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MODELS OF INPUT CHOICE UNDER TECHNOLOGICAL UNCERTAINTY

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In this paper simple models are developed in order to examine firm behavior when a random variable appears in the firm's production function. Specifically, our objective is to study the effect on input choice of changes in the entrepreneur's attitude toward risk, of changes in the subjective probability distribution of the random variable, and of changes in product and input price.

First, the incorporation of random variables in the production function and the nature of the subjective probability beliefs are discussed. Then a model which employs the assumption of expected utility maximization is developed. A variation of the chance constrained programming model of production is also presented. Finally several theoretical and empirical implications are derived from the analysis.

1. Introduction

Two types of uncertainty which confront the producer have been distinguished in the recent literature. The first is uncertainty imposed by the inability to predict future product and input prices (price uncertainty).[1] The second is uncertainty due to the inability to predict output quantities given input levels (technological uncertainty).[2] In what follows we are concerned mainly with technological uncertainty, since this type of uncertainty has not been extensively analyzed in the literature and since it appears to be an important form of uncertainty in agriculture.

Traditional economic theory assumes that the production function is a single valued function which gives the maximum quantity of the output for any given combination of inputs /5/, /13/.

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[1] For analysis of product price uncertainty under perfect competition see, for instance, Tisdell /32/, Penner /25/, and McCall /19/. For the oligopolistic case see Hymans /14/ and Mills /22/; for the monopolist see Dhrymes /8/. Factor price uncertainty has not been extensively studies.

[2] Three works comprise virtually all of the literature concerned with technological uncertainty in the context of the theory of the firm, those of Tintner /31/, Walters /35/, and Moses /23/. In the mathematical programming literature, of course, technological uncertainty is also incorporated into programming models of risk.

In agricultural production, however, actual output depends on meteorological and other variables which cannot be controlled by the decision maker. This means that a given combination of inputs employed according to a given technology, will yield different quantities of output depending on the outcome of some random phenomena. In such a situation a single valued production function does not exist, but rather each combination of inputs leads to a probability distribution of outputs. Thus output is a random variable and can be expressed as a function of the input variables and certain random variables representing solar radiation, soil moisture, etc.,

$$(1) \qquad Q = f\,(X, \beta\,)$$

where X is a vector of inputs, and β is a vector of random variables, and f denotes the production function. We will see below that we need not be specific about the sources of production uncertainty, but we must specify the way in which random variables are entered in the production function.

Initially we assume that the random variables, β , are parameters of the production function which assume different values in different production periods, having a joint probability density function $g\,(\,\beta\,)$ distributed independently of X. This approach is suggested by experimental data on the response of grain yield to nitrogen.[3] Such an assumption has also been used in the

[3] Fuller /11/, fit separate production functions to the fertilizer trials data for each of eleven years. Seasonal fertilizer production parameters were also estimated at the International Rice Research Institute /15/.

mathematical programming literature concerned with incorporating risk into programming models.[4]

If we know the joint density function of β and the functional form of the production function, we can derive the conditional density function of output, given the input levels.[5] A knowledge of the conditional density function of output is unnecessary, however, for finding the conditional mean and conditional variance of output. For it can be shown that the expectation of a function of a random variable with respect to the probability law of this function is equal to the expectation of the function with respect to the probability law of the random variable.[6] Hence we can write the conditional mean and the variance of output as

$$(2) \qquad E(Q|X) = \int_0^{\infty} Qh(Q|X)dQ$$

$$= \int_0^{\infty} f(X,\beta)g(\beta)d\beta$$

$$(3) \qquad Var(Q|X) = \int_0^{\infty} (Q - E(Q|X))^2 h(Q|X)dQ$$

$$= \int_0^{\infty} (f(X,\beta) - E(f(X,\beta)))^2 g(\beta)d\beta$$

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[4]See, for instance, van de Panne and Popp /36/, Miller and Wagner /21/, and Sengupta /29/. This approach is more general than the approach taken in econometric work which commonly postulates an additive or multiplicative random term in the production function. We allow the random terms to be coefficients or exponents in the production relationship.

[5]See Parzen /24/, pp. 308-338.

[6]Again, see Parzen /24/, pp. 344-346.

where h is the conditional density function of output given the input levels and β cannot assume negative values. Thus, under our assumption of random production parameters, both the mean and the variance of output become functions of the input levels. The mean and variance of output are not independent of input levels, although the mean and variance of β are.

Similarly, the conditional mean and variance of profit given input levels and input and output prices can be expresse as

$$(4) \qquad E(\pi | X,p,c) = \int_0^\infty \pi(X,p,c,\beta)g(\beta)d\beta = \int_0^\infty pf(X,\beta)g(\beta)d\beta - c'X = pE(Q|X) - c'X$$

$$(5) \qquad Var(\pi | X,p,c) = \int_0^\infty (\pi(X,p,c,\beta) - E(\pi|X,p,c))^2 g(\beta)d\beta = \int_0^\infty p^2(f(X,\beta) - E(f(X,\beta)))^2 g(\beta)d\beta = p^2 Var(Q|X)$$

where p is the price of the product and c is a vector of input prices. Again, the mean and variance of profit depend on input levels although the mean and variance of β do not.

As we have seen, a characteristic of agricultural production functions is that output in a given production period depends on both the input levels in that time period and on the outcome of certain random phenomena. This means, of course, that profit, output, and marginal product are random variables, and that choice criteria used under certainty cannot be applied to select the optimum level of output. The decision maker faces the problem of choosing an action which maximizes his objective function under uncertainty.

We assume that the decision maker has a complete and consistent preference ordering over the set of possible actions. Von Neumann and Morgenstern /34/ have shown that if preferences over sets of probabilistic outcomes (prospects) are complete and consistent in the sense that they satisfy certain axioms, the preference ordering can be represented by an expected utility function. This implies that the rational decision maker will choose that prospect which maximizes expected utility. This expected utility equals the sum of the utilities of each of the outcomes contained in the prospect weighted by their probabilities.

The expected utility approach assumes that the expectations of decision makers concerning future outcomes can be expressed as a probability distribution. This appears to restrict the applicability of the expected utility approach to cases where the actual probabilities are known by the decision maker. However, Savage /28/ has shown that these probability weights may be interpreted either as objective probabilities derived from limiting values of relative frequencies <u>or</u> as subjective probabilities derived from personal beliefs concerning the probabilities of the outcomes. The personal beliefs are obtained from partial knowledge of the probabilities based on past experience or mere hunches. Savage demonstrates that if the preference ordering satisfies certain consistency conditions then it is possible to specify a utility function, <u>and</u> a probability distribution which satisfies the axioms of probability, so that an expected utility function describes the preference ordering.

Clearly probabilities assigned by the decision maker need not conform to the axioms of probability. Nevertheless, we assume that the probabilities assigned satisfy these axioms so that the theorems derived in the mathematical theory of probability can be utilized. This does not imply, however, that the probabilities must be interpreted as relative frequencies. They may alternately be regarded as subjective probabilities derived from preferences satisfying the Savage conditions.

Knight /17/ has introduced a distinction between risk and uncertainty. A situation is characterized by risk if the probabilities of the outcomes are known. If the probabilities are not known the situation is described as uncertain. Such a distinction disappears, however, when a subjectivist interpretation of probability is allowed.[7] If individuals make

[7] For critical discussions of Knight's view see Arrow /1/, and Shubik /30/.

decisions in the face of uncertainty by applying subjective beliefs about the probabilities of the outcomes, their behavior under uncertainty will not be qualitatively different from that under risk. Given his preference function and the cost of acquiring additional information about the probability distribution, the decision maker can choose the optimal amount of uncertainty in a manner consonant with modern statistical decision theory.[8] Thus decision theory, by utilizing a preference function, enables the decision maker to transform Knightian uncertainty into a situation of risk. In what follows we conform to the convention which uses the term uncertainty to refer to any situation where the outcome cannot be predicted with certainty.

We will not be concerned with expectation formation in what follows. I assume that the expectations concerning possible outcomes are given, and search and learning behavior is not considered. Obviously, economic considerations are involved in the collection of information and, as suggested above, decision theory provides a formal framework for analyzing this problem.

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[8] See Fishburn /10/, or other decision theory text.

2. Expected Utility Approach

It has been shown by Penner /25/, McCall /19/, and Fellner /9/, that when the source of risk is due to price uncertainty the risk averter, under perfect competition, will always produce less on average than will the expected profit maximizer. We will show that under technical uncertainty risk aversion does not imply a lower level of output. Additional assumptions concerning the nature of the dispersion of output are necessary to guarantee the negative effect of uncertainty on output.

We assume the farmer chooses input levels in order to maximize expected utility of profit in the next period. It is assumed that he has single valued expectations concerning product price (independent of the production decisions), and that the price of the input is known at the time the input decisions are made. The firm has negligible influence on price and can sell any quantity at the market price. The production function is specified as in equation (1) with a single input and a single random parameter, but the analysis can easily be extended to many variables by interpreting X as a vector of inputs and β as a vector of random parameters.

In this case expected utility may be written

$$(6) \qquad EU = \int_0^\infty U(pf(X,\beta) - cX)g(\beta)d\beta$$

$$= EU(pf(X) - cX).$$

We assume that U is monotone increasing, twice differentiable, and strictly concave, $U'' < 0$. That is, there is aversion to risk everywhere.

The first order condition for expected utility maximization is

$$\frac{dEU}{dX} = E\left[U'.(pf'(X) - c)\right] = 0 . \tag{7}$$

The second order condition is

$$\frac{d^2EU}{dX^2} = E\left[U'.pf''(X) + U''.(pf'(X) - c)^2\right] < 0, \tag{8}$$

clearly satisfied if U and f are strictly concave.

The first order condition may alternatively be written as[9]

$$EU'.E(pf'(X) - c) + Cov(U',pf'(X) - c) = 0. \tag{9}$$

Dividing by EU'

$$pE(f'(X)) - c + \frac{Cov(U', pf'(X) - c)}{EU'} = 0. \tag{10}$$

At optimum, expected marginal value product plus a term due to risk is equated to factor price. This term may be positive, negative or zero even if there is aversion to risk. If the covariance between marginal utility and marginal profit is zero, then the term due to risk is zero and the risk averter simply equates expected marginal value product to factor price. This reproduces the case of an expected profit maximizer. Thus, the risk averter does not necessarily apply a lower level of the

[9] Given two random variables X and Y , $E(XY) = E(X)E(Y) + Cov(X,Y)$. See Parzen /24/, page 356.

variable input than does the expected profit maximizer. Only if the covariance between marginal utility and marginal profit is negative does risk aversion imply a lower level of input and thus a lower level of output on average than in the case of neutral risk preference.

The covariance between U' and π' is a measure of the change in the dispersion of profit as the input level changes. To see this, write the variance of profit as

(11) $$Var(\pi) = E(\pi^2) + E^2(\pi).$$

Then

(12) $$\frac{dVar(\pi)}{dX} = 2E(\pi . \pi') - 2E(\pi)E(\pi')$$

$$= 2Cov(\pi,\pi').$$

Hence, for $U''<0$ negative $Cov(U',\pi')$ implies that $Cov(\pi,\pi')$ is positive, which in turn implies that $dVar(\pi)/dX$ is positive. Thus, the term due to risk is negative if there is risk aversion and if an increase in the level of the input causes an increase in the dispersion of profit.

If marginal utility is high (relative to its mean) when marginal profit is high (relative to its mean), then the covariance will be positive. Conversely, if low values of marginal utility are associated with high values of marginal profit, then the covariance will be negative. This relationship can be illustrated graphically. Figure 1 shows profit curves for two values of β. In this case the covariance is negative because U' decreases when π' increases (the utility function is concave, remember). In Figure 2 the covariance is positive since U' assumes a large value when π' assumes a large value. It is obvious that in Figure 1 the dispersion of profit about the mean increases as the input level increases, while in Figure 2 the dispersion diminishes.

The first order conditions can also be interpreted in terms of a risk premium. Pratt /26/ defines the risk premium, $\tilde{\pi}(z)$, as an amount which would make the decision maker indifferent between receiving the random amount z and the certain amount $E(z) - \tilde{\pi}(z)$. For the firm facing uncertainty, the random prospect is profit, so we have

(13) $$EU(\pi) = U(E\pi - \tilde{\pi}(\pi)).$$

Differentiating with respect to X

(14) $$\frac{dEU}{dX} = U'(E\pi - \tilde{\pi}(\pi)).(E\pi' - \frac{d\tilde{\pi}(\pi)}{dX}) = 0.$$

Dividing by $U'(E\pi - \tilde{\pi}(\pi))$ which is positive

(15) $$\frac{dEU}{dX} = E\pi' - \frac{d\tilde{\pi}(\pi)}{dX} = 0 .$$

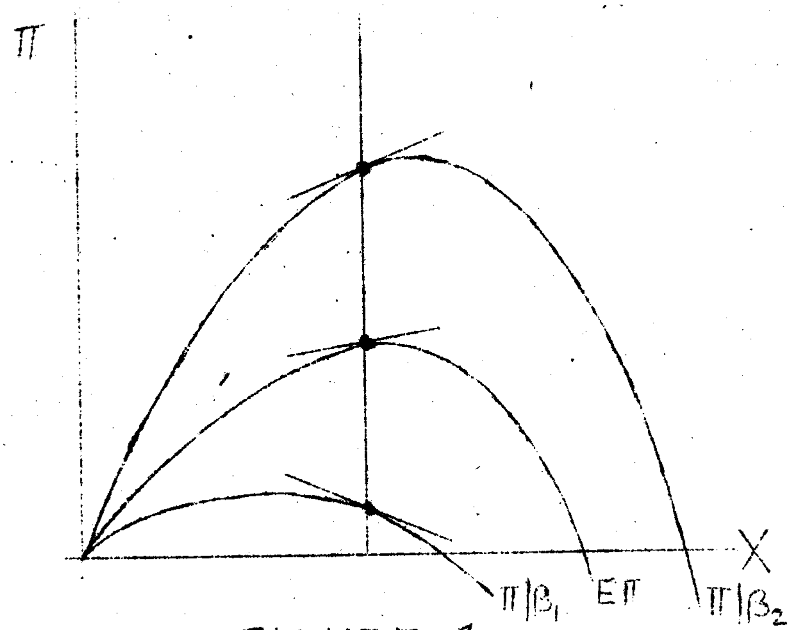

FIGURE 1.

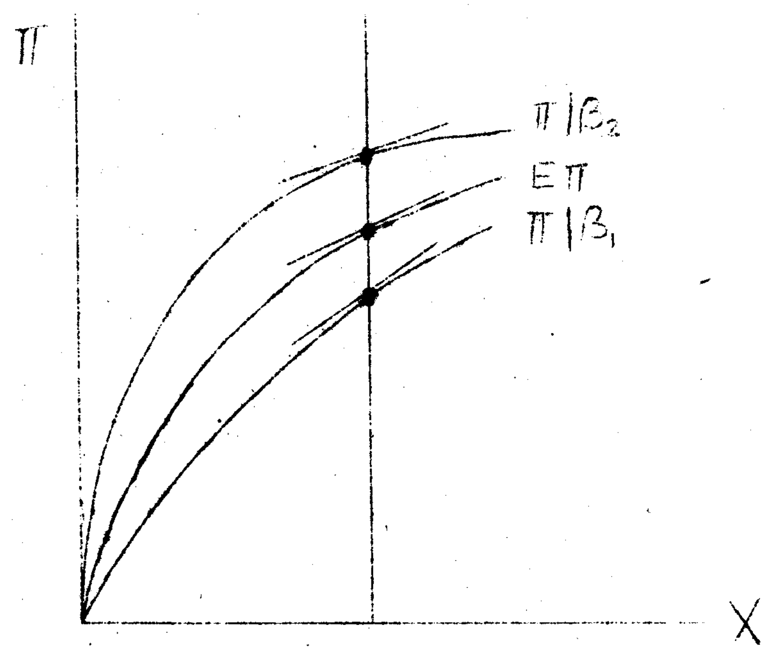

FIGURE 2.

Comparing with the first order condition we get

$$(16) \qquad \frac{d\bar{\pi}(\pi)}{dX} = \frac{Cov(U',\pi')}{EU'} .$$

If $Cov(U',\pi')$ is negative, an increase in the use of the factor causes an increase in the risk premium. Thus $Cov(U',\pi')/EU'$ can be interpreted as an additional marginal cost due to the attitude toward risk of the firm. At optimum an increase in the use of the factor can cause expected profit to increase, but expected utility falls because of the increase in the risk premium demanded by the firm.

We are interested in determining the effect on choice of input level of (a), a change in aversion to risk, and (b) a change in the uncertainty faced by a firm.

Using the Pratt/Arrow /26//2/ measure of absolute risk aversion, $r(\pi) = U''(\pi)/U'(\pi)$, it is possible to show that the optimal input level decreases as the risk aversion index decreases, provided of course, that $Cov(U',\pi')$ is negative.[10]

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[10] This demonstration essentially follows Baron /3; p. 91/, who proved the statement for the case of demand uncertainty.

Consider two utility functions U_1 and U_2, with $r_1(\pi) > r_2(\pi)$ for all π indicating that U_1 is everywhere more risk adverse than U_2. The maximization of expected utility is equivalent to maximizing

(17) $$\frac{1}{U'(\pi_o)} EU(\pi)$$

where π_o is a constant defined below. Differentiating EU_1 and EU_2 and subtracting gives

(18) $$\frac{dEU_1}{dX} - \frac{dEU_2}{dX} = \int_0^{\beta_o} \pi' \left(\frac{U_1'(\pi)}{U_1'(\pi_o)} - \frac{U_2'(\pi)}{U_2'(\pi_o)} \right) g(\beta) d\beta + \int_{\beta_o}^{\infty} \pi' \left(\frac{U_1'(\pi)}{U_1'(\pi_o)} - \frac{U_2'(\pi)}{U_2'(\pi_o)} \right) g(\beta) d\beta$$

where β_o is defined such that $\pi_o' = \pi'(X_o, \beta_o) = 0$. Now for U concave and $\pi > \pi_o$ Pratt /26/ shows that $\frac{U_1'(\pi)}{U_1'(\pi_o)} < \frac{U_2'(\pi)}{U_2'(\pi_o)}$; for $\pi < \pi_o$, $\frac{U_1'(\pi)}{U_1'(\pi_o)} > \frac{U_2'(\pi)}{U_2'(\pi_o)}$. Thus the term in brackets in the first integral is positive while in the second integral the bracketed term is negative. Let X_0 be the optimal input for U_2, $dEU_2/dX_0 = 0$, and evaluate the integral at X_0. Assuming positive $Cov(U', \pi')$ (which means that π' increases as β increases), the first integral is negative since $\pi' < 0$ for $\beta < \beta_o$ and the term in brackets is positive. Using the same assumption about $Cov(U', \pi')$, the second integral is negative because $\pi' > 0$ for $\beta > \beta_o$ and the bracketed term is negative. But, since $dEU_2/dX_0 = 0$, $dEU_1/dX_0 < 0$. By the second order condition a decrease in the input level is required for the optimum conditions to be satisfied. Thus a global increase in risk aversion implies an increase in the absolute value of Cov (U', π')/EU' and by additional implication an increase in the risk premium.

We now consider the effect on input level of a change in the parameters of the distribution of the random variable. Specifically, we consider the effect of a change in the variance, while the mean of the distribution remains unchanged. In general, the sign of the effect is difficult to determine.

Using Taylor's formula expand expected utility about the mean value of profit.

$$(19)\qquad EU(\pi) = U(E\pi) + \frac{1}{2}\int_0^{\infty} U''(E\pi)\cdot(\pi - E\pi)^2 g(\beta)d\beta + \text{higher order terms}$$

Differentiating with respect to X and integrating gives the first order condition

$$(20)\qquad \frac{dEU(\pi)}{dX} = U'(E\pi)\cdot E\pi' + \frac{1}{2}U'''(E\pi).Var(\pi) + \frac{1}{2}U''(E\pi).\frac{dVar(\pi)}{dX} + \text{higher order terms} = 0.$$

Assume that the higher order moments of the distribution are of smaller order than Var(π). By allowing a small increase in Var (π) we can examine the change in the first order condition in order to determine the direction of change in X from X_o for the necessary condition to be satisfied

$$(21)\qquad \Delta\frac{dEU(\pi)}{dX_o} = \frac{1}{2}U'''(E\pi)\,\Delta Var(\pi) + \frac{1}{2}U''(E\pi)\,\Delta\frac{dVar(\pi)}{dX}$$

since U', U'', and U''' are constants. $E\pi'$ is also a constant since an unchanged $E\pi$ implies an unchanged $E\pi'$ evaluated at X_0. The sign of $U'''(E\pi)$ is unknown without further behavioral assumptions. If we assume with Pratt /26/ and Arrow /2/ that the measure of risk aversion $r(\pi) = -U''(\pi)/U'(\pi)$ is a decreasing

[illegible]

(22) [illegible]

[illegible]

(24) $$d(\partial EU/\partial X) = \frac{\partial^2 EU}{\partial X^2} dX + \frac{\partial^2 EU}{\partial X \partial c} dc = 0.$$

Thus

(25) $$\frac{dX}{dc} = - \frac{\partial^2 EU}{\partial X \partial c} \bigg/ \frac{\partial^2 EU}{\partial X^2} .$$

$\partial^2 EU/\partial X^2$ is negative by the second order condition, so dX/dc will have the same sign as $\partial^2 EU/\partial X \partial c$. $\partial^2 EU/\partial X \partial c$ is

(26) $$\frac{\partial^2 EU}{\partial X \partial c} = E\left(U'' \frac{\partial \pi}{\partial c} \cdot \frac{\partial \pi}{\partial X} + U' \frac{\partial^2 \pi}{\partial X \partial c}\right)$$

$$= -E(U''.X\frac{\partial \pi}{\partial X} + U')$$

$$= -EU' - XE(U')\cdot E\left(\frac{\partial \pi}{\partial X}\right) - XCov\left(U'', \frac{\partial \pi}{\partial X}\right) .$$

The first term is negative, the second is positive, the third term is indeterminate in sign. If, as above, we assume that U''' is positive, then $Cov(U'', \partial\pi/\partial X)$ is positive. In spite of this assumption the sign of the whole expression cannot be determined in general since the second term may be larger in magnitude than the sum of the first and last terms.

This ambiguous result is explained when we recognize that an increase in c causes the optimal level of use of X to decrease, but that a reduction in level of X diminishes the dispersion of profit. This, in turn, may increase the level of use of X. This is clear if we write by analogy with the Slutsky equation of consumer behavior theory

(27) $$\frac{dX}{dc} = \left(\frac{\partial X}{\partial c}\right)_{\substack{risk= \\ const}} + \left(\frac{\partial X}{\partial (Cov/EU')}\right)_{\substack{c \; = \\ const}} \cdot \left(\frac{\partial (Cov/EU')}{\partial c}\right) .$$

The total effect on input level of a change in factor price is composed of the sum of two effects which may be interpreted as a "substitution effect" and a "risk effect". The first term in equation (27) represents the "substitution effect" and indicates the change in input level which would occur if the decision maker were an expected profit maximizer or if risk did not vary as input level varied. The second term represents the "risk effect" and is the rate of change in input level brought about by the change in the risk term as c varies. The "substitution effect" is always negative. The "risk effect" may be positive, zero, or negative since we found that an increase in the variance of profit may cause input level to increase, remain the same, or to decrease. Note, however, that the "substitution effect" in the case of uncertainty does not refer to the rate of change in a decision variable as factor price changes where utility remains constant, but rather refers to the same rate of change while risk remains unaltered.

The result and the interpretation for a change in p are similar to those for a change in c. Again, differentiate the necessary condition totally using the implicit function rule

(28) $$\frac{dX}{dp} = \frac{\partial^2 EU}{\partial X \partial p} \bigg/ \frac{\partial^2 EU}{\partial X^2} .$$

The denominator in equation (28) is negative by the second order condition, so the sign of dX/dp is the same as the sign of the numerator. The numerator is

(29)
$$\frac{\partial^2 EU}{\partial X \partial p} = E\left(U' \cdot \frac{\partial^2 \pi}{\partial X \partial p} + U'' \cdot \frac{\partial \pi}{\partial p}\ \frac{\partial \pi}{\partial X}\right)$$
$$= E\left(U' \cdot \frac{\partial^2 \pi}{\partial X \partial p}\right) + E\left(U'' \cdot \frac{\partial \pi}{\partial p}\ \frac{\partial \pi}{\partial X}\right)$$
$$= EU' E\left(\frac{\partial^2 \pi}{\partial X \partial p}\right) + \mathrm{Cov}\left(U', \frac{\partial^2 \pi}{\partial X \partial p}\right)$$
$$+ EU''. E\left(\frac{\partial \pi}{\partial p}\ \frac{\partial \pi}{\partial X}\right) + \mathrm{Cov}\left(U'', \frac{\partial \pi}{\partial p}\ \frac{\partial \pi}{\partial X}\right).$$

The sign of this expression cannot be determined without further specification. As in the previous case a change in p leads to a change in X. But a change in X causes a change in the dispersion of profit which leads to a change in X of indeterminate sign.

3. Quadratic Utility Function

For illustration of the above results assume the utility function of farmers is quadratic in the value of profit and that the farmer attempts to maximize expected utility.[11]

(30)
$$U(\pi) = \pi - a\pi^2 \quad a>0.$$

In order to assure a positive marginal utility of profit we assume that profit satisfies the condition $\pi < \frac{1}{2a}$. The stipulation that

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[11]A quadratic utility function is often postulated in the theoretical literature concerning behavior under uncertainty.

a be positive implies that the decision maker displays an aversion to risk. The use of a quadratic utility function may be rationalized as an incomplete Taylor's series expansion of a more general utility function. Unfortunately a quadratic function implies that risk aversion, in the Pratt-Arrow sense, increases as profit increases.

We suppose that the price and production specification are as given in the previous section.

The expected value of the utility function can be written in terms of the moments of probability distribution of profit.

(31)
$$EU = \int_{0}^{\infty} (\pi(X,\beta) - a\,(\pi(X,\beta))^2)g(\beta)d\beta$$
$$= E(\pi) - aE(\pi^2)$$
$$= E(\pi) - a(Var(\pi) + E^2(\pi)).$$

The first order condition for a maximum is

(32)
$$\frac{dEU}{dX} = \frac{dE\pi}{dX} - a\left(\frac{dVar(\pi)}{dX} + 2E\pi\frac{dE\pi}{dX}\right) = 0$$
$$= (1-2aE\pi)\frac{dE\pi}{dX} - a\frac{dVar(\pi)}{dX} = 0$$
$$= pEf'(X) - c - \frac{ap^2}{1-2aE\pi} \cdot \frac{dVar(f(X))}{dX} = 0.$$

Such a function has been employed in the analysis of portfolio selection by Tobin /33/ and Markowitz /20/, in the study of a multiproduct monopolist by Dhrymes /8/, and in consideration of the precautionary demand for saving by Leland /8/.

Or, in order to maintain symmetry with the previous derivation of the first order condition, we may write

(33)
$$\frac{dEU}{dX} = pEf'(X) - c + \frac{Cov(U',\pi')}{EU'} = 0$$
$$= pEf'(X) - c - \frac{2aCov(\pi,\pi')}{1-2aE\pi} = 0$$
$$= pEf'(X) - c - \frac{2ap^2Cov(f(X),f'(X))}{1-2aE\pi} = 0.$$

$1-2aE\pi$ is positive by our assumption concerning the range of π, so the term due to risk is negative on the condition that $dVar(f(X))/dX$ is positive.

The second order condition is

(34)
$$\frac{d^2EU}{dX^2} = (1-2aE\pi)\frac{d^2E\pi}{dX^2} - 2a\left(\frac{dE\pi}{dX}\right)^2 - a\,\frac{d(dVar(\pi)/dX)}{dX} < 0.$$

Compliance with this condition is assured by our assumption concerning the range of π, the sign of a, the concavity of $f(X)$ and the method of entering β in $f(X,\beta)$.

Differentiating equation (32) gives us the comparative static results. An increase in risk aversion implies an increase in the value of a. Using the implicit function rule

(35)
$$\frac{dX}{da} = -\frac{\partial^2 EU}{\partial X\partial a}\Big/\frac{\partial^2 EU}{\partial X^2}.$$

But the second order condition implies that $\partial^2 EU/\partial X^2$ is negative, so we need only determine the sign of the numerator.

(36)
$$\frac{\partial^2 EU}{\partial X\partial a} = -\,2E\pi(pEf'(X) - c) - p^2\,\frac{dVar(f(X))}{dX} < 0.$$

Hence, an increase in risk aversion decreases the optimal level of use of the production factor.

We also want to know the response of equilibrium input to an increase in technological uncertainty, measured by the variance of output. The sign of this response is identical to the sign of $\partial^2 EU/\partial X \partial Var(f(X))$.

$$(37) \qquad \frac{\partial^2 EU}{\partial X \partial Var(f(X))} = - ap^2 \frac{d(dVar(f(X))/dX)}{dVar(f(X))} < 0.$$

if the derivative of the "marginal variance" is positive. This will be the case if the random variable is a production function parameter. Thus, an increase in uncertainty produces a decrease in the equilibrium level of input.

We can also attempt to determine the response of optimum output to change in c and p. As we expect from the results in the general case, the signs of the responses cannot be unambiguously determined. To determine the sign of dX/dc we need to determine the sign of

$$(38) \qquad \frac{\partial^2 EU}{\partial X \partial c} = (2aE\pi - 1) + 2aX(pEf'(X)-c) \gtrless 0.$$

Without further specification the response is not determinate.

Similarly, the sign of dX/dp is the same as the sign of

$$(39) \qquad \frac{\partial^2 EU}{\partial X \partial p} = (1 - 2aE\pi)(E(f'(X)) - (2aE(f(X))(pE(f'(X)) - c) - 2ap \frac{dVar(f(X))}{dX} \gtrless 0.$$

Again, the result is ambiguous in the absence of further specification.

4. Price and Technological Uncertainty

In this section we allow both product price and output to be random variables. The input price is assumed to be known with certainty. The farm firm is assumed to act so as to maximize expected utility.

The first order condition is

$$\frac{dEU}{dX} = E(U' \cdot (pf'(X) - c)) = 0$$

$$= EU' \cdot (E(pf'(X)) - c) + Cov(U', pf'(X)) = 0$$

$$= EU' \cdot (E(p) \cdot Ef'(X) + Cov(p, f'(X)) - c) + Cov(U', pf'(X)) = 0.$$

Dividing by EU'

$$\frac{dEU}{dX} = E(p) \cdot Ef'(X) + Cov(p, f'(X)) - c + \frac{Cov(U', pf'(X))}{EU'} = 0.$$

$E(p) \cdot Ef'(X) + Cov(p, f'(X))$ is simply expected marginal value product. Thus we have again that expected marginal value product plus a term due to risk is equated to factor price. This term may be positive, negative, or zero. In this case a negative $Cov(U', f'(X))$ does not guarantee a negative risk term.

An interesting question is what additional assumptions are sufficient to guarantee the negativity of $Cov(U, pf'(X))$. A positive correlation between p and f'(X) will suffice, but this assumption is not very plausible since we expect output and product price to be negatively correlated. In general, the result is indeterminate.

5. Constant Risk Term

Some writers have utilized a constant risk deduction in order to introduce uncertainty (and risk aversion) into models of production under certainty /7/. Our analysis suggests that this common practice is inappropriate, except in certain special cases. The assumption of a constant risk term implies that $Cov(U',\pi')/EU'$ is constant for all values of X. That is, the derivative with respect to X vanishes. Differentiating gives

$$\frac{\partial(Cov(U',\pi')/EU')}{\partial X} = \frac{\partial Cov(U',\pi')/\partial X}{EU'} - \frac{Cov(U',\pi')\cdot\partial EU'/\partial X}{(EU')^2} = 0.$$

Rearranging we obtain

$$\frac{\partial Cov(U',\pi')/\partial X}{Cov(U',\pi')} = \frac{\partial EU'/\partial X}{EU'} .$$

Thus, if $\partial Cov/\partial X$ and $\partial EU'/\partial X$ both equal zero, or if the relative increases in the dispersion of profit and expected marginal utility are equal, then the risk term remains constant.

Note that the risk term is a marginal concept rather than an average or total concept. Also note that subtracting a constant risk allowance gives no information concerning the magnitude of the appropriate allowance. In our model the allowance clearly depends on the variance and expected value of profit and on the risk aversion contained in the utility function.

6. Baumol Criterion

An alternative to the measurement of risk by the "risk premium" demanded is to measure risk in terms of the chance of "disaster". Thus the risk contained in a particular random prospect might be represented by

$$P(z<z_0)$$

where P is probability and z_0 is a minimal acceptable level of the random variable.

Several economists have argued that outcomes below some disaster level are of particular concern to the firm, and they suggest that the chance of such outcomes should enter explicitly into the firm's decision criterion. Roy /27/ argues that the firm will seek to minimize the probability of income falling below the stipulated disaster level. Katoaka /16/ suggests that the decision maker choose a probability level and maximize the lower allowable limit of income subject to the constraint that the probability of income falling below the lower limit is less than the preassigned probability. Thus Roy advocates minimizing $P(z<z_0)$ for a given z_0. Katoaka, on the other hand, assumes $P(z<z_0)$ is given and recommends the maximization of z_0.

Both of the above criteria are concerned with the minimization of risk. A more pausible criterion may be the maximization of expected income subject to a chance constraint requiring income to be greater than some lower bound with a stipulated probability.

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The discovery of the optimal solution then becomes a chance constrained programming problem /6/.

In this section we will formulate a model of production which is a variation of the chance constrained programming model. We cannot use a chance constrained programming approach because the constraint

$$P(\pi<\pi_0) = H(\pi_0)$$

where $H(\pi)$ is the cumulative distribution function of profit, cannot be shown to be convex in X, unless we unncessarily restrict the form of $H(\pi)$. Nor can we show that the Chebyshev upper bound on the probability

$$P(\pi<\pi_0) \leq \frac{Var(\pi)}{(E(\pi) - \pi_0)^2}$$

is a convex function of X. However, Baumol /4/ has suggested a closely related criterion which requires the maximization of expected profit subject to a constraint on $E(\pi) - k\sigma(\pi)$, where $\sigma(\pi)$ denotes the standard deviation of profit. This amounts to a constraint on the square root of the probability (rather than the probability) that income falls below some minimum level. If $d^2\sigma(\pi)/dX^2$ is non-negative, then $E(\pi) - k\sigma(\pi)$ is a convex function of X.

Adopting the Baumol criterion the problem can be written

(40) $$\max \quad E(\pi)$$

subject to

(41) $E(\pi) - k\sigma(\pi) \geq \pi_0$

$X \geq 0.$

The problem is to maximize a concave function subject to concave inequality constraints. Form the Lagrangian function

(42) $L(X,\lambda) = E\pi - \lambda(\pi_0 - E(\pi) + k\sigma(\pi)).$

Necessary and sufficient conditions for a constrained maximum are[12]

(43) $$\frac{\partial L}{\partial X} = \frac{\partial E\pi}{\partial X} - \lambda\left(-\frac{\partial E\pi}{\partial X} + k\frac{\partial\sigma(\pi)}{\partial X}\right) \leq 0$$

(44) $$\frac{\partial L}{\partial \lambda} = \pi_0 - E(\pi) + k\sigma(\pi) \leq 0$$

(45) $$\frac{\partial L}{\partial X}\cdot X = 0$$

(46) $$\frac{\partial L}{\partial \lambda}\cdot \lambda = 0$$

(47) $\lambda \geq 0.$

The constraint (44) holds as an equality since we assume it is binding if it is imposed. It follows that λ is positive. The "marginal risk" term $(- \partial E\pi/\partial X + k\partial\sigma(\pi)/\partial X)$ must also be positive for the firm to be on its risk-expected value production possibility curve. That is, since an increase in X increases the expected value of profit, at optimum an increase in X must also lead to an increase in risk. If this is not the case, the expected profit-risk combination will be dominated by one with a higher expected profit and a lower risk.

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[12] See Hadley /12/, Chapter 6.

Condition (43) means that if expected marginal profit is less than marginal risk times the imputed cost of risk then zero production is optimal. If positive production is optimal, the firm will equate expected marginal value product minus marginal risk times its imputed cost to factor price. This gives a result similar to that of the expected utility approach.

The comparative static results are obtained by totally differentiating the constraint (44). An increase in risk aversion is indicated by either an increase in π_0 or an increase in k. The latter follows because an increase in k implies a decrease in the probability of profit falling below π_0. Differentiating totally with respect to π_0, we get

$$(48) \qquad \left(- \frac{\partial E\pi}{\partial X} + k \frac{\partial \sigma(\pi)}{\partial X}\right) \frac{dX}{d\pi_0} = -1.$$

Since the expression in brackets is positive, $dX/d\pi_0$ must be negative.

Differentiating totally with respect to k, we find

$$(49) \qquad \left(- \frac{\partial E\pi}{\partial X} + k \frac{\partial \sigma(\pi)}{\partial X}\right) \frac{dX}{dk} = -\sigma$$

so dX/dk is negative. Hence, an increase in risk aversion leads to a decrease in input level and thus a lower level of output on average.

The effect of an increase in uncertainty on optimal input is given by differentiating the constraint with respect to σ . We have

(50) $$\left(-\frac{\partial E\pi}{\partial X} + k\,\frac{\partial\sigma(\pi)}{\partial X}\right)\frac{dX}{d\sigma} = -\,k,$$

which implies that $dX/d\sigma$ is negative. An increase in the uncertainty faced by the firm, the mean profit level remaining unchanged, implies a decrease in optimal input and in expected output.

We are also interested in finding the change in input level as product and input prices change. Again, total differentiation gives

(51) $$\left(-\frac{\partial E\pi}{\partial X} + k\,\frac{\partial\sigma(\pi)}{\partial X}\right)\frac{dX}{dp} = Ef(X)$$

and

(52) $$\left(-\frac{\partial E\pi}{\partial X} + k\,\frac{\partial\sigma(\pi)}{\partial X}\right)\frac{dX}{dc} = -X.$$

This yields $dX/dp>0$ and $dX/dc<0$. An increase in product price increases the level of use of the input, while an increase in the factor price decreases the optimal input level. Thus, in this case we get determinate results under uncertainty which conform to the results derived in the case of certainty.

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