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MONEY IN A MODIFIED NEOCLASSICAL GROWTH MODEL

by

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MONEY IN A MODIFIED NEOCLASSICAL GROWTH MODEL*

by   DELANO P. VILLANUEVA**

I. *INTRODUCTION*

Two outstanding features have in the past characterized neoclassical growth models: (i) *technical change is exogenously given both in total amount and in bias (usually as Harrod-neutral)*, and (ii) *the models run in real terms, i.e., portfolio behavior and the existence of money are ignored.* A recent model by John Conlisk [2] relaxes feature (i) by introducing a technical change frontier endogenously positioned by amounts of capital and labor allocated to a productivity sector; but his model retains feature (ii). Recent models by James Tobin [8][9] and Harry Johnson [4][5] relax feature (ii) by introducing a simplified monetary sector; but their models retain feature (i).

The model of this paper relaxes both features (i) and (ii). It may thus be viewed as a synthesis of the models mentioned above. It

incorporates a real balance effect into a modified neoclassical growth model with the objective of getting monetary parameters into the act. The major conclusion is that the equilibrium per capita growth rate of output is now sensitive to portfolio and saving behavior, population growth rate, depreciation rate, form of the production function, and perhaps most important (in the extended model), monetary policy, with the sensitivities having the expected signs.

Section II develops the model. Section III presents the reduced model and specifies stability conditions. Section IV compares the model's sensitivities with those of the basic Solow-Swan and Tobin-Johnson models (henceforth referred to as S-S and T-J models, respectively). Section V analyzes the model's speed of adjustment. Section VI takes up the issue of technical change bias. Section VII explores some extensions. Section VIII suggests a rough empirical test procedure to discriminate between the T-J model and the modified model. Section IX concludes.

II. *THE MODEL*

The model consists of 5 equations in 5 variables Q, K, L, S, and (M/P), explained immediately below.[1]

(1) $\qquad Q = F(K,L)$ $\qquad$ (Production Function)

(2) $\qquad \dot{K} = \alpha S - \delta K$ $\qquad$ (Capital Growth Equation)

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[1] A dot over a variable denotes time derivative, e.g., $\dot{K} = dK/dt$.

(3) $\qquad \dot{L} = \beta(1-\alpha)S + nL \qquad$ (Labor Growth Equation)

(4) $\qquad S = s_1 Q - s_2(M/P) \qquad$ (Saving Equation)

(5) $\qquad (M/P) = bQ \qquad$ (Money Equation)

Q, K, L, S, and (M/P) are output, capital, labor, saving, and real money stock; F(K,L) is a unit-homogeneous, neoclassical production function with $F_1(k,1)$, $F_2(1,1/k) > 0$, where $k = K/L$; α, δ, β, n, s_1, s_2, and b are structural parameters.

Mathematically, Eqs. (2) and (3) are of the form $\dot{Y} = aX + bY$. The term aX may be interpreted as an endogenous growth component, since it makes $\dot{Y}$ depend on X [which is equal to total saving S in the the model]. The term bY may be interpreted as an exogenous growth component, since it does not depend on X; if a = 0 (no endogenous growth component), then $\dot{Y}/Y = b$ (Y grows exogenously at a constant rate b.). If K, L, and the structural parameters are appropriately interpreted, the model allows for endogenous technical change, exogenous technical change, or any mixture of the two.

K and L are measured in *efficiency-corrected* or *technical change-augmented* units. The endogenous growth components of $\dot{K}$ and $\dot{L}$ capture any endogenous capital-augmenting and labor-augmenting technical change, while the exogenous growth components capture any exogenous technical change (if any). Under these interpretations, the parameter α is not entirely a savings parameter; it is partly a technology parameter, i.e., it reflects any induced capital-augmenting technical change. The

depreciation rate δ should be made smaller by the rate of exogenous

capital-augmenting technical change (if any). The parameter $\beta(1-\alpha)$

reflects any induced labor-augmenting technical change and any

endogenous population growth responses. The parameter n should be

interpreted as an exogenous population growth rate adjusted for any

exogenous labor-augmenting technical change.

Of course, there is an issue of separating the growth rate of

capital and labor into a term attributable to increases in natural units

and a term attributable to increases in the efficiency of those units.

For labor, a man-hour may be chosen as the natural unit. Then the

efficiency component is calculated as a residual, i.e., difference

between the growth of labor and the rate of increase in man-hours. For

capital, defining a natural unit is as problematical as the issue of

what the concept of *aggregate capital stock* is.

In Eq. (2), α is the fraction of total savings S applied to increasing

K and δ the constant rate of depreciation. In Eq. (3), $(1-\alpha)$ is the

fraction of total savings S applied to increasing L, β a parameter which

translates L-oriented savings (in \$) into new L (in L-units, whatever

they are) and n a constant.[2]

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[2]One possible derivation of Eq. (3) is as follows: Let (i) L = yN,
where N is the exogenous population base adjusted for any exogenous
labor-augmenting technical change and y an endogenously determined
technical change multiplier. Let the growth rate of N be equal to n, i.e.,
(ii) $\dot{N}$ = nN. Postulate a technical change function (iii) $\dot{y}$ = $\beta(1-\alpha)$S/N.
In words, (iii) says that endogenous labor-augmenting technical change
increases proportionately with saving per head. The total time derivative
of (i) is given by (iv) $\dot{L}$ = $\dot{y}$N + y$\dot{N}$, which, using (i) through (iii),
reduces to Eq. (3).

Eq. (4) is the saving function of the Patinkin type. Finally, Eq. (5) is the equilibrium condition for the simplified monetary sector, where M is nominal money stock and P the absolute price level. The nominal money stock M and its growth rate $\dot{M}/M$ are assumed to be exogenous variables subject to control by the monetary authorities. This specification implies that, given the equilibrium growth rate of the real money stock, the equilibrium growth rate of the price level is determined. The following are reasonable restrictions on the structural parameters of the model:

(6) $$0 < b, \alpha < 1; \; s_1, s_2, (s_1 - s_2 b), \beta > 0$$

III. *REDUCED MODEL AND STABILITY ANALYSIS*

By successive substitutions, Eqs. (1) through (5) reduce to the differential equation

(7) $\dot{k}/k = \alpha(s_1 - s_2 b)F(1, 1/k) - \beta(1-\alpha)(s_1 - s_2 b)F(k, 1) - (n+\delta)$

the slope of which is given by

(8) $d(\dot{k}/k)/dk = -\alpha(s_1 - s_2 b)k^{-2}F_2(1, 1/k) - \beta(1-\alpha)(s_1 - s_2 b)F_1(k, 1)$

$$< 0.$$

For the model to have a unique stable equilibrium, not only must Eq. (7) have a negative slope throughout the relevant range of k but also must intersect the k-axis at some positive k, say, k*. The following intersection conditions, although not assured, are plausible:

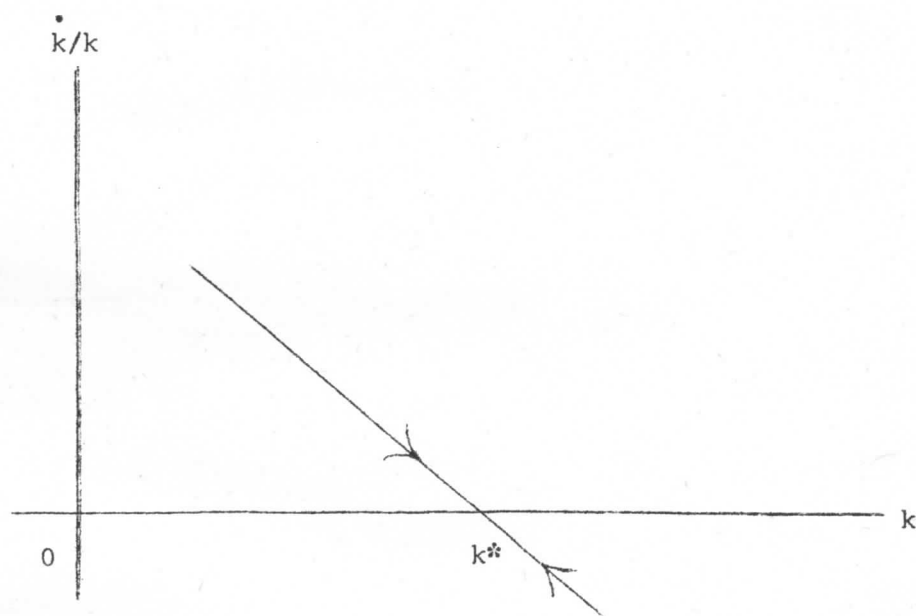(9) $\alpha(s_1 - s_2 b)F(1, 1/k) - \beta(1-\alpha)(s_1 - s_2 b)F(k, 1) - (n+\delta)$

> 0 for k small enough
< 0 for k large enough

for which it is sufficient that

$$E(k) \begin{cases} \geq 1 \text{ for } k \text{ small enough} \\ \leq 1 \text{ for } k \text{ large enough} \end{cases}$$

where $E(k)$ is the elasticity of substitution.

Given the above slope and intersection conditions, Eq. (7) graphs as --

where k^* is equilibrium capital intensity. The directional arrows may be filled in by inspection from Eq. (7).

IV. EQUILIBRIUM

The following are stable equilibrium values of some important variables of the model:

$k \to k^*$ is the root of $\alpha(s_1 - s_2 b)F(1, 1/k^*) - \beta(1-\alpha)(s_1 - s_2 b)F(k^*, 1) = n + \delta$

r (interest rate) $\to r^* = F_1(k^*, 1)$

$(Q/K) \to (Q/K)^* = F(1, 1/k^*)$

$$(\dot{Q}/Q) \rightarrow (\dot{Q}/Q)^* = \alpha(s_1-s_2)F(1,1/k^*) - \delta = \beta(1-\alpha)(s_1-s_2b)F(k^*,1) + n$$

$$(\dot{Q}/Q-n) \rightarrow (\dot{Q}/Q-n)^* = \beta(1-\alpha)(s_1-s_2b)F(k^*,1)$$

In order to compare this model with S-S and T-J, consider the sensitivities of the equilibrium per capita growth rate of output with respect to some selected parameters. It turns out that both S-S and T-J models are special cases of the present model.

$$(\dot{Q}/Q-n)^*$$

	b	s_1	s_2	β	δ	n	F
S-S[3]	0	0	0	0	0	0	0
T-J[4]	0	0	0	0	0	0	0
MODIFIED MODEL	−	+	+	+	−	−	+

Broadening the S-S and T-J models yields this important result: the equilibrium per capita growth rate of output is now sensitive to all structural parameters b, s_1, s_2, β, δ, n, and the form of the production function F, with the sensitivities having the expected signs.

The explanation of per capita growth of output is fundamentally different in the model of this paper from that in the S-S and T-J models. The latter's source of per capita growth is exogenous Harrod-

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[3]Parametric restrictions are $\alpha = 1$; $s_2 = b = \beta = 0$; $s_1 = s = S/Q$. See [6][7].

[4]Parametric restrictions are $\alpha = 1$; $s_2 = \beta = 0$; $s_1 = s[1-b(\lambda+n)(1/s-1)]$, where λ is the growth rate of exogenous Harrod-neutral technical change. See [9][10].

neutral (or labor-augmenting) technical change, while the present model explains per capita growth endogenously.

It is true that the S-S model differs from the T-J model on the sensitivities of equilibrium *levels* of the variables with respect to monetary parameters. Whereas in the S-S model, equilibrium *levels* are independent of *monetary* factors, these equilibrium *levels* are jointly determined by both *real* and *monetary* phenomena in the T-J model. The interesting contrast is between the present model and the T-J model since both include a simplified monetary sector. In the T-J model, equilibrium growth *rates* and *per capita* growth *rates* are insensitive to *monetary* behavior. In this paper's model, these equilibrium *rates* are jointly determined by both *real* and *monetary* behavior.

V. *SPEED OF ADJUSTMENT*

The speed of adjustment issue can be analyzed in at least two ways. (1) A full scale numerical analysis. This requires knowledge of the explicit form of the production function. The idea is to arrive at a numerical solution to the $\dot{k}$ equation. (2) A measure of speed of adjustment in the vicinity of $k = k^*$ is the absolute size of $d\dot{k}/dk$ at $k = k^*$. Less explicit results are expected if this approach is used. However, given information on F or the parameters, the analysis yields useful conclusions without resort to a full scale numerical analysis. This paper adopts the second approach.

Recall that $k^* = \phi(b,\ldots) =$ root of

(10) $\alpha F(1,1/k^*) - \beta(1-\alpha)F(k^*,1) - (n+\delta)/(s_1-s_2 b) = 0$

where (...) denotes other parameters on which k^* depends. An increase in b tends to make the left side of Eq. (10) negative; hence, a decrease in k^* is needed to compensate. Thus $\partial k^*/\partial b = \phi_1 < 0$.

The reduced model [Eq. (7)] can be rewritten as

(11) $\dot{k} = \alpha(s_1-s_2 b)f(k) - \beta(1-\alpha)(s_1-s_2 b)f(k)k - (n+\delta)k$

where $f(k) = F(k,1)$.

From Eq. (11),

(12) $(d\dot{k}/dk)^{*5} = \alpha(s_1-s_2 b)f'(k^*) - \beta(1-\alpha)(s_1-s_2 b)[f(k^*) + k^* f'(k^*)] -$

$(n+\delta)$

$= u(k^*,b,...)$

$= u[\phi(b,...),b,...]$

$= v(b,...)$

(13) $\partial(d\dot{k}/dk)^*/\partial b = u_1 \phi_1 + u_2$

where

(14) $u_1 = (s_1-s_2 b)[\alpha-\beta(1-\alpha)k^*]f''(k^*) - \beta(1-\alpha)(s_1-s_2 b)2f'(k^*)$

(15) $u_2 = s_2\{\beta(1-\alpha)f(k^*)[1+\pi(k^*)] - \alpha f'(k^*)\}$

where $\pi(k^*) = k^* f'(k^*)/f(k^*) = k^* F_1(k^*,1)/F(k^*,1) =$ marginal productivity share of capital.

Eq. (13) gives the absolute size of $(d\dot{k}/dk)^*$ as b changes. It is a measure of local speed of adjustment.

If we are willing to assume that $f''(k^*) < 0$, and $\alpha > \beta(1-\alpha)k^*$, then $u_1 < 0$, and, hence, $u_1\phi_1 > 0$. It is very likely that

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[5]$(d\dot{k}/dk)^* = d\dot{k}/dk$ evaluated at $k = k^*$.

$\beta(1-\alpha)f(k^*)[1+\pi(k^*)] > \alpha f'(k^*)$, so that $u_2 > 0$. We conclude, on these assumptions about f and some of the parameters, that $\partial(dk/dk)^*/\partial b > 0$, or that an increase (decrease) in b will increase (decrease) the speed of adjustment of capital intensity at any moment of time to its asymptotic equilibrium value.

VI. *TECHNICAL CHANGE BIAS*

We shall now discuss an important parameter which may be subject to policy control: α. It may be recalled that α is the fraction of total saving devoted to increasing the capital stock. The choice of α is a choice of bias between the increment αS to capital and the increment $\beta(1-\alpha)S$ to labor.[6] There are, at least, two criteria for choice: short-run and long-run. The short-run criterion considers the maximization of the instantaneous growth rate of output for currently given capital and labor. The long-run criterion considers the maximization of the long-run equilibrium growth rate of output. Presumably, α goes to a constant in long-run equilibrium. Assuming that it does, we may ask the golden rule type of question: what α maximizes $(Q/Q)^*$?

The long-run choice may be formulated as

(16) $\quad \max_{\alpha, k^*} \quad \beta(1-\alpha)(s_1-s_2b)f(k^*) + n$

subject to

$\alpha f(k^*)/k^* - \beta(1-\alpha)f(k^*) - (n+\delta)/(s_1-s_2b) = 0.$

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[6]This is analogous to Conlisk's choice of bias between technology-augmented components of $\dot{K}$ and $\dot{L}$. See [2].

The first-order conditions are the constraint in Eq. (16) itself and the condition

(17) $$\beta(1-\alpha)f'(k^*)\partial k^*/\partial\alpha = \beta f(k^*),$$

where $\partial k^*/\partial\alpha$ is derived by implicitly differentiating the constraint with respect to α, i.e.,

(18) $$\partial k^*/\partial\alpha = (1+\beta k^*)/\{(\alpha/k^*)[1-\pi(k^*)] + \beta(1-\alpha)\pi(k^*)\}.$$

Plugging Eq. (18) into Eq. (17) and solving for α yields

(19) $$\alpha = \pi(k^*).$$

Heuristically, condition (19) says that if the fraction of total saving devoted to increasing capital is unequal to its marginal productivity share, then it would pay society to alter that fraction to conform with capital's marginal productivity share. In particular, if α exceeds $\pi(k^*)$, then society is allocating resources to capital in a proportion that is more than capital's marginal productivity share. Therefore, there exists an incentive to decrease α in order to increase long-run equilibrium growth rate of output. The opposite argument is true when α is below $\pi(k^*)$.

VII. *SOME EXTENSIONS*

Role of Monetary Policy

Strictly speaking, the model so far specified implies no useful role for monetary policy [measured by the behavior of $(\dot{M}/M)$]. This is perhaps partly due to the assumption that velocity is a constant. A logical extension would be to regard velocity (or its reciprocal, b) as a function of the relative rate of change in the absolute price level.

Recall that the equilibrium growth rate of real money stock in the model is given by

(20) $(\dot{M}/M) - (\dot{P}/P) = \beta(1-\alpha)(s_1-s_2 b)f(k^*) + n,$

implying that the equilibrium growth rate of the price level is

(21) $(\dot{P}/P)^* = (\dot{M}/M) - \beta(1-\alpha)(s_1-s_2 b)f(k^*) - n.$

Thus, at a given level of equilibrium capital intensity k^*, the only endogenous variable that monetary policy can affect is the equilibrium rate of change in the price level. Now if $b = \psi(\dot{P}/P)^*$, where $\psi' < 0$, then monetary policy can, in principle, affect the equilibrium growth rate of output.

Let us trace the effects of an expansive monetary policy. Starting from initial equilibrium, suppose that an exogenous increase in the growth rate of nominal money supply occurs. *Ceteris paribus*, the observed growth rate of the price level increases. Since $\psi' < 0$, wealth-holders reduce their desired cash-income ratio. Consequently, equilibrium per capita growth rate of output increases [from the result that $\partial(\dot{Q}/Q-n)^*/\partial b < 0$]. A restrictive monetary policy has opposite effects.

Under the above conditions, the following theorem is true: *If the equilibrium growth rate of real money stock is achieved by either deflation or expansion of nominal money supply at some rate not equal to the growth rate of the economy, then both equilibrium levels and equilibrium growth rates of all variables are appropriately higher or*

lower than they would otherwise be.[7]

Unemployment

The aggregate production function [Eq. (1)] may be modified to allow for unemployment. The modified production function may be

(1)' $Q = F(eK, eL)$

where e is a constant rate of employment in percentage terms, say, 95 per cent. It need not be identical for both K and L; making it so is just to simplify the model. The extended model is a constant-rate-of-unemployment model, where 1-e is the unemployment rate.

The reduced model is now given by

(7)' $\dot{k}/k = \alpha(s_1-s_2 b)eF(1,1/k) - \beta(1-\alpha)(s_1-s_2 b)eF(k,1) - (n+\delta)$

the slope of which is given by

(8)' $d(\dot{k}/k)/dk = -\alpha(s_1-s_2 b)k^{-2}eF_2(1,1/k) - \beta(1-\alpha)(s_1-s_2 b)eF_1(k,1)$

$$< 0$$

and the intersection conditions by

(9)' $\alpha(s_1-s_2 b)eF(1,1/k) - \beta(1-\alpha)(s_1-s_2 b)eF(k,1) - (n+\delta)$

> 0 for k small enough,
< 0 for k large enough.

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[7]PROOF: If a constant growth rate of nominal money stock is assumed, i.e., $\dot{M}/M = m$, then, in equilibrium, the price level changes at the rate $m - [\beta(1-\alpha)(s_1-s_2 b)f(k^*) + n]$. A special case arises when the nominal money supply is kept constant, in which case $m = 0$, and the price level falls at the rate $[\beta(1-\alpha)(s_1-s_2 b)f(k^*) + n]$. Assuming that $m \neq [\beta(1-\alpha)(s_1-s_2 b)f(k^*) + n]$, then $(\dot{P}/P)^* \neq 0$. This suggests that if the growth rate of output exceeds the growth rate of nominal money supply, the real return on a unit of money rises, and vice-versa. Thus, the desired cash-income ratio adjusts accordingly, and so do all equilibrium *levels* and growth *rates* of all variables.

Equilibrium per capita growth rate of output is positively related to the employment rate e (hence, negatively related to the unemployment rate 1-e), as given by

$$(\dot{Q}/Q-n) \rightarrow (\dot{Q}/Q-n)^* = \beta(1-\alpha)(s_1-s_2b)eF(k^*,1).$$

VIII. *A ROUGH EMPIRICAL TEST PROCEDURE*

The equilibrium growth rate expressions for the T-J and the modified models are

$$
\begin{array}{lll}
(22) & G = \lambda+n & \text{(T-J MODEL)} \\
& G = \alpha(s_1-s_2b)F(1,1/k^*) - \delta & \text{(MODIFIED MODEL)} \\
& = \beta(1-\alpha)(s_1-s_2b)F(k^*,1) + n &
\end{array}
$$

where G is the equilibrium growth rate and λ the **rate of exogenous** labor-augmenting technical change. Given the function F, if there exists a series of observations on G, $(\lambda+n)$, α, s_1, s_2, b, β, δ, and n, it is a simple statistical decision to choose between the two equations in Eq. (22).

Since the models to be tested are long-run, aggregate models, aggregate data over long periods on a set of cross-country observations must be used. As expected, only proxy variables are available. It may well be that no proxies for certain parameters are available. For example, G may be approximated by the average growth rate over a decade or so of real national product. s_1 and n may be roughly measured, respectively, by the ordinary gross savings-income ratio and ordinary population growth rate. The gross savings-cash ratio and cash-income ratio can be used to roughly measure, respectively, s_2 and b. Proxies

for α, β, λ, and δ may not be available. Under such circumstances, the data may be used to fit the following regression

(23) $$G = a_0 + a_1 s_1 + a_2 b + a_3 s_2 + a_4 n + u$$

where u is assumed to be a random, nonautocorrelated error term with zero mean. Both equations in Eq. (22) predict that $a_4 > 0$. However, the T-J model also predicts that $a_1 = a_2 = a_3 = 0$, whereas the modified model predicts that $a_1 > 0$, $a_2 < 0$, and $a_3 > 0$. If the data is not too crude, if the proxies are not too bad, and if the omitted variables enter the constant and error terms in a well-behaved manner, then estimates of a_1, a_2, and a_3 can be used to decide which of the two models is closer to the truth.

IX. CONCLUSION

This paper has monetized a modified neoclassical growth model and contrasted its properties to those of the T-J model.

(a) In the T-J model, the characterizing assumption that labor-augmenting technical change is exogenous is mainly responsible for the rather surprising result that money and portfolio behavior do not affect equilibrium growth rate and per capita growth rate of output. The modified model relaxes such an assumption.

(b) The equilibrium growth rate of the modified model depends on all structural parameters b, s_1, s_2, β, δ, and n; whereas the equilibrium growth rate of the T-J model depends only on the parameter $(\lambda + n)$, where λ is the exogenous rate of Harrod-neutral technical change.

(c) The source of per capita growth in the T-J model is exogenous labor-augmenting technical change, while the modified model explains per capita growth endogenously.

(d) Under certain plausible assumptions about the aggregate production function and some of the parameters, the desired cash-income ratio can affect the model's speed of adjustment. It was shown that an increase (decrease) in the desired cash-income ratio is likely to increase (decrease) the speed of adjustment. It follows that monetary policy, through its influence on the rate of change in the price level and the latter on the desired cash-income ratio, can also affect the speed of adjustment.

(e) In order to maximize the long-run equilibrium growth rate of output, the fraction α of total savings devoted to increasing the capital stock must be equated to its marginal productivity share.

(f) In the extended model, the effects of monetary policy and of unemployment on the equilibrium growth rate of the model are allowed for.

(g) Finally, a rough empirical test procedure was sketched to discriminate between the T-J model and the modified model.

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