. Borch, "Mean-Standard Deviation Analysis: Comment," American Economic Review, this issue.
- S. C. Tsiang, "The Rationale of the Mean-Standard Deviation Analysis, Skewness Preference, and the Demand for Money," American Economic Review, June 1972, 62, 354-71.
- \_\_\_\_\_, "Risk, Return, and Portfolio Analysis: Comment," J. Polit. Econ., June 1973, 81.