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INTERTEMPORAL OPTIMIZATION IN AN AGRARIAN MODEL WITH Z-ACTIVITIES

by

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I. Introduction

This paper examines the optimal characteristics of an agrarian growth model in which two production activities are distinguished, viz., one producing the usual agricultural (consumption) goods F, and the other consisting of labor using Z-activities directed toward substitute consumption for industrial goods and augmentation of agricultural capital. To quote from Hymer and Resnick, who originally developed and cogently argued for the relevance of such production schema in developing agrarian economies, Z-activities represent "a variety of processing, manufacturing, construction, transportation and service activities to satisfy the needs for food, clothing, shelter, entertainment and ceremony" [4, p.493] as well as investment activities like "metal working, . . . , manufacture and repair of tools and implements, . . . , fence repairing, . . . , transport and distribution" [4, p.493n]. For analytical convenience, the output of these nonagricultural activities has been collectively termed Z-goods.

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An earlier paper [2] has presented a descriptive growth model of the small, open economy in which the average saving propensity is assumed to remain constant over time; the dynamic implications of Z-goods production are examined, complementing the static, partial equilibrium analysis by Hymer and Resnick. Here, we drop the assumption of rigid saving pattern and investigate the problem of intertemporal utility maximization in the agrarian economy having the same production structure.

Most optimal growth models explicitly or implicitly assume that the planning authority controls directly the allocation and/or valuation of resources in the economy. This would not seem a reasonable assumption for nonsocialist countries. In the present model it is in fact possible to choose from a number of possible instrument variables. Since there is only one objective (target) variable -- maximum discounted sum of per capita utilities over time -- only one policy variable need be specified. As will be indicated below, one good candidate is the imposition of import quota on industrial consumption goods.

II. Statement of the Problem

We seek to maximize the utility functional

$$(1) \qquad \int_0^T u(q_F(t), q_Z(t), q_C(t))e^{-\delta t}dt \quad 0 < T \leq \infty$$

subject to the following conditions:

(2) $$\tilde{s}(t) \equiv \rho(t) + (\mu+\lambda) \ \rho(t) = x_Z(t) - q_Z(t),$$

(3) $$q_F(t) = x_F(t) - p_C q_C(t),$$

(4) $$x_Z(t) = \alpha l_Z(t), \qquad x_F(t) = l_F(t) g_F(\tilde{\rho}(t)),$$

(5) $$l_F(t) + l_Z(t) = 1, \quad \rho(t) = \tilde{\rho}(t) l_F(t),$$

(6) $$\rho(0) = \rho_0, \qquad \rho(T) > \rho_T \text{ for } T < \infty,$$

where $u(\ )$ is per capita utility at time t, δ the positive discount rate, $\tilde{s}$ is gross investment per capita and, following the notations in [2]:

q_F, q_Z, q_C = per capita consumption of F-goods; of Z-goods; of C-goods

x_F, x_Z = per capita output of F-goods; of Z-goods

l_F, l_Z = ratio to the total labor force of employment in F-production; in Z-production

$\tilde{\rho}$, ρ = ratio of the stock of agricultural capital to labor employment in F-production; to the total labor force

p_C = world market (domestic) price of industrial consumption goods in terms of F-goods

μ = constant rate of depreciation of capital

λ = constant growth rate of population (labor force).

All variables in (2) to (5) are nonnegative.

The definition of the gross rate of capital accumulation is given in equation (2). Since consumption of industrial consumption goods is itself the amount imported and trade balance is assumed, consumption of F-goods is equated in (3) to the difference between F-output and the value of C-consumption (in terms of F). Equations (4) define the production relations in F and Z where, as may be recalled, $\alpha > 0$, $g_F' > 0$, and $g_F'' < 0$. The first relation in (5) states that labor is fully employed; the second is definitional. Finally, (6) specifies the given initial endowment and the range of terminal values of the capital-labor ratio.

Eliminating x_F and x_Z , the following Hamiltonian expression may be formed from the above:

$$H = \{u(q_F,q_Z,q_C) - v_1[\dot{\rho} - \alpha l_Z - q_Z + (\mu+\lambda)\rho] - v_2[q_F - l_F g_F(\tilde{\rho}) + p_C q_C] - v_3(l_F + l_Z - 1) - v_4(\tilde{\rho} l_F - \rho)\} e^{-\delta t} , \quad (7)$$

where the v_i's (i=1,2,3,4) are the auxiliary variables corresponding to the constraints (2), (3) and (5). For brevity, the time reference on each variable in (7) has been omitted.

Certain restrictive assumptions on the form of the utility function have to be introduced to make further analysis possible: (1) It is twice continuously differentiable, has positive marginal utilities (u_F, u_Z, u_C) and strictly concave; (2) the marginal utility in the consumption of each commodity approaches infinity as the quantity consumed decreases to zero[1] and approaches zero as consumption increases to infinity; and (3) Z- and C-goods are substitutable to each other in consumption in the Edgeworth-Pareto sense, but are each independent with respect to F-goods. These conditions may be written formally as follows:

$$(8)\quad \begin{aligned} &u_F = u_F(q_F) > 0, \quad u_{FF} < 0, \quad u_{FZ}, u_{FC} = 0, \\ &u_Z = u_Z(q_Z, q_C) > 0, \; u_{ZZ} < 0, \; u_{ZC} < 0, \quad u_{ZF} = 0, \\ &u_C = u_C(q_Z, q_C) > 0, \; u_{CC} < 0, \; u_{CZ}(=u_{ZC}) < 0, \; u_{CF} = 0, \\ &u_{ZZ}u_{CC} - u_{ZC}^2 > 0, \; \lim_{q_i \to 0} u_i = \infty \text{ and } \lim_{q_i \to \infty} u_i = 0 \; (i=F,Z,C), \end{aligned}$$

where u_{ij} is the derivative of the marginal utility of the ith commodity with respect to the consumption of the jth commodity. Notice that assumption (3), represented by the last two relations in (8), rules out specialization of the agrarian economy in the production of either F- or Z-goods, as well as specialization in the consumption of any of the three commodities, for optimality; the only boundary condition that is relevant, therefore, is the nonnegativity of gross investment.

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[1]This represents a continuous formulation of the existence of minimum levels of consumption of the three types of commodities.

II. Derivation of the Optimal Solution Path

Ignoring for the moment the condition that gross investment has to be nonnegative, the first order conditions necessary and sufficient[2] for optimality are given by the following Euler equations:

(9) $\frac{\partial H}{\partial q_F} = 0: \quad v_2 = u_F(q_F);$ (10) $\frac{\partial H}{\partial q_Z} = 0: \quad v_1 = u_Z(q_Z, q_C);$

(11) $\frac{\partial H}{\partial q_C} = 0: \quad p_C v_2 = u_C(q_Z, q_C);$ (12) $\frac{\partial H}{\partial \tilde{\rho}} = 0: \quad v_4 = g_F'(\tilde{\rho}) v_2;$

(13) $\frac{\partial H}{\partial 1_F} = 0: \quad v_4 \tilde{\rho} = g_F(\tilde{\rho}) v_2 - v_3;$ (14) $\frac{\partial H}{\partial 1_Z} = 0: \quad v_3 = \alpha v_1;$

(15) $\frac{\partial H}{\partial \rho} = \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{\rho}}\right): \quad \dot{v}_1 = (\mu + \lambda + \delta) v_1 - v_4 .$

Noting (9) and (10), the variables v_2 and v_1 may be interpreted as the shadow (accounting) price of F- and Z-goods, respectively. Also, from (11), $p_C v_2$ represents the shadow price of industrial consumption goods. Substituting from (12) into (13) gives

(16) $$v_3 = (g_F - \tilde{\rho} g_F') v_2 ,$$

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[2]Sufficiency of the Euler equations may be established along the line of reasoning given in Uzawa /7, pp. 4-5/. Since the production function $g_F(\tilde{\rho})$ and the utility function $u(\ )$ in (7) are strictly concave, the optimum solution (if it exists) is also unique.

and hence v_3 stands for the shadow price of labor. Each price variable must be nonnegative.

Defining $p_Z = \frac{v_1}{v_2}$ as the price of Z-goods relative to the price of F, it is seen from (14) and (16) that

$$p_Z = g_F(\tilde{\rho}) - \tilde{\rho} g_F'(\tilde{\rho}) = w(\tilde{\rho}), \tag{17}$$

which defines the wage rate (w) in terms of F-goods as the value of the marginal productivity of labor in each of the two activities.

To the original static equations (3) - (5) may therefore be added the optimum conditions (17) and

$$\begin{aligned} p_Z u_F(q_F) &= u_Z(q_Z, q_C) \\ p_C u_F(q_F) &= u_C(q_Z, q_C), \end{aligned} \tag{18}$$

which equates the marginal rate of substitution between each pair of commodities to the price ratio. Together they determine uniquely the variables $\tilde{\rho}$, l_F, l_Z, x_F, x_Z, q_F, q_Z, and q_C, given ρ and p_Z. Eliminating x_F, x_Z, l_F and l_Z and taking total differentials, the following matrix equation may be formed

$$\begin{bmatrix} a_0 & & 0 & p_C\tilde{\rho} \\ -\tilde{\rho} g_F'' & 0 & 0 & 0 \\ 0 & -p_Z u_{FF} & u_{ZZ} & u_{ZC} \\ 0 & -p_C u_{FF} & u_{ZC} & u_{CC} \end{bmatrix} \begin{bmatrix} d\tilde{\rho} \\ dq_F \\ dq_Z \\ dq_C \end{bmatrix} = \begin{bmatrix} g_F & 0 \\ 0 & \alpha \\ 0 & u_F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d\rho \\ dp_Z \end{bmatrix} \tag{19}$$

where

$$a_0 = q_F - \tilde{\rho} g_F' + p_C q_C \quad \text{which, using (3) and the second relation in (4) reduces to}$$

(19a)

$$= l_F(g_F - \tilde{\rho} g_F') > 0.$$

Notice that

(20) $$\frac{\partial \tilde{\rho}}{\partial \rho} = 0 \text{ and } \frac{\partial \tilde{\rho}}{\partial p_Z} = \frac{-\tilde{\rho}\, g_F''}{\alpha} > 0,$$

as (17) would readily show. It is proved in the Appendix that

(21)
$$\frac{\partial q_F}{\partial \rho} > 0, \quad \frac{\partial q_Z}{\partial \rho} > 0, \quad \frac{\partial q_C}{\partial \rho} > 0 \quad \text{for all } \rho,$$
$$\frac{\partial q_F}{\partial p_Z} < 0, \quad \frac{\partial q_Z}{\partial p_Z} < 0, \quad \frac{\partial q_C}{\partial p_Z} > 0 \quad \text{for all } p_Z,$$

provided that <u>both</u> Z- and C-goods are non-inferior. If Z-goods are assumed inferior as was done in /‾2_7, then $\frac{\partial q_Z}{\partial \rho} > 0$ and the sign of $\frac{\partial q_Z}{\partial p_Z}$ becomes ambiguous. To simplify the exposition, non-inferiority of Z-goods will be assumed from hereon, in which case uniqueness of the short-run values of the variables is also assured.

The dynamic equation (15) may be written, noting (16) and

(22) $$\dot{v}_1 = \left[(\mu+\lambda+\delta) - \frac{\alpha g_F'(\tilde{\rho})}{g_F(\tilde{\rho}) - \tilde{\rho} g_F'(\tilde{\rho})}\right] v_1 .$$

Thus, $\dot{v}_1 = 0$ if and only if $\tilde{\rho} = \tilde{\rho}^*$, where $\tilde{\rho}^*$ is the capital intensity in F-production for which[3]

$$\text{(23)} \qquad \frac{g_F'(\tilde{\rho}^*)}{g_F(\rho^*) - \tilde{\rho} g_F'(\rho^*)} = \frac{\mu+\lambda+\delta}{\alpha} .$$

The determination of ρ^* is shown graphically in Figure 1.

Since a one-to-one correspondence exists between $\tilde{\rho}$ and p_Z, $\dot{v}_1 = 0$ implies $p_Z = p_Z^* =$ constant. Therefore[4],

$$\text{(24)} \qquad \left.\frac{dv_1}{d\rho}\right|_{\dot{v}_1=0} = \left.\frac{dv_1}{d\rho}\right|_{p_Z=p_Z^*} = \frac{\partial v_1}{\partial \rho} = u_{ZZ}\frac{\partial q_Z}{\partial \rho} + u_{ZC}\frac{\partial q_C}{\partial \rho} \text{ using (10),}$$

$$< 0, \text{ from (8) and (21).}$$

The dynamic equation (2), on the other hand, requires that for $\dot{\rho} = 0$,

$$\text{(25)} \qquad \alpha l_Z(\rho,p_Z) = q_Z(\rho,p_Z) + (\mu+\lambda)\,\rho .$$

Eliminating l_Z in (25) by (5) and differentiating with respect to ρ , we obtain

$$\text{(26)} \qquad \left.\frac{dp_Z}{d\rho}\right|_{\dot{\rho}=0} = \frac{\alpha + \left[(\mu+\lambda) + \frac{\partial q_F}{\partial \rho}\right]\rho}{\left[\alpha - (\mu+\lambda)\,\rho - q_Z\right]\frac{\partial \tilde{\rho}}{\partial p_Z} - \tilde{\rho}\frac{\partial q_Z}{\partial p_Z}} > 0$$

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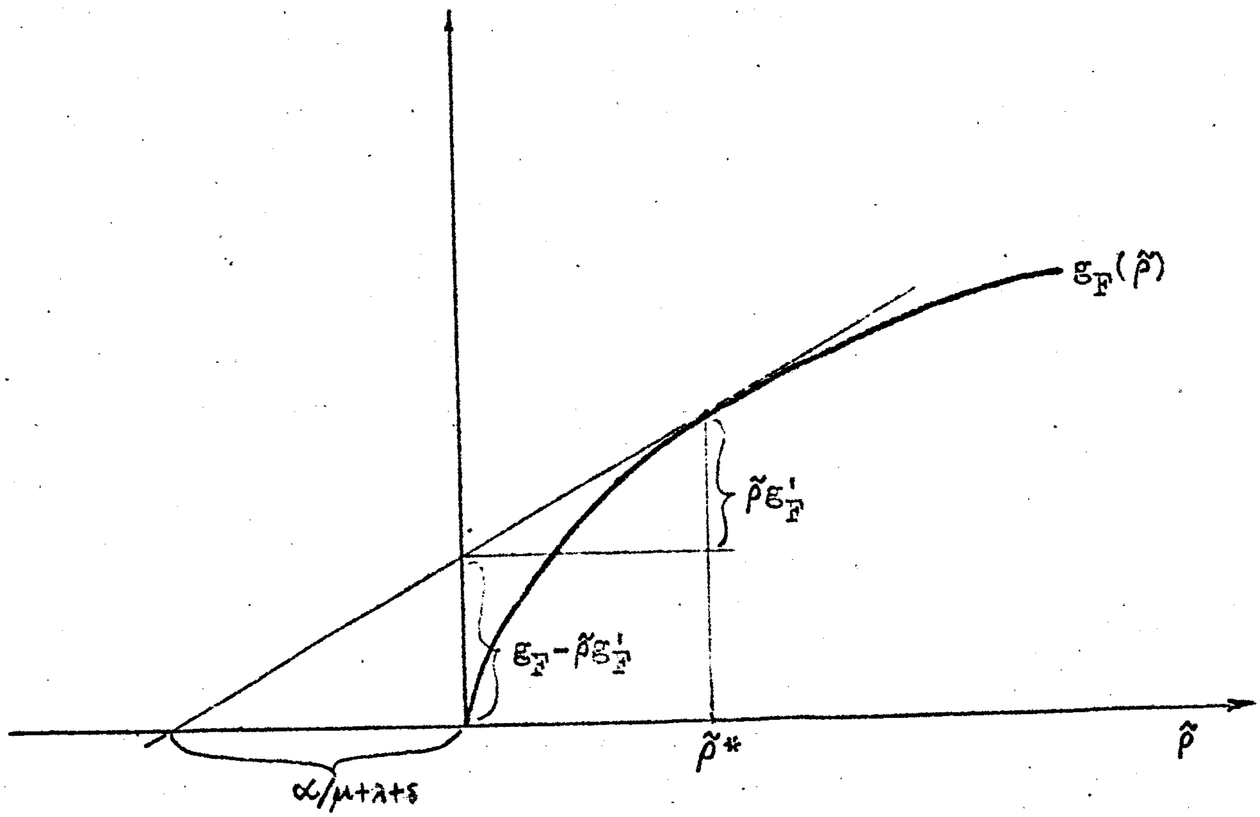
[3]Equation (23) states that the rate of change of the shadow price of Z-goods will be zero when the ratio of factor shares (capital to labor) is equal to the sum of the gross rate of growth of capital and the discount rate divided by the average productivity of labor.

[4]Notice that if Z-goods were assumed inferior the sign of $\left.\frac{dv_1}{d\rho}\right|_{\dot{v}_1=0}$ would have been ambiguous.



$g_F(\tilde{\rho})$
$\tilde{\rho} g_F'$
$g_F - \tilde{\rho} g_F'$
$\tilde{\rho}^*$
$\tilde{\rho}$
$\alpha/\mu+\lambda+\delta$

FIGURE 1

noting that the bracketted expression in the denominator is zero by (25).

Recalling that v_1 is uniquely determined by ρ and p_Z, we have

$$(27)\qquad \left.\frac{dv_1}{d\rho}\right|_{\dot{\rho}=0} = \frac{\partial v_1}{\partial \rho} + \frac{\partial v_1}{\partial p_Z}\left.\frac{dp_Z}{d\rho}\right|_{\dot{\rho}=0}$$

where

$$(27a)\qquad \frac{\partial v_1}{\partial p_Z} = \frac{\partial (p_Z u_Z)}{\partial p_Z} = u_F + u_{FF}\frac{\partial q_F}{\partial p_Z} > 0 \quad \text{by (10), (18)}$$

and (21).

Therefore, noting (24),

$$\left.\frac{dv_1}{d}\right|_{\dot{\rho}=0} > \left.\frac{dv_1}{d\rho}\right|_{\dot{v}_1=0}$$

The typical structure of solution paths (trajectories) to the pair of differential equations (2) and (22) is illustrated in the phase diagram of Figure 2. The slopes of the curves $\dot{\rho} = 0$ and $\dot{v}_1 = 0$ are consistent with the inequalities (24) and (28), which together imply that the $\dot{\rho} = 0$ curve may either

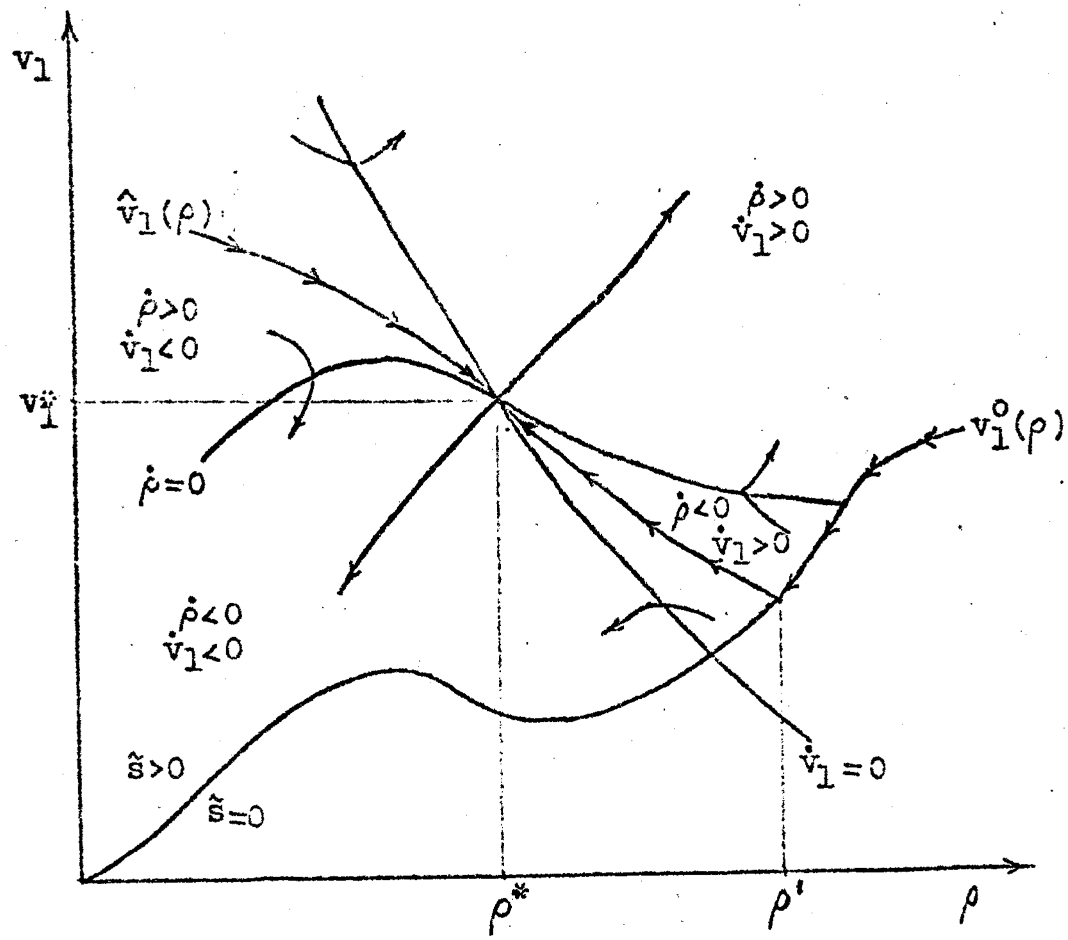

v_1
$\hat{v}_1(\rho)$
$\dot{\rho}>0$
$\dot{v}_1>0$
$\dot{\rho}>0$
$\dot{v}_1<0$
v_1^*
$v_1^o(\rho)$
$\dot{\rho}=0$
$\dot{\rho}<0$
$\dot{v}_1>0$
$\dot{\rho}<0$
$\dot{v}_1<0$
$\tilde{s}>0$
$\tilde{s}=0$
$\dot{v}_1=0$
ρ^*
ρ'
ρ

FIGURE 2

be upward or downward sloping at any point but is always less steep than the $\dot{v}_1=0$ curve where it has a negative slope. Notice from (22) and Figure 10 that $\dot{v}_1 < 0$ if and only if $\tilde{\rho} < \tilde{\rho}^*$. Since $\frac{\partial v_1}{\partial p_Z}$ and $\frac{\partial \tilde{\rho}}{\partial p_Z}$ are both positive, the rate of change in v_1 is greater (less) than zero at points lying above (below) the $\dot{v}_1=0$ curve. Likewise, ρ tends to increase (decrease) if (ρ, v_1) lies above (belcw) the $\dot{\rho} = 0$ curve. Therefore, the general direction of the trajectories (indicated in the figure by arrowheads) remains the same whether the $\dot{\rho} = 0$ curve has a negative or positive slope at any point.

At the stationary point (ρ^*, v_1^*) where the two curves intersect,

(29) $$\rho = \frac{x_Z^* - q_Z^*}{\mu + \lambda} \quad \text{from (2),}$$

which in conjunction with (23) and the relevant static equations may be used to determine ρ^* and v_1^* together with the other variables.

It is clear from the phase diagram that (ρ^*, v_1^*) is a saddle-point. Let the two branches of the solution path converging to this point be represented by $\phi_1(\rho)$. For any arbitrary capital-labor ratio $\bar{\rho}$, there exists a shadow price of Z-goods $v_1(\bar{\rho})$ for which the trajectory passing through $(\bar{\rho}, \phi_1(\bar{\rho}))$ converges to (ρ^*, v_1^*). The function $\phi_1(\rho)$ is seen to be a monotonically decreasing function of ρ.

Thus far, the constraint that gross investment must be non-negative has been ignored. The boundary condition, which is $\tilde{s}(t) = 0$, implies from (2),

(30) $$x_Z(t) = q_Z(t).$$

Since x_Z and q_Z are uniquely determined by ρ and p_Z, we have

(31) $$\left.\frac{dp_Z}{d\rho}\right|_{\tilde{s}=0} = \frac{\partial q_Z/\partial p_Z - \partial x_Z/\partial p_Z}{\partial x_Z/\partial \rho - \partial q_Z/\partial \rho} > 0 ,$$

using (21) and noting from (4) and (5) that

(32) $$\frac{\partial x_Z}{\partial \rho} = \frac{\alpha\partial}{\partial \rho}\left(1-\frac{\rho}{\tilde{\rho}}\right) = \frac{-\alpha}{\tilde{\rho}} < 0 ,$$
$$\frac{\partial x_Z}{\partial p_Z} = \frac{\alpha\rho}{\tilde{\rho}^2}\left(\frac{-\tilde{\rho}g_F''}{\alpha}\right) > 0.$$

In analogy to (27), therefore,

(33) $$\left.\frac{dv_1}{d\rho}\right|_{\tilde{s}=0} = \frac{\partial v_1}{\partial \rho} + \frac{\partial v_1}{\partial p_Z} \cdot \left.\frac{dp_Z}{d\rho}\right|_{\tilde{s}=0} \gtrless 0,$$

where, by (24), the first term in the RHS is the slope of the $\dot{v}_1 = 0$ curve.

Such boundary is represented in Figure 2 by the curve labelled $v_1^o(\rho)$; it passes through the origin and may be positively or negatively sloped at any point but is always less steep than the $\dot{v}_1 = 0$ curve where the latter case holds. Gross investment is positive at points above the $v_1^o(\rho)$ curve. If (ρ, v_1) lies below the boundary curve, the level of investment must be taken to be zero. In such case,

$$(34) \qquad \rho = -(\mu+\lambda)$$

with $\dot{v}_1$ as given in (22).

Notice from Figure 11 that $\tilde{s}(t) \geq 0$ implies nonnegatively of $v_1(t)$ and that $\tilde{s} = 0$ if $v_1 = 0$. Thus the transversality condition

$$(35) \qquad v_1(T)e^{-\delta T}\left[\rho(T) - \rho_T\right] = 0$$

in the Pontryagin solution[5] to the finite horizon problem may be replaced by the strcnger condition that

$$(36) \qquad \tilde{s}(t)\left[\rho(T) - \rho_T\right] = 0$$

which means that the inequality $\rho(T) > \rho_T$ may hold even when $v_1 = 0$ provided the optimum gross investment at the end of the planning period is zero.

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[5]See Pontryagin, et al. [5, pp. 49-50]; equation (35) has the interpretation that at the end of the planning period the target requirement $\rho(T) \geq \rho_T$ must either hold with equality or the present value of the shadow price of Z-goods be zero.

The stage is now set for the specification of the optimal solution path. Consider first the special case where $T = \infty$ and the terminal capital-labor ratio is left free. If the initial capital-labor ratio happens to be less than ρ' (at which $\hat{v}_1(\rho)$ intersects $v_1^o(\rho)$), then the unique optimal path is some portion of $\hat{v}_1(\rho)$ converging to the stationary state (ρ^*, v_1^*).[6] An initial value of ρ greater than ρ' means that the agrarian economy in the optimal path will first consume the entire output of Z-goods, gradually lowering its capital-labor ratio along $v_1^o(\rho)$ until ρ' is reached, after which the economy will approach the stationary state along $\hat{v}_1(\rho)$.

For finite horizon planning, the turnpike theorem rigorously proved by Cass [3] is applicable in the present case: Optimal growth takes place within an arbitrarily small neighborhood of the saddle-point (ρ^*, v_1^*) except possibly over some initial or terminal phase[7]. Figures 3 and 4 illustrate the optimal paths associated with $\rho_0 < \rho_T < \rho^*$ and $\rho_0 < \rho^* < \rho_T$, respectively. Two different patterns of optimum capital accumulation and valuation emerge from the two cases. In Figure 3, where the capital-labor ratio at the end of the planning period is less than the stationary value, the optimal path (represented by

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[6] Cf. Uzawa [7].

[7] See also Samuelson [6].

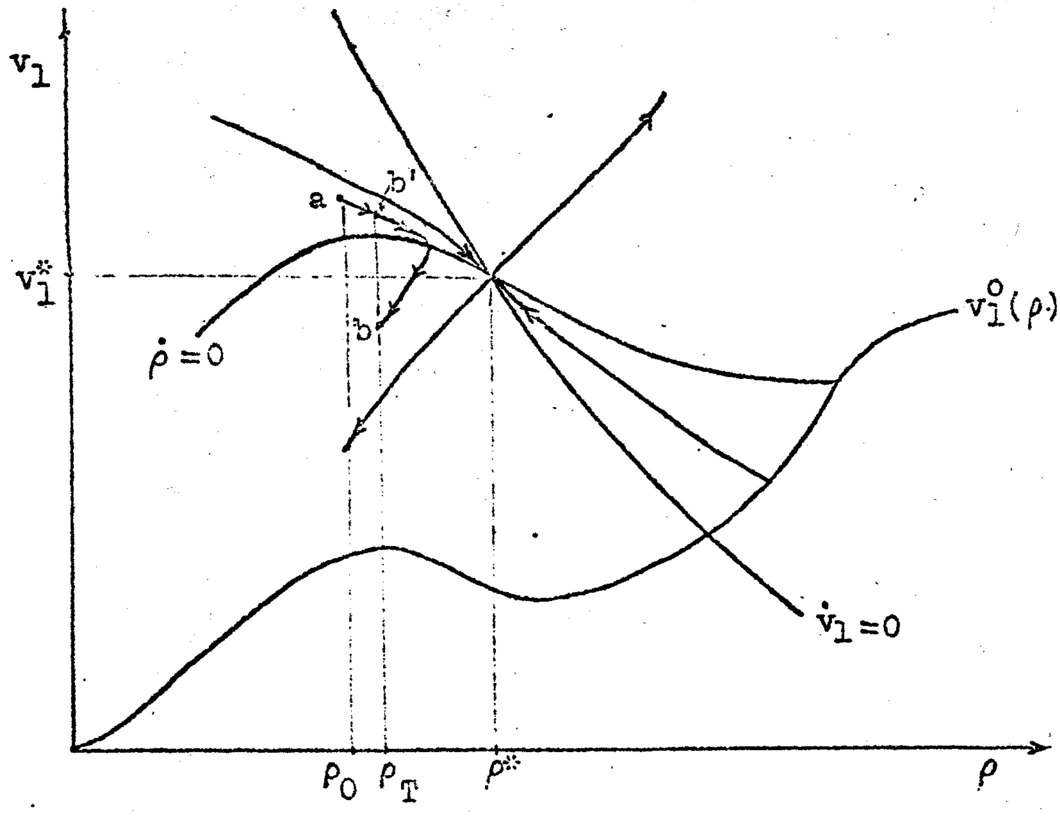

FIGURE 3

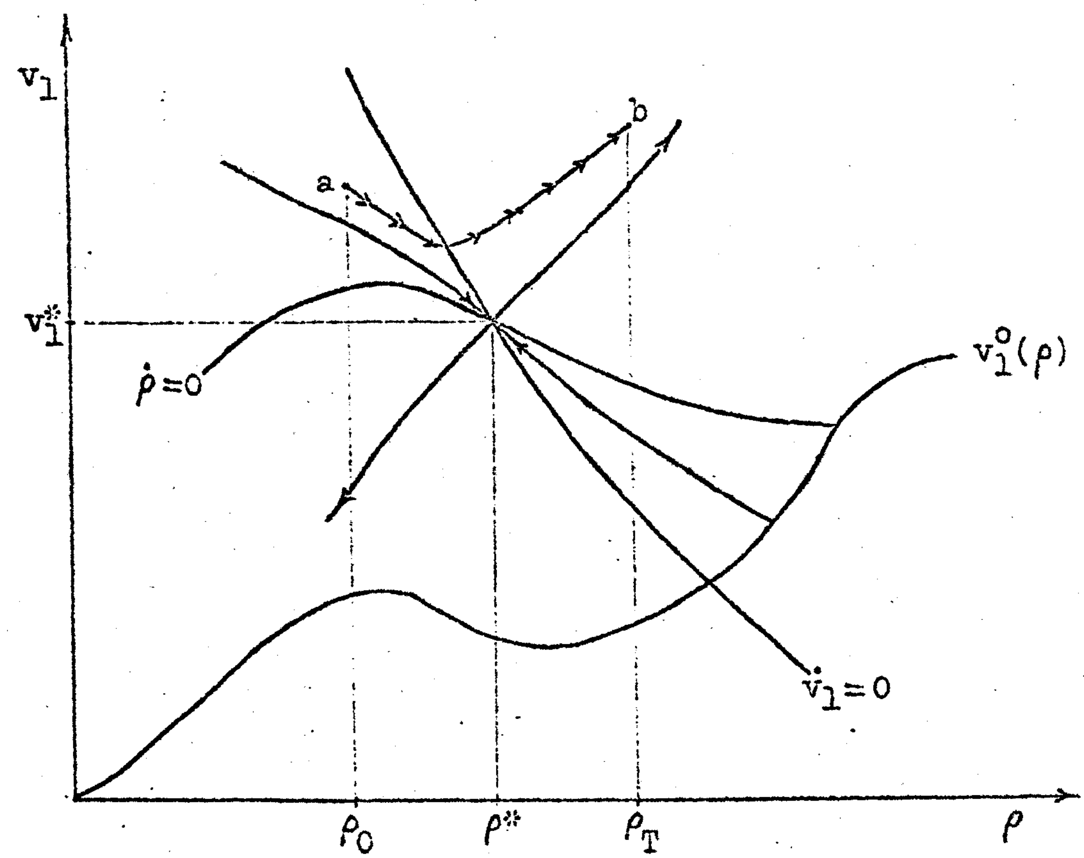

FIGURE 4

the curve a-b) is seen to be characterized by an increasing capital-labor ratio which reaches a peak beyond ρ_T and then decreases to the required terminal value at point b; the shadow price of Z-goods meanwhile decreases continuously. The optimal path for the case where $\rho_T > \rho^*$, as depicted in Figure 4 by the sequence of arrows from a to b is characterized by a steadily increasing capital-labor ratio and a valuation of Z-goods which first decreases toward the stationary level (until the $\dot{v}_1 = 0$ curve is reached) and then increases along the northeast trajectory. As drawn, the optimal growth path in each case does not entail zero investment at any time during the period of optimization.

One important assumption is implicit in the foregoing description of the optimal patterns of capital accumulation and valuation, namely, that it is feasible for the economy with initial capital-labor ratio $\rho(0) = \rho_0$ to reach ρ_T in the specified time period T along the optimal path indicated in Figures 3 and 4. Suppose such path is infeasible. Then the optimal program consists of choosing an initial value of v_1 such that the trajectory just reaches ρ_T at the time T; in Figure 3, for example, the optimal path may be represented only by the segment a-b', which indicates a continuously increasing capital-labor ratio.

IV. The Optimal Program and Choice of Instruments

Having obtained the optimal program of capital accumulation and valuation, the implied pattern of the other variables may be determined using relevant static equations of the model. From (10) and the first relation in (18),

$$v_1 = p_Z u_F(q_F), \tag{37}$$

which may be differentiated with respect to t to yield

$$a_1 \frac{dp_Z}{dt} = a_2 \frac{d\rho}{dt} + \frac{dv_1}{dt}, \tag{38}$$

where

$$a_1 = u_F + p_Z u_{FF} \frac{\partial q_F}{\partial p_Z} > 0 \quad \text{and} \quad a_2 = -p_Z u_{FF} \frac{\partial q_F}{\partial \rho} > 0.$$

From (38) it is possible to determine the time profile of the relative price of Z-goods (with respect to F) corresponding to the optimum pattern of capital accumulation and valuation.

Labor allocation over time is described by

$$\tilde{\rho} \frac{dl_F}{dt} = \frac{d\rho}{dt} + l_F \frac{d\tilde{\rho}}{dt} \quad \text{from (5)},$$

$$= \left(1 + \frac{\alpha l_F}{\tilde{\rho} g_F''} \cdot \frac{a_2}{a_1}\right)\frac{d\rho}{dt} + \frac{\alpha}{\tilde{\rho} g_F''} \cdot \frac{1}{a_1} \cdot \frac{dv_1}{dt} \quad \text{using (17)}$$

and (38); and the dynamic behavior of the consumption variables are given by

$$\frac{dq_i}{dt} = \frac{\partial q_i}{\partial \rho} \cdot \frac{d\rho}{dt} + \frac{\partial q_i}{\partial p_Z} \cdot \frac{dp_Z}{dt} \quad (i=F,Z,C) \text{ which, using (38) gives}$$

(40)

$$= \left(\frac{\partial q_i}{\partial \rho} + \frac{\partial q_i}{\partial p_Z} \cdot \frac{a_2}{a_1}\right)\frac{d\rho}{dt} + \frac{\partial q_i}{\partial p_Z} \cdot \frac{1}{a_1} \cdot \frac{dv_1}{dt} .$$

Thus the optimum values of p_Z, l_F, q_F, q_Z and q_C through time are determinable. It also follows that optimization in the agrarian economy can be brought about by appropriately controlling (at least) one of the variables. In principle it matters little whether the planning authority makes use of resource allocation and/or any of the shadow prices and consumption variables as instruments in pursuing the goal of optimal growth. In practice regulating the level of importation (consumption) of industrial consumption goods seems both feasible and administratively least burdensome.[8]

The correspondence observed above between the number of target variables and of policy instruments conforms to the well-known proposition in the Tinbergen-Theil theory of economic policy. An intertemporal optimization problem with a single objective (maximum discounted sum of per capita utilities) has been shown to require only one instrument variable.

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[8] It may be noted that the imposition of import quota on industrial consumption goods is commonly used in less developed countries as an ad hoc solution to short-term balance of payments difficulties.

APPENDIX

Derivation of (21)

Applying Cramer's rule on the matrix equation (19) to solve for the partial derivatives,

$$\text{(i)}\quad D_o \frac{\partial q_Z}{\partial \rho} = \tilde{\rho} g_F g_F''(u_{ZZ}u_{CC} - u_{ZC}^2)$$

$$\text{(ii)}\quad D_o \frac{\partial q_Z}{\partial \rho} = \tilde{\rho} g_F g_F'' u_{FF}(p_Z u_{CC} - p_C u_{ZC})$$

$$\text{(iii)}\quad D_o \frac{\partial q_C}{\partial \rho} = \tilde{\rho} g_F g_F'' u_{FF}(p_C u_{ZZ} - p_Z u_{ZC})$$

$$\text{(iv)}\quad D_o \frac{\partial q_F}{\partial p_Z} = a_o \alpha (u_{ZZ}u_{CC} - u_{ZC}^2) + p_C u_F u_{ZC} \tilde{\rho} g_F''$$

$$\text{(v)}\quad D_o \frac{q_Z}{\partial p_Z} = a_o \alpha u_{FF}(p_Z u_{CC} - p_C u_{ZC}) + u_F \tilde{\rho}^2 g_F''(u_{CC} + p_C^2 u_{FF})$$

$$\text{(vi)}\quad D_o \frac{\partial q_C}{\partial p_Z} = -a_o \alpha u_{FF}(p_C u_{ZZ} - p_Z u_{ZC}) - u_F \tilde{\rho}^2 g_F'' u_{CZ}$$

where

$$\text{(vii)}\quad D_o = \tilde{\rho}^2 g_F'' \left[(u_{ZZ}u_{CC} - u_{ZC}^2) + p_C^2 p_Z u_{FF}(p_C u_{ZZ} - p_Z u_{ZC})\right].$$

Recalling the assumptions concerning signs already made, viz., (4a) and (8), the following conditions are seen to be sufficient for the inequalities in (21) to hold:

$$\text{(viii)}\quad p_C u_{ZZ} - p_Z u_{ZC} < 0 \quad \text{and} \quad p_Z u_{CC} - p_C u_{ZC} < 0.$$

Given that C-goods are non-inferior, (viii) will be true if and only if Z-goods are also non-inferior. This is proved in what follows.

Consider the short-run allocation of income y to the consumption of the three kinds of commodities F, Z and C. Utility maximization subject to the budget constraint

$$q_Z + p_Z q_Z + p_C q_C = y \tag{ix}$$

requires that

$$u_F = \frac{p_Z}{p_Z} = \frac{u_C}{p_C} \tag{x}$$

from which the following matrix equation of partial derivatives may be formed:

$$\begin{bmatrix} -p_Z u_{FF} & u_{ZZ} & u_{ZC} \\ 0 & (p_Z u_{ZZ} - p_C u_{CC}) & -(p_Z u_{CC} - p_C u_{ZC}) \\ 1 & p_Z & p_C \end{bmatrix} \begin{bmatrix} \partial q_F/\partial y \\ \partial q_Z/\partial y \\ \partial q_C/\partial y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{xi}$$

Application of Cramer's rule gives

$$D_1 \frac{\partial q_F}{\partial y} = -u_{ZZ} \, (p_Z u_{CC} - p_C u_{ZC}) - u_{ZC}(p_C u_{ZZ} - p_Z u_{CZ}) \tag{xii}$$

$$= -p_Z(u_{CC} u_{ZZ} - u_{ZC}^2)$$

$$D_1 \frac{\partial q_Z}{\partial y} = -p_Z u_{FF}(p_Z u_{CC} - p_C u_{ZC}). \tag{xiii}$$

(xiv) $D_1 \frac{\partial q_C}{\partial y} = -p_Z u_{FF}(p_C u_{ZZ} - p_Z u_{CZ})$,

where D_1 is the determinant of the transformation matrix in (xi) given by

(xv) $$D_1 = -(p_C u_{ZZ} - p_Z u_{CZ})(u_{ZC} + p_Z p_C u_{FF}) - (p_Z u_{CC} - p_C u_{ZC})(u_{ZZ} + p_Z^2 u_{FF}).$$

From (xiii) and (xiv)

(xvi) $$\frac{\partial q_Z/\partial y}{\partial q_C/\partial y} = \frac{p_Z u_{CC} - p_C u_{ZC}}{p_C u_{ZZ} - p_Z u_{ZC}} .$$

Since C-goods are non-inferior, $\frac{\partial q_C}{\partial y} > 0$ by definition. It immediately follows that, for the inequalities in (viii) to hold, it is necessary and sufficient that $\frac{\partial q_Z}{\partial y}$ in (xvi) be greater than zero, i.e., Z-goods are also non-inferior . . . . Q. E. D.

RMB:abg

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