Institute of Economic Development and Research
SCHOOL OF ECONOMICS
University of the Philippines

Discussion Paper No. 71-27

6 December 1971

NEOCLASSICAL THEORY OF DISTRIBUTION AND GROWTH WITH CAMBRIDGE CRITIQUES

by

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Neoclassical Theory of Distribution and Growth with Cambridge Critiques

Introduction

One often hears critiques of neoclassical theory out of context from the neoclassical theories they are supposed to be attacking. One also hears several versions and interpretations of neoclassical theory so that it is difficult to say which one the criticisms are to apply to and whether those criticisms apply generally to all versions. In this paper I attempt to lay out a concise, clear, and exact interpretation of the neoclassical parable designed to explain income distribution and growth. The subsequent criticisms are then made in the context of that exposition.

The paper is organized as follows. First the neoclassical parable is developed for a one sector world and its normative implications are discussed. Then Samuelson's Surrogate Production function is assessed as an attempt to justify extension of the parable to a world with heterogenous capital goods. In the following section, the inverse relationship between capital per man and the rate of profit is shown to break down for a two-sector world even when the capital-labor ratio is the same in both sectors. Allowing differences in the capital-labor ratios in each of two sectors results in a breakdown of the inverse relation between the rate of profit and value of capital per man. Furthermore an infinity of techniques and approximate smoothness of the wage-profit frontier is shown not to imply approximation of that relationship.

The order of models is from specific to general. I was tempted to state the general equation for the factor-price frontier first and then to derive the Samuelson and Hicks models as special cases. For obvious pedagogical reasons however, it seems wiser to put the easier material first.

The main purpose of this paper is to focus attention on and to clearly explain what seems to be the essence of a mass of scattered literature. As often happens in attempts of this sort, however, certain new results and interpretations emerge in the process. Among these I feel that the contrast of the Neoclassical and

Cambridge views of the parable world and the comments and conjectures regarding continuity of the wage-profit frontier are the most interesting.

Numerous short essays are appended to the paper. These are felt to be important to understand the neoclassical position yet aren't necessary for the major points developed in this paper.

Finally, I have attempted to keep the exposition as simple and intuitively appealing as possible without sacrificing rigor. Usually, I hope, this has meant that the presentation here is easier to grasp than the original source of the concept and the various other sources which have summarized the original. At times, however, it has been necessary to supplement the literature where an idea is sound but has not been satisfactorily explained and when an idea is unsound due to a lack of regard for the foundations of calculus. The latter applies specifically to conditions for continuity of the wage-profit frontier. For the most part, however, these comments are confined to footnotes and can easily be avoided by readers inclined to do so.

Summary of the "Cambridge Controversy"

The lesson of the neoclassical parable is that technical change aside, growth in per capita income is due to increases in the aggregate capital-labor ratio. The capital-labor ratio is determined by population growth and the average propensity to save. The capital-labor ratio also explains distribution of income. Capital and labor are each paid their marginal products which are also a function of the capital-labor ratio.

The moral of the story is that if policy instruments are available to regulate the savings rate, then savings should be increased, increasing the capital-labor ratio until the marginal product of capital equals the growth rate of population. This strategy will maximize per capita consumption.[1] Alternatively, if there is a second asset available which is subject to government regulation, then its rate-of-return should be set equal to the population growth rate in order to induce the amount of capital formation which will maximize per capital consumption.[2]

It is now clear, due to so-called "Cambridge Critiques",[3] that the parable results do not necessarily apply to a world of heterogeneous outputs and capital goods. Specifically, there are two factors which confound the relationship between output per man and capital per man when we compare steady-state growth paths associated with different techniques. The first factor is that each technique has its own relationship of prices to factor payments. Thus comparing different steady-states involves the use of different weights to combine heterogeneous goods; this may be called the price effect. The second factor is that output per head in each steady state is equal to the maximum wage which is dependent only on the technical coefficients of the technique used and is independent of the profit rate. This may be called the composition effect.

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[1] E. Phelps, "The Golden Rule of Accumulation," AER, September 1965.

[2] J. Tobin, Money and Economic Growth," Econometrica, October 1965.

[3] G. Harcourt, "Some Cambridge Controversies in Theory of Capital," JEL, June 1969.

The Neoclassical Theory of Production and Distribution as Embodied in the Solow-Swan One Asset Growth Model

In this hypothetical economy, one and only one scarce and (perfectly) durable commodity, say shmoos, is produced which can be consumed or used in conjunction with labor to produce more shmoos. Aggregate production of the shmoos can be described by the function,

$$Q \equiv F(K,L),$$

where F is linearly homogeneous, K is homogeneous real capital and L is the homogeneous labor stock. Since there is no depreciation, any shmoos saved will be employed producing more shmoos so long as their marginal product is positive,[4] i.e., savings always equals investment, thus

$$\dot{K} \equiv sQ$$

Homogeneous labor grows at an exogenously constant rate, i.e.,

$$\frac{\dot{L}}{L} = n$$

and is always employed so long as its marginal product is positive. Nobody, labor owners, shmoo owners, or producers colludes; all strive to sell to the highest bidder; and producers maximize profits. At any instant a Walrasian equilibrium[5] prevails wherein factors are supplied at zero opportunity cost to their owners and allocated to their best use via competition among producers who hire factors until their marginal products equal their rentals.

Given the foregoing setting we shall see that the economy is run by the very variables which are not explained, but set exogenously in order to simplify the analysis--the saving rate, s, and the growth rate of labor, n.

First express per capital output as

$$\frac{Q}{L} = \frac{F(K,L)}{L} = F(k,1) \equiv f(k)$$

where k is the capital-labor ratio. By the identify above,

$$\frac{\partial F}{\partial K} > 0 \text{ and } \frac{\partial^2 F}{\partial K^2} < 0 \text{ are translated as}$$

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[4] If there is depreciation, define Q as net product and the analysis above will be appropriate.

[5] See Appendix, "The Timeless Equilibrium of Walras."

$$f'(k) > 0 \text{ and } f''(k) < 0.$$

The growth rate of the capital-labor ratio is equal to the growth rate of capital minus the growth rate of labor:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{sQ}{K} - n = \frac{sf(k)}{k} - n$$

$$\frac{d\frac{\dot{k}}{k}}{dk} = \frac{s[f'(k)k - f(k)]}{k^2} - 0$$

$$= \frac{sf'(k)}{k} - s\frac{f(k)}{k^2} < 0$$

since

$$f'(k) < \frac{f(k)}{k}$$

Thus the linear homogeneity assumption implies that a rise in k will cause a fall in the growth rate of k, and a fall in k will cause a rise in the growth rate of k, i.e., as times passes, $\frac{\dot{k}}{k}$ tends to zero. This asymptotic equilibrium condition,

$$\frac{\dot{k}}{k} = \frac{sQ}{K} - n = 0,$$

can be written

$$\frac{\dot{K}}{K} = n$$

i.e., capital grows apace with labor, hence "balanced growth".

The "golden rule" of this economy is to save until

$$f'(k) \equiv F_K = n.$$

This will maximize the path of sustainable consumption per head.[6]

To complete the neoclassical parable, note that the conditions:

$$w = f(k) - kr \text{ (zero profit)}^7 \text{ and}$$

$$r = f'(k) \text{ (marginal cost pricing)},$$

are implied by the assumption of Walrasian equilibrium where r and w are the rental rates of capital and labor respectively. Combining the two equations:

$$w = f(k) - kf'(k).$$

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[6]Per capital consumption $\equiv c \equiv f(k) - \frac{sQ}{L}$ which for balanced growth becomes $c = f(k) - \frac{nK}{L} = f(k) - nk$. To find the k which maximizes sustainable consumption per head, set $\frac{\partial c}{\partial k} = f'(k) - n = 0$ thereby obtaining the solution $f'(k) = n$.

[7]To see this, rewrite $Q - wL - rK = 0$ as $w = \frac{Q}{L} - \frac{rK}{L} = f(k) - rk$.

$$w = f(k) - kf'(k).$$

Taking the derivatives of these three equations with respect to r, k, and k respectively, we have:

$$\frac{dw}{dr} = f'(k)\frac{dk}{dr} - r\frac{dk}{dr} - k = -k < 0$$

$$\frac{dr}{dk} = f''(k) < 0$$

$$\frac{dw}{dk} = f'(k) - f'(k) - kf''(k) = -kf''(k) > 0$$

The "wage rate", w, varies directly with the capital-labor ratio; the "interest rate", r, varies inversely with it; and w and r vary inversely with each other. The simple neoclassical theory of production and distribution is complete.

Normative Implications of Neoclassical Theory

1) The moral of this story is to use policy instruments to increase aggregate saving. This will increase k and thereby per capital consumption up until the golden age limit. This normative facet of simple neoclassical models stands in stark contrast to that of the Harrod and Domar models (see Appendix, "Harrod's Warning"), which point the finger of suspicion at excessive thrift.

2) Another moral of the neoclassical story, which is never preached by high priests of the MIT school, but which is always lurking in the background just the same, is that the payments to capital and labor are earned rewards and are somehow justified. It seems unrealistic to assume that proponents of the "just-reward" proposition have overlooked the well-known doctrine that positive theories cannot produce normative results. More reasonable is the assumption that they have made the implicit value judgement that what is earned is deserved. But even given that value judgement, the just-reward conclusion doesn't follow.

Firstly, one can no more attribute the marginal product of labor to a single man than to a single machine. Where capital and labor are combined in production, attempting to determine how much output was produced by labor and how much by capital is akin to asking, "Which of your eyes gives you two dimensional vision?" It is impossible to identify one and only one correct

answer. That labor produces its marginal product times the stock of labor employed, and likewise for capital, is just as arbitrary as saying labor produced all of the output.

Secondly, the result that factors are paid their marginal products obtains only under the special assumption that a Walrasian equilibrium prevails at any fixed point in time that one could pick. In that world, profit never exceeds interest. But if that were so there would be no inducement for positive investment, yet the neoclassical model has investment going on continuously. Profits have generally exceeded interest rates since the industrial revolution, and any theory which ignores that cannot claim a relevant explanation of distribution. (See Appendix, "Profit--Surplus or Reward?")

Samuelson's One Sector Surrogate Production Function

Samuelson [17] has attempted to extend the realism of the neoclassical parable[8] by suggesting that the parable results apply to a more complex model involving fixed proportions and heterogeneous capital goods.

Available production techniques in this world are described in a book of blueprints, each page giving the fixed-proportion requirements for a particular technique. Each process uses one type of capital good and produces a homogeneous consumption good (or a fixed-proportions market basket) and the particular capital good used in that process. Depreciation is assumed to be zero.[9] The model fits a world of shmoos where n types of shmoos are produced, and each type can be used as a consumption or capital good. If used for the latter all types are distinguishable; for the former they are not. There is one homogeneous primary output called "labor". Symbolically, let

$$T = \{i, | 1,2,3,\ldots n\}$$

be the set of n techniques. Each technique T_i is a matrix of the

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[8]For a similarly motivated attempts, see the Appendices, "Introducing Investment into the Neoclassical Model: Tobin's Two-Asset World" and "Technical Change in Neoclassical Models."

[9]Samuelson allows a positive depreciation rate, but his results are essentially the same as the ones presented here.

form

$$\begin{vmatrix} \alpha_i^L \\ \alpha_i^K \end{vmatrix}$$

where α_i^L and α_i^K are the capital and labor coefficients for the ith technique.

Defining r as the own interest rate for all capital goods (shmoos per shmoo), w_i as the wage (shmoos per unit of labor) for process i,

$$w_i = \frac{1}{\alpha_i^L} - \frac{\alpha_i^K}{\alpha_i^L} r$$

where $\frac{\alpha_i^K}{\alpha_i^L}$ is the capital-labor ratio for process i and $\frac{1}{\alpha_i^L}$ is the output produced by one unit of labor and $\frac{\alpha_i^K}{\alpha_i^L}$ units of capital.

This equation is plotted in figure 1 below and is called the factor-price frontier for process i. Notice that the absolute value of the slope of this graph is the capital-labor ratio, $\frac{\alpha_i^K}{\alpha_i^L}$.

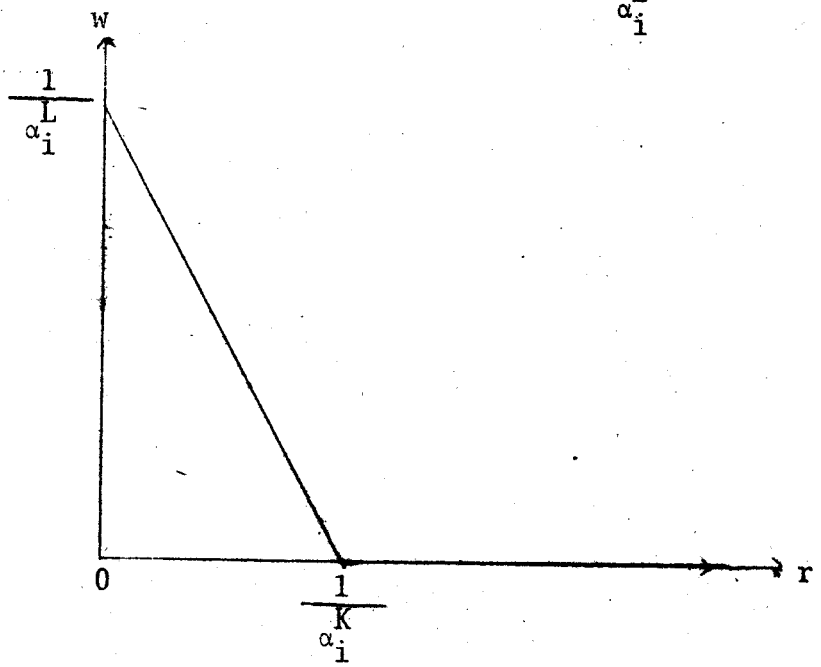

figure 1

Assuming the techniques in set T (the pages in the book-of-blueprints) are ordered by capital intensity, i.e., 1,2,3,...n, we can graph the factor-price frontiers for all the processes in the book as shown in figure 2 (for a hypothetical book with four pages).

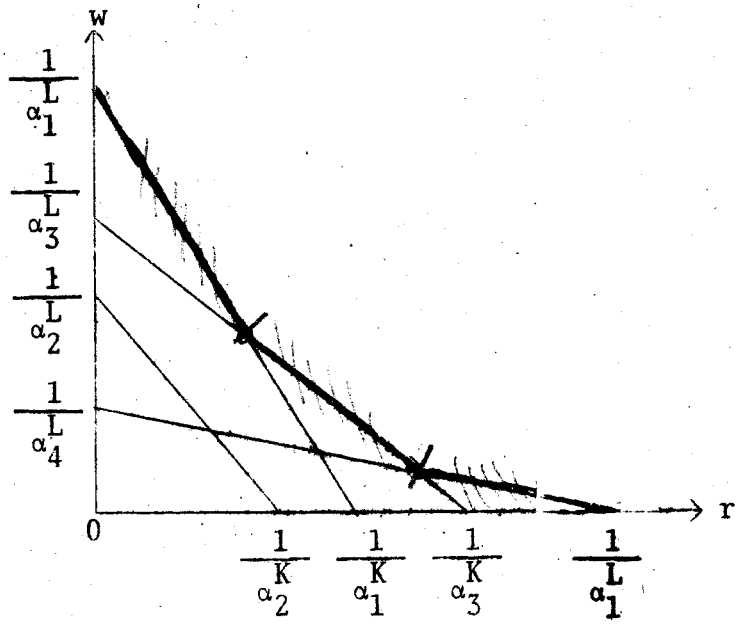

figure 2

The heavily-shaded envelope of these curves is the grand factor-price frontier. Its equation may be written:

$$w = \text{Max } w_i(r) \forall i \epsilon T$$

In the parable world the grand factor-price frontier is given by:

$$w = f(k) - kr \text{ where}$$
$$r = f'(k)$$

This is derived graphically in figure 3. The Marshallian elasticity of the curve is

$$\frac{-r}{w} \cdot \frac{dw}{dr} = \frac{r}{w} k = \frac{rK}{wL} = \frac{\alpha}{1-\alpha} \quad \text{(a constant)}$$

for the Cobb-Douglas case, ergo the convex (to the origin) shape of the frontier.

Now notice that the envelope frontier in the discrete case can be thought of as an approximation of the continuous parable frontier. For the envelope the inverse relationship between w and r obtains; in fact except at switch points (the "corners", associated with more than one technique) $\frac{dw}{dr} = -k$,

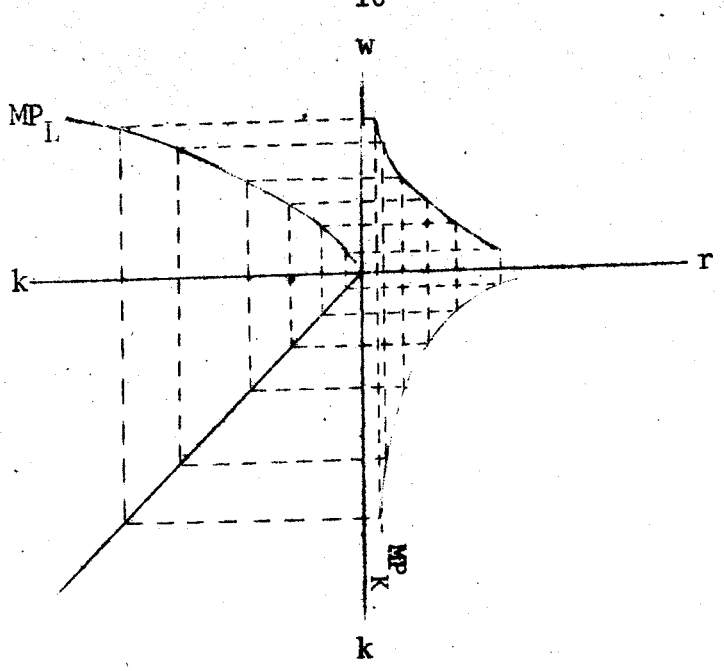

figure 3

just as in the continuous case. Furthermore as k decreases, w increases and r decreases--similar again to the parable case.

It is possible to define expansion of the book-of-blueprints in such a way that the envelope becomes identical to the surrogate production function in the limit. We can define a (countably) infinite paged book such that there is one dominant technique for every rational k within a certain range of k.[10] For such a book the monotonic (decreasing) relationship between k and r and the

[10] Let $T^{\infty} = \{i \mid 1,2,3,\ldots\infty\}$, $\frac{\alpha_i^K}{\alpha_i^L} \neq \frac{\alpha_j^K}{\alpha_i^L}$ $\forall\, i,j \in T^{\infty} \ni i \neq j$, such that T^{∞} is in one-to-one correspondence with the set of rational numbers in the closed interval from k* to k** and where $\frac{\alpha_i^K}{\alpha_i^L}$ is equal to the corresponding rational number in the latter set for all $i \in T^{\infty}$. Assert further that one and only one w,r combination corresponding to one technique appears on the grand frontier, i.e., $w(r_g) = w_i(r_g)$ implies $w(r_h) \neq w_i(r_h)$, $r_h \neq r_g, \forall\, i \in T^{\infty}$ and $w(r_g) \neq w_i(r_g) \forall\, r_g \neq r_h$ implies $w(r_h) = w_i(r_h) \forall\, i \in T^{\infty}$.

inverse relationship between r and w hold for all rational k within the specified range. For each rational k in the range and the associated w and r, define

$$g(k) \equiv w + kr.$$

Then the surrogate production function can be defined as

$$f(k) \equiv \overline{g(k)} \text{ (the adherance}^{11} \text{ of } g(k)).$$

But $\overline{g(k)}$ is indistinguishable from

$$w(r) = \text{Max } w_i(r)$$

for all $i \epsilon T^{\infty}$. Thus Samuelson's conjecture that it is possible to show "rigorous equivalence" between the surrogate production function and an envelope of process factor-price frontiers for a special set of techniques is correct.

A Cambridge Interpretation of the Neoclassical Parable Based on "Historical Time"

With this new tool, the factor-price frontier, at our disposal, we can now give an alternate interpretation of the neoclassical model. The interpretation above relies heavily on differential calculus to show the tendency toward balanced growth. Time in that model is continuous.

Now consider the same model except that time is discrete, i.e., t refers to an interval (say a year) rather than a point in time. This model is summarized below:

$$f_t(k_t) \equiv F_t(k_t, 1) = \frac{Q_t}{L_t}$$

$$K_t = K_{t-1} + sQ_{t-1}$$

$$L_t = L_o e^{nt}$$

Since time derivatives are undefined here, we cannot derive equilibrium from the stability analysis used above. Instead define equilibrium as steady-state growth:

$$\frac{Q_t - Q_{t-1}}{Q_{t-1}} = \frac{K_t - K_{t-1}}{K_{t-1}} = \frac{L_t - L_{t-1}}{L_{t-1}}$$

This condition is satisfied if

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[11] For an explanation of "adherance" see e.g. Debreu, Theory of Value, Chapter One.

$$\frac{sQ_{t-1}}{K_{t-1}} = n.$$

The equation for the wage-profit curve[12] of this economy,

$$w \equiv f(k) - rk,$$

follows directly from the accounting identity

$$f(k) \equiv w + rk.$$

Note that:

$$f'(k) = \frac{dw}{dr} \cdot \frac{dr}{dk} + k\frac{dr}{dk} + r$$

$$= -k\frac{dr}{dk} + k\frac{dr}{dk} + r$$

$$= r$$

since

$$\frac{dw}{dr} = -k.$$

The equation for the wage-profit frontier can now be written:

$$w = f(k) - f'(k)k$$[13]

and it follows that

$$\frac{dw}{dk} = -kf''(k).$$

---

[12]In the Cambridge approach, the process factor-price frontier is called the "wage-price curve". The grand factor-price frontier is called the "wage-price frontier". The different terminology reflects a disagreement on the nature of r. In the neoclassical approach r is the rental rate of capital, in units of capital per unit of capital employed, or simply the own rate of interest. This view implicitly views the management of an enterprise renting its capital from separate owners or from themselves. In the Cambridge approach, profit is simply defined as all the return to capital whether or not any payment is needed to keep the capital in its present use. The former view is correct for the parable world of perfectly mobile and consumable capital shmoos. Whether it is reasonable in other models depends on the behavioral assumptions regarding managers (not on legal or physical distinctions between owners and managers). The latter view is a tautology and is always correct. In either view, r is equal to the marginal product of capital. (See Appendix, "Profit--Surplus or Reward?")

[13]Since w and r are functions of k, and k is constant in steady-state growth, we can also interpret equilibrium to mean that state which maintains a constant ruling wage and profit rate over time. Indeed that is Joan Robinson's [16] definition of equilibrium.

Summarizing we have the same results as in the conventional interpretation of the parable:

$$\frac{dw}{dr} = -k < 0$$

$$\frac{dr}{dk} = f''(k) < 0$$

$$\frac{dw}{dk} = -kf''(k) > 0$$

These results are illustrated in figure 4 below.[14]

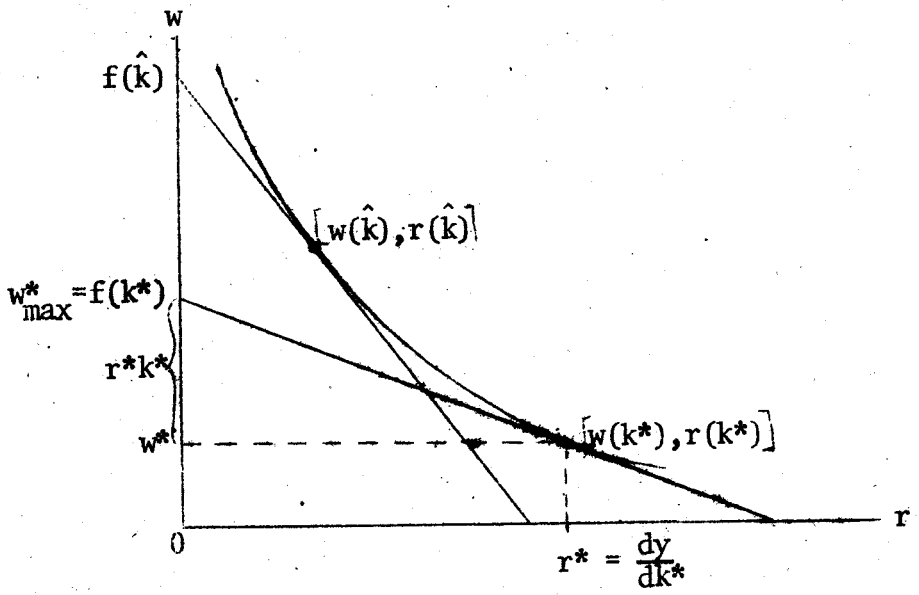

figure 4

The neoclassical theory of distribution is illustrated on the vertical axis for an arbitrary capital-labor ratio, k^*.[15] Comparing the equilibrium points, note that

$$r(\hat{k}) < r(k^*)$$

implies

$$w(\hat{k}) > w(k^*),\ f(\hat{k}) > f(k^*) \text{ and } \hat{k} > k^*.$$

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[14]The diagram can also be used to show how the Surrogate Production Function f(k) can be derived from the wage-profit frontier.

[15]There are two ways to derive r^*k^*. First, since the line from f(k) through the point (w^*,r^*) is tangent at the latter point, the slope of that line is $\frac{dw}{dr} = -k^* = \frac{x}{-r^*}$ where x is the distance between w^* and the point labeled f(k). Therefore $x = r^*k^*$ and we have $f(k^*) = r^*k^* + w^*$ represented correctly on the graph. Second, noting the tangent is also the wage-profit curve for the technique associated with $k^*, w^*_{max} = f(k^*) \equiv w + rk^*$ where r is 0.

These are the essential relationships of the neoclassical parable.

The important point here is that the neoclassical results obtain without the assumption of Walrasian equilibrium. Marginal cost pricing follows as a consequence of the technology (linearly homogeneous production function). What is lost is the ability to perform stability analysis based on differential calculus. What is gained is a model of "historical time".[16] The discrete time periods can be placed "end-to-end" to arrive at a model of a long-run. This cannot be done in the conventional interpretation. No matter how many timeless, Walrasian-equilibrium points we stack end-to-end we still have a point in time, not an interval.

The Wage-Profit Frontier for a Two Sector Model with Linear Techniques[17]

One can think of the model above as describing a world where only one good is produced, but there are many uses for the good. The model also describes a world where many goods are produced, but all relative prices are equal to one.

Now consider a model which is identical to Samuelson's except that it describes a world wherein consumption goods are distinguishable from capital goods in production as well as in use. Capital goods, however, are distinguishable from one another only in use. In other words, the relative prices of capital goods are always one, but those of capital and consumption goods generally are not.

There are n linear techniques in this world, each of which produces the capital shmoos or the consumption good, shoos. The wage-price curve for the ith technique is:

$$w_i(r) = \frac{1}{\alpha_i LC} - rk_i p_i$$

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[16]The term is frequently found in the works of Joan Robinson.

[17]The model here is due to Hicks [9], but our exposition of the model is different than his. The explanation here is designed to clarify the relationship between the Hicks case and the Samuelson case discussed above.

where

$$k_i = \frac{\alpha_i^{KM}}{\alpha_i^{LM}} = \frac{\alpha_i^{KC}}{\alpha_i^{LC}}$$

$$p_i = \frac{\alpha_i^{KM}}{\alpha_i^{KC}} = \frac{\alpha_i^{LM}}{\alpha_i^{LC}} \quad ,$$ [18]

and the α_i's are the input coefficients for shmoos (M) and shoos (C) for the ith technique. Note that by adding the requirement that $p_i = 1$, $i = 1$ to n, gives us Samuelson's special (one-sector) case.

The wage-profit frontier is given by

$$w \equiv \text{Max } w_i(r) \; \forall \; i \varepsilon T$$

The wage-profit curves and frontier are illustrated below in figure 5.[19]

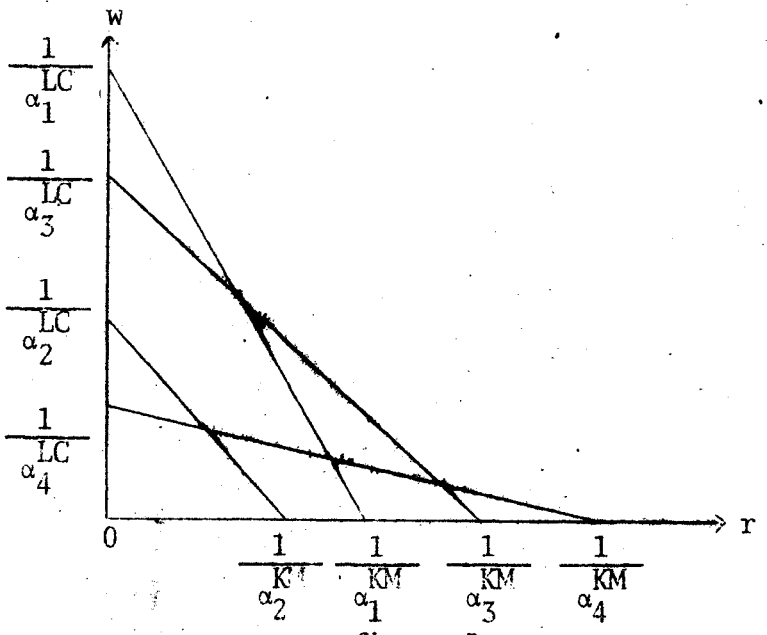

figure 5

[18]Harris [7] p. 12 has noted that that this condition implies a labor theory of value. One should not fail to notice, however, that a "capital theory of value" is also implied. It is somewhat misleading to state one but not the other since it is precisely the equality of capital and labor input-coefficient-ratios that allows us to state price in terms of either of the ratios.

[19]To construct the graph rewrite w_i as $\frac{1}{\alpha_i^{LC}} - r \frac{\alpha_i^{KM}}{\alpha_i^{LM}} \cdot \frac{\alpha_i^{LM}}{\alpha_i^{LC}}$

$$= \frac{1}{\alpha_i^{LC}} - r \frac{\alpha_i^{KM}}{\alpha_i^{LC}}$$

Which of the neoclassical results hold for this case? Since the slope of w is:

$$\frac{dw}{dr}(i) = -k_i p_i$$

except at switch points, the inverse relationship between w and r is preserved. Since the frontier becomes flatter as r increases, there is an inverse relationship[20] between r and kp, the value of capital per man.[21] This does not mean that there is necessarily an inverse relationship between r and k however. If the percentage change in p is greater than the percentage change in r and in the same direction, then k will also change in that direction. In other words, an increase in the capital-labor ratio may involve a switch to a technique for which shmoos become so cheap that the value of capital per man decreases thus implying an increase in r. In conclusion, there are conceivable book-of-blueprints that invalidate the neoclassical results even in the absence of reswitching.

Reswitching

We saw above that taking one step toward reality away from the Samuelson model gets the neoclassical parable into trouble. We now take a second step, and become more engulfed in a quagmire. Specifically, we now relax the assumption that the capital-labor ratios in both sectors are identical. As a consequence we can no longer express the aggregate capital-labor ratio as the

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[20]Because of the existence of switch points representing two or more techniques, this relationship cannot be expressed as a function.

[21]It is tempting to argue further (as does Harris [7] pp. 15-18) that the "marginal product of capital" in this model equals the rate of return on capital, r. First define $V \equiv kp$ as the value of capital per man. From the accounting identity, $y \equiv w + rv$,

$$\frac{dy}{dv} = \frac{dw}{dr} \cdot \frac{dr}{dv} + v\frac{dr}{dv} + r = -v\frac{dr}{dv} + v\frac{dr}{dv} + r = r; \text{ Q. E. D.}$$

But this is a misapplication of calculus. Whether capital is defined in real or value terms, its marginal product is undefined for a world where the wage-profit frontier is not rigorously equivalent to the Surrogate Production Function. This is because derivatives are undefined at switch points and at non-switch points both k and pk are constant.

capital-labor ratio in either sector. Likewise the price of the manufactured good is equal to neither the capital nor the labor input-coefficient ratio. We shall see that in this world, there is not necessarily an inverse relationship between r and the value of capital per man.

In order to derive the wage-profit curve we need to first state the price equations:

$$p_i = \alpha_i^{LM} w + p_i \alpha_i^{KM} r$$

$$1 = \alpha_i^{LC} w + p_i \alpha_i^{KC} r$$

where w is the uniform wage rate (in shoos), r is the uniform rate-of-return on capital (shmoos per shmoo), p_i is the price of the manufactured good for the ith technique (in shoos) and 1 is the price of the consumption good for all techniques. Solving the second equation for p_i:

$$p_i = \frac{1 - \alpha_i^{LC} w}{\alpha_i^{KC} r}$$

Now substituting this expression for p_i in the first equation and solving for w:

$$w(i) = \frac{1 - \alpha_i^{KM} r}{\alpha_i^{LC} + (\alpha_i^{LM}\alpha_i^{KC} - \alpha_i^{LC}\alpha_i^{KM})r}$$

which is the equation of the wage-profit curve. Notice that the term in parentheses is the determinant of the technique matrix, $|T_i|$. The Hicks Model of the previous section is a special case of this one; equality of the capital-labor ratios in both sectors imply that $|T_i| = 0$, so the equation above collapses to the wage-profit curve of the Hicks case. Returning to the frontier,

$$w_i = \frac{1 - \alpha_i^{KM} r}{\alpha_i^{LC} + |T_i| r} \quad \text{for all } i \in T$$

$$\frac{dw_i}{dr} = \frac{-\alpha_i^{KM}\alpha_i^{LC} + |T_i|}{(\alpha_i^{LC} + |T_i| r)^2}$$

$$= - \frac{\alpha_i^{LM} \alpha_i^{KC}}{(\alpha_i^{LC} + |T_i|r)^2}$$

$$= - \frac{\alpha_i^{LM} \alpha_i^{KC}}{(\alpha_i^{LC} + |T_i|r)^2} < 0, \text{ for all } i \varepsilon T$$

i.e., the wage-profit curves are negatively sloped.

$$\frac{d^2 w_i}{dr^2} = \frac{2|T_i|(\alpha_i^{LC} + |T_i|r)\alpha_i^{LM}\alpha_i^{KC}}{(\alpha_i^{LC} + |T_i|r)^4} .$$

Since

$$(1 - \alpha_i^{KM}) \geq 0$$

for a non-negative wage,

$$\alpha_i^{LC} + |T_i|r > 0$$

follows from the equation above for w_i. Thus

$$\frac{d^2 w_i}{d^2 r} \lesseqgtr \text{ if } |T_i| \lesseqgtr 0 \text{ for all } i$$

The wage-profit curve is either straight, convex, or concave throughout. It cannot be convex in one section and concave in another.

The wage-profit frontier can therefore be composed of straight, convex, and concave segments but must be negatively sloped throughout. An example is illustrated below in figure 6. Notice that α_1

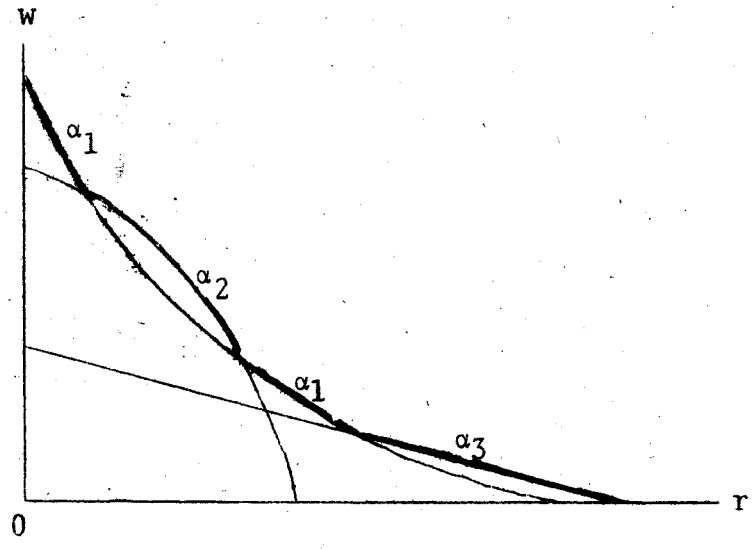

figure 6

dominates in the lowest range of r, is replaced by α_2 at higher levels of r, and then dominates again at still higher levels of r. This phenomenon is known as "reswitching" and makes it impossible to order techniques uniquely by the rate of profit.[22] More poignantly, there is no unique mapping from the set of techniques $\{i|i = 1,2,3,\ldots n\}$ to r on the closed interval $[0,1]$, i.e., r is not a well-defined function of i.

Since each wage-profit curve must be straight, convex, or concave throughout, we can see from the diagram that if either

$$r_{max}(i) > r_{max}(j) \text{ and } w_{max}(i) < w_{max}(j) \text{ or}$$

$$r_{max}(i) < r_{max}(j) \text{ and } w_{max}(i) > w_{max}(j)$$

for any $i \neq j$ technique, where e.g. $w_{max}(i)$ is the value of w_i when $r_i = 0$, then there is no reswitching.[23]

A Neoclassical Reply

The obvious reply to all of this has been made by Solow himself:

> I have never thought of the macroeconomic production function as a rigorously justifiable concept... It is either an illuminating parable, or else a mere device for handling data, to be used so long as it gives good empirical results, and to be abandoned as soon as it doesn't, or as soon as something better comes along.[24]

In other words the parable wage-profit frontier and the relations that follow from it seem to provide a useful approximation of

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[22]This seems to be a tautology since, as near as I can make out from the literature, a ranking of techniques is defined to be unique when, once a technique has come into use over a continuous closed interval on the wage-price frontier, it never comes into use again, i.e., there is no reswitching.

[23]Essentially the same condition has been stated by Madrid, "The Theory of Growth in a Two-Sector Cambridge Production Model," [1970] (mimeo).

[24]Solow, "Review of Capital and Growth," American Economic Review, V. 55, [December 1966], pp. 1257-60.

of reality.

A Cambridge Retort

Pasinetti [12] has shown, however, that even if the parable frontier is a good approximation of the grand frontier corresponding to the techniques actually avaialable, the neoclassical results are not necessarily approximately correct.

> For vicinity of any two techniques on the scale of variation of the rate of profit does not imply closeness of the total values of their capital goods. It is therefore not true that, as the number of techniques becomes larger and larger, the differences in the values of capital goods per man of any two neighboring techniques necessarily become smaller and smaller. These differences might well remain quite large, no matter how infinitesimally near to each other two techniques are on the variation of the rate of profit. In other words, continuity in the variation of techniques, as the rate of profit changes, does not imply continuity in the variation of values of capital goods per man and of net outputs per man.[25] [p.253]

Pasinetti's point (made verbally) is illustrated in figure 7 for the two-sector case described above (under "Reswitching").

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[25]This final sentence is meaningless without a statement of the necessary conditions for continuity of the wage-profit frontier. Harris [7], p. 22, states

> The larger the number of techniques, the smaller is the segment contributed by each to the frontier. In the limit, with an infinite number of techniques, the frontier becomes a smooth curve tangent at each point to one wage-profit curve. There is then continuous variation of techniques in relation to the profit rate.

This also seems to be what Pasinetti had in mind in the statement above, but it is incorrect. For any set of ordered techniques $\{i|i = 1,2,3,...n\}$, $n = \infty$ does not imply continuity of the frontier. A stronger statement can be made: It is impossible for any such infinite set to be in one-to-one correspondence with the set of points comprising a continuous wage-profit frontier. (See the discussion of real and irrational numbers, infinity, adherance, and continuity in the first chapter of Debreu [2].)

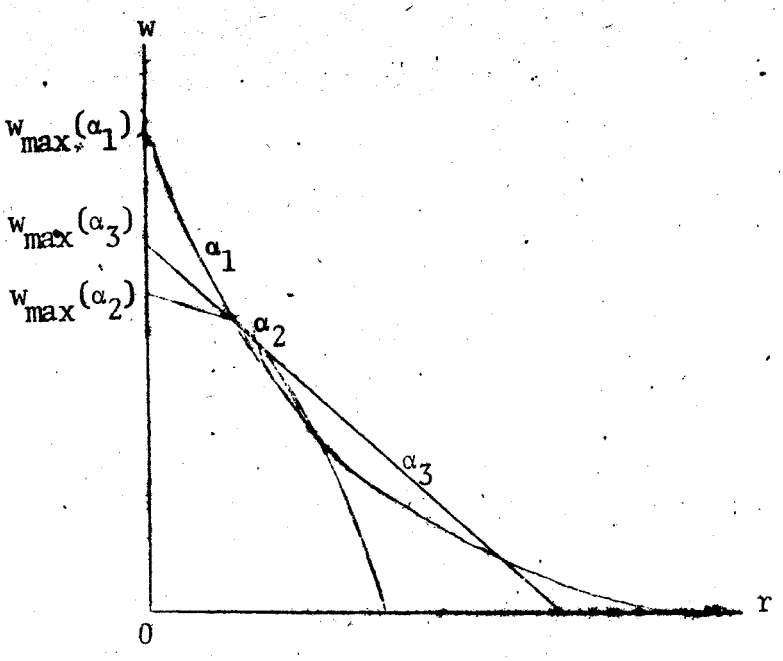

figure 7

Within an arbitrarily small range of r, three techniques are employed in succession, α_1, α_2, α_3, and the slopes of their respective curves are arbitrarily close to one another. Output per man for each technique is equal to the wage per unit of labor for that technique where r = 0 and is shown on the vertical axis as $w_{max}(i)$ for i = 1,2,3. Note the wide disparity in output per man for each technique and that the ranking of $w_{max}(i)$ on the vertical axis does not correspond to the order that the techniques appear on the frontier. This phenomenon may be called the composition effect as it derives from the different capital-labor ratios for the three techniques. There is also a price effect involved. The relationship of $w_{max}(i)$ for i = 1,2,3 shown above depends on an arbitrary selection of numeraire. Using the other good as numeraire, the shapes of the wage-price frontiers can change dramatically, changing again the relationship of output per head for each technique.[26]

The Neoclassical Parable and Paradigm

The criticism reviewed above was aimed at showing that the neoclassical parable relations are inappropriate for analysis of

[26]This was explained in another context by Sraffa [21] Chapter 6.

real-world phenomena. What does this imply about the usefulness of the general neoclassical approach to economic analysis? Samuelson sums up the essence of the neoclassical paradigm as follows:

> ...if we are to understand the trends in how incomes are distributed among different kinds of labor and different kinds of property owners, both in the aggregate and in the detailed composition, then studies of changing technologies, human and natural resources availabilities, taste patterns, and all the other matters of microeconomics are likely to be very important.[27]

Whether or not this philosophy is reasonable has not been the subject of the Cambridge critiques reviewed here. The critiques have shown the pitfalls of aggregation, not the limitations of microeconomics.

Criticism of the parable is not necessarily criticism of the paradigm. Indeed the neoclassical parable is weak largely because it ignores many facets of microeconomics. The role of demand in determining output, investment, and factor allocation is ignored. All economic differences among sectors are assumed away. A corollary is that there is no possibility for different factor intensitities in different sectors. Finally adjustment costs, information costs, transportation costs, and transactions costs are left out. One result of this is instantaneous market clearing.

The Cambridge reliance on the wage-profit frontier and steady-state growth also has severe limitations. Factor scarcity, and the interplay of supply and demand of factors have no role in determining factor payments. Output composition can shift from the labor intensive sector to the capital intensive sector with absolutely no effect on the wage or profit rate. The reason for this apparent weakness is that the model only claims relevance for characterizing steady-state equilibria.

The Cambridge steady-state equilibrium seems to be superior to the Walrasian timeless equilibrium for a growth model simply because it provides a model of historical time. Still its limita-

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[27] Samuelson [17], p. 193.

tions remind us of Joan Robinson's suggestion to

> ...give up the idea of equilibrium and exhibit an economy blundering on from one situation to another (as happens in the history of the world we live in) following no simple predictable path.[28]

Conclusion

The neoclassical parable applies to a special case with severely limited generality. In a world with two or more economically distinct sectors, there is in general no inverse monotonic relationship between aggregate capital per man and the profit rate. Furthermore such a relationship does not necessarily tend to be correct as the wage-profit frontier represents more and more techniques. The neoclassical theory of distribution is seen to be dependent on the existence of a linear homogeneous production function and does not generalize to a world of discrete techniques. There is no book-of-blueprints even one with infinite pages that contains all the techniques represented by an aggregate production function, and there is no set of ordered techniques that can be put in one-to-one correspondence with the set of points comprising a continuous wage-profit frontier.

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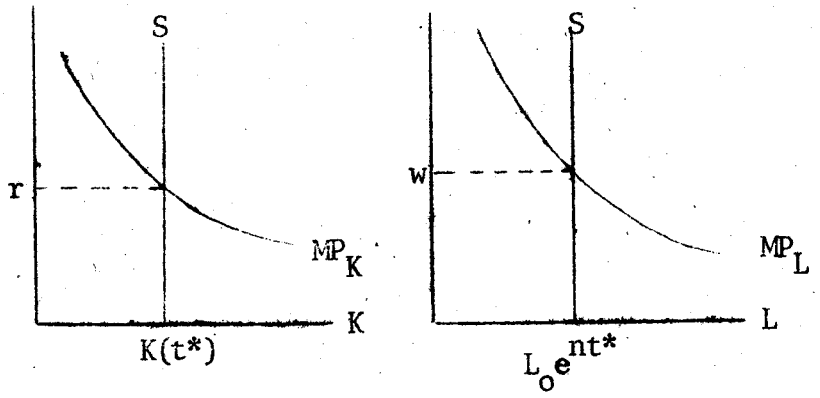
[28]Joan Robinson, "Equilibrium Growth Models," AER, V. 51, [June 1961], p. 361.

APPENDICES

The Timeless Equilibrium of Walras

Attainment of Walrasian equilibrium at any point in time is an implicit assumption of the conventional neoclassical parable. The parable equilibrium for time t* can be illustrated as follows:



There are two severe disadvantages of building a growth model around this concept. First, you cannot get a model of time by taking the union of a number of timeless points. Second, the assumption of perfectly inelastic factor supplies is at odds with any reasonable assumptions about individual behavior. Even leaving questions of leisure aside, the model ignores the very real possibility in the parable world of consuming ones capital.

Harrod's Warning

Harrod's model extends the Keynesian "paradox of thrift" to the long run. Harrod's long-run is not long enough for all of us to be dead or even for all of us to be employed. It is just long enough that short-run Keynesian fluctuations in output, caused by differences in ex ante saving and investment, are dominated by the tendency of capital accumulation to cause growth. Harrod follows Keynes in abandoning marginalism and Walrasian equilibria and in incorporating a high degree of aggregation, using single functions to represent total output and consumption. He adds the assumption that investment tends to increase the capital stock at the same rate as output. The assumption that savings equals investment closes the system.

The economy is run by the exogenous average propensity to save, s, and the capital-output ratio at time zero, $c \equiv \frac{K_o}{Q_o}$. Since investment is assumed to maintain the capital-output ratio, $\frac{\dot{Q}}{Q} = \frac{\dot{K}}{K}$. This is called the "warranted growth rate", g_w. Saving, sQ, becomes investment or $\dot{K}$ so the warranted growth rate is

$$g_w = \frac{\dot{K}}{K} = \frac{sQ}{K} = \frac{sQ}{cQ} = \frac{s}{c} \quad .$$

So long as the growth rate of output proceeds at g_w, ex ante saving will be just matched by ex ante investment. Once we are off that magic path we are in the roller-coaster world of the accelerator--almost anything can happen. And even if we stay on the g_w track, we will eventually bump into the bounds of full-employment (provided $\frac{s}{c} > \frac{\dot{L}}{L}$) and this will knock us off. Thus while rapid population growth was the scourge of affluence of Malthus and Ricardo, the opposite becomes the problem for Harrod.

Profit--Surplus or Reward?

These are two issues underlying this question.

1. Whether profit is deserved
2. Whether profit is necessary for accumulation

The two questions are confused both by the neoclassical apologists and by their opponents. Joan Robinson writes of the neoclassical system

> ...interest and profits are the necessary supply price for capital, without which it would not be forthcoming. Wages, interest and profit are grouped together as the reward of human efforts and sacrifices ...and a moral justification is provided for interest and profit.[29]

When a landlord resells a sack of rice to his starving tenant for twice (or half!) the price that the tenant sold it to him, we cannot say that the payment to the landlord is morally justified, regardless of the landlord's reservation price. Likewise if profit is necessary for accumulation we cannot conclude that profit is deserved. That the participants of the surplus-reward debate do not understand this distinction may explain the highly emotional tone of the quarrel.

Morals aside then, is profit a surplus or a payment? If capital is liquid, then interest is the payment necessary (not sufficient) to restrain individuals from consuming or reselling their capital. Even if capital is not liquid, interest is the necessary expected payment to induce investment. (If expected profits on new capital are less than the interest on money, lending is superior to investment.)

But net investment and profits in excess of the money interest rate have been the rule since the industrial revolution. It seems reasonable to call this difference surplus, thereby defining surplus as profit minus the necessary payment to induce investment.

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29 Joan Robinson, Essay on Marxian Economics, p. 61.

The reason that surplus persists over time [31] is that the sufficient expected-profit-rate (that which will induce investment or the "supply price of capital") contains a subjective element which "must obviously be influenced very much by the past experience of capitalists, so that the level of profits which they feel to be sufficiently attractive to justify enterprise is largely based on a conventional view of what it is reasonable to expect." [32]

Conclusions

There is no economic theory that morally justifies profit. Profit consists of a necessary payment (interest and a surplus). Neoclassical theory, by providing a theory of interest, does not provide a theory of profit.

[31] Attempt to prove that the differential is a payment for risk taking have been ad hoc. Robinson (Ibid. p. 71) suggests "reluctance to expose wealth to risk is essentially subjective, and there is no method to discover the laws of its operation, except by begging the question, and using the actual level of profits to measure the cost of risk-bearing.

[32] Ibid. p. 71.

Introducing Investment into the Neoclassical Model: Tobin's Two-Asset World

A Common critique of the neoclassical parable is that no account is taken of entrepreneurs' willingness to invest. Tobin takes account of the investment motive by adding an alternative asset, government debt, to the Solovian system described above. In the parable world of shmoos there is only one thing to save, and all shmoos set aside as savings are intended acquisitions of capital goods, i.e., ex ante investment. In Tobin's world, wealth-owners will not hold shmoos when government debt offers a higher rate of return. Thus the identity between ex ante saving and ex ante investment is broken.

In this economy there are two policy problems. One is to make sure intended investment is equal to the change in capital stock required to stay on a given growth path. The other is to attain the particular capital-labor ratio which will maximize per capita consumption.

To understand the first problem, let k^* be the unique solution of the balanced growth condition given the rate of saving, s.[32] Since $f'(k^*)$ is the rental rate of capital for this balanced growth path, the first problem is solved by not allowing the rate of return on government debt, r_b, to exceed $f'(k^*)$. Monetary policy in the Tobin world is presumed to be capable of this requirement.

Fiscal policy in this world includes tax cuts (issuing government debt) and taxing (recalling government debt). Changing the size of the government debt is supposed to affect real savings since government debt is a substitute for capital goods in satisfying the wealth motive. Thus fiscal policy can be used to generate the real savings and thereby the capital-labor ratio

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[32] From the derivation of balanced growth on page 2, this is expressed by the equation $\frac{sf(k^*)}{k^*} \equiv n$.

that maximizes per capita consumption, i.e., that satisfies $f'(k) = n$.

In summary, policy makers have two instruments, monetary and fiscal policy, to achieve golden age balanced growth. They are to use fiscal policy to assure $f'(k) = n$ and monetary policy to assure $r_b \leq n$. Other than that they are to allow (promote?) the competitive operation of markets. The Tobin model thereby purports to increase the reality of the simple neoclassical parable by relaxing the assumption of identical investment and saving. It simultaneously purports to provide the policy instrument for effecting the Golden Age.

The model has also been used as a resolution of the Harrod dilemma (see Appendix, "Harrod's Warning"). Once the economy hits full employment, which makes it impossible to stay on the warranted growth path, the government increases the national debt, decreasing real savings until the warranted growth rate, $\frac{s}{c}$, equals the natural growth rate, n.

Some aspects of Tobin's model remain unclear, however. What is the monetary policy designed to regulate r_b if it is not changing the size of government debt, in which case it is indistinguishable from fiscal policy. There seems to be only one policy instrument--the size of the government debt. If that is so then attainment of golden age growth may be impossible.

Furthermore, the specification of the investment motive is of questionable significance since there are no entrepreneurs in the system--only wealth-owners. Presumably, Tobin has a Tobin-Brainard[33] world in mind wherein wealth-owners, acting through their financial intermediaries, determine the rate of capital accumulation, but this needs to be spelled out in context of the simple two-asset model. It is also not clear what Tobin's debt has in common with real-world money.

Finally, the Tobin model doesn't solve the Harrod dilemma; it assumes the problem away. Harrod's accelerator is replaced by wealth-owner's marginal comparison of assets, and the Keynsian world of unemployment is replaced by a return to an unconnected sequence of Walrasian equilibria.

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[33] [22], pp. 383-400.

Technical Change in Neoclassical Models

"Solow's first paper on growth grew out of a critique of Harrod and Domar."[34] And once Solow and others had gained some considerable recognition and praise for the normative implications of their works, they were obliged to show that the neoclassical parable was a useful descriptive tool as well.

Here there arose an immediate difficulty, for according to the "stylized facts of capitalism,"[35] k, Q, and c had risen together but the capital-output ratio had remained constant--a prediction of Harrod-Domar, but in disobedience of neoclassical rules. In addition it became evident that relating growth of output to growth of k left a large unexplained residual.

Both problems are solved by invoking technical change as a deus ex machina. The capital-output ratio remained stable, according to the new rationale,[36] because the higher k, associated with a higher w, induced labor-saving technical change. This increased labor productivity just enough, low and behold, to offset the tendency of a rising k to increase the capital-output ratio.

As all econometricians know, the best way to raise R^2 when dealing with time-series data, is to include time among the regressors. Solow[37] did that and called time, "technical change," i.e., technical change is thereby defined as any change in per capita output which is correlated with time and not explained by

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[34] Henry Wan, Economic Growth, Harcourt, Brace & Jovanovich, Inc., New York, 1971, p. 10.

[35] These facts have been subject to dispute particularly regarding the measurability of capital stock (independently of the rate of profit) and the criterion for "constancy".

[36] See Fellner [3].

[37] "Technical Change and the Aggregate Production Function," Review of Economics and Statistics, 1957.

k. But as Arrow commented,[38] such "trend projections...are basically a confession of ignorance;" they do not explain technical change nor do they increase our understanding of the nature and causes of economic growth.

There are two kinds of technical change which are easily accomodated into the simple neoclassical story--disembodied technical change and labor augmenting technical change in the context of Cobb-Douglas production functions. The first increases the marginal products of both inputs by the same factor and therefore leaves relative factor shares constant.[39] In labor-augmented technical change, $L(t)$ is replaced by $L^*(t) = L(t)e^{\lambda t}$ where λ is the rate of labor augmentation. Balanced growth thus proceeds at the rate $n + \lambda$, and both marginal products increase at the rate $e^{\lambda t}$ so that factor shares remain unchanged.

In retrospect much of the neoclassical theory of technical change seems to be derived from a desire to preserve the normative results of the neoclassical parable rather than a desire to understand the nature and causes of technical change itself.

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[38]"The Economic Implications of Learning by Doing," RESTUD, June 1962, p. 155.

[39]For $Q = A(t)K^{\alpha}L^{1-\alpha}$, the relative factor share of labor to capital is

$$\frac{[A(t)F_L]L}{[A(t)F_K]K} = \frac{F_L L}{F_K K} = \frac{1-\alpha}{\alpha}$$

where $A(t)$ is the disembodied technical change and the terms in brackets are the new marginal products.

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