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Imitation Dynamics with Spatial Poisson-Distributed Review and Mutation Rates

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Using spatial Poisson processes, we re-interpret the Imitation Dynamics and show how the spatial clustering of players affects both the selection of strategies and the speed by which they replicate. We find that the more clustered are the players, (a) the faster the evolution of strategies and (b) if some players have inherent preferences for a strategy, the greater the probability of mutation into, and selection of, that preferred strategy.

Key Words: evolutionary games, learning dynamics, local interaction games, spatial clustering

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1. INTRODUCTION

We present some Imitation Dynamics with mutation behavior in which the spatial clustering of agents plays a role in the speed with which an evolutionary game is played and the probable strategy that is selected. The literature has shown that the clustering of players can indeed matter - studies of local interaction games and contagion effects show how strategies spread across the population as players interact with their immediate 'neighbours' or network.¹ Here, we focus on the

¹See, for instance, Ellison (1993, 2000), Blume (1995), Anderlini and Ianni (1996), Morris (2000), Lee and Valentinyi (2000).

Imitation Dynamics of Weibull (1995) and Bjornerstedt and Weibull (1993), from which the canonical Replicator Dynamic (RD) can be derived, but show how it can deviate from the RD when the rate at which players are drawn and the probability of switching between strategies depend on how players are clustered. Specifically, we show that when players are close to each other, the probability of interactions increases and, hence, the opportunity to review and change strategies increases, which speeds up the replication/growth of strategies. More importantly, any inherent bias can also spread faster in a highly clustered environment, which can then override 'rational' behavior and obtain equilibria that are different from the RD.

Section 2 presents the model and obtains the results, while Section 3 concludes with a comparison of relevant results from the literature.

2. THE MODEL

Let there be a large population N of agents randomly drawn to play an evolutionary game, and let each pure strategy $h \in K$ of the game be thus associated with payoff-type $u(e^h, e^h)$. The state of the population is described by vector $x = x_1, \dots, x_K$, where x_h is the proportion of the population adopting strategy h . The expected payoff of playing h when the population is in state x is thus $u(e^h, x(t))$.

Suppose that each player is pre-programmed to adopt a pure strategy, so that there are K player-types corresponding to each strategy, but when a player is drawn to play, she gets to 'review' her strategy and may switch to another type. Let the rates at which player-types i, h review their strategy be denoted by $r_{i,h}$, and let p_h^i and p_i^h denote the probability of switching from type/strategy h to i and i to h , respectively. The total outflow of strategy i is thus given by:

$$\sum_{h \neq i} x_i r_i(x) p_i^h(x) \quad (1)$$

while total inflow is:

$$\sum_{h \neq i} x_h r_h(x) p_h^i(x) \quad (2)$$

The net flow, or growth of, strategy i is the difference between (2) and (1):

$$\dot{x}_i = \sum_{h \in K} x_h r_h(x) p_h^i(x) - r_i(x) x_i. \quad (3)$$

What determines the rates of review and probabilities of switching? In Weibull and Bjornerstedt and Weibull, the intuition is straightforward in that less successful strategies are reviewed more often and experience greater outflow. Such success is, in turn, determined by the expected payoffs of strategies. Here we further specify that given the expected payoff, a strategy is reviewed more often in an environment in which players are more clustered, to posit the idea that the more neighbors a player has, the more likely she interacts with them to play, which provides the opportunity for re-evaluation of her strategy. To the extent, then, that the rate of review increases in this manner, the speed of the evolution or growth of strategies also increases with greater clustering of players. In addition, if some players have an inherent bias or preference toward a specific strategy, the strength of the bias may be stronger in more clustered environments since it would be more likely for a biased player to meet another biased player. The effect of this would be to speed up the growth of their preferred strategy in the population in spite of its expected payoff, that is, to induce mutation away from more 'rational' strategies that otherwise have higher expected payoffs.

To show this formally, let the N agents be located within space or environment S and let r.v. $X(S)_1$ count the number of players that are randomly drawn to play, which is assumed to be a spatial Poisson process. Thus, the probability that a player is drawn to play is given by:

$$\delta = P_r[X(S)_1 = N] = \frac{e^{-\lambda_1 V(S)} [\lambda_1 V(S)]^N}{N!} \quad (4)$$

where $V(S)$ is the size, or in the 3-D case, the volume, of space S , λ_1 is the intensity parameter of the spatial process which captures the extent to which the players are spatially clustered, and $\lambda V(S)$ the mean occurrence in $V(S)$. The review rates of strategies can be assumed to be linearly decreasing in payoffs (as in Weibull) so that less successful strategies are reviewed more often, but to the extent that players have to be drawn in order to review, the review rates are also influenced by probability δ . Restricting the number of strategies to two, $K = \{i, j\}$, let the review rates of pure strategies i and j be then given by:

$$r_i(x) = \delta[\alpha - \beta u(e^i, x)] \quad (5a)$$

$$r_j(x) = \delta[\alpha - \beta u(e^j, x)] \quad (5b)$$

where $\beta > 0$ and $\alpha/\beta \geq \max[u(e^i, x), u(e^j, x)]$ to ensure all review rates are non-negative.

Now, when a player reviews her strategy, she has the opportunity to keep it or to switch. Suppose, as in Weibull and Bjornerstedt and Weibull, the reviewer imitates the 'first man on the street' such that $p_j^i = x_i$ and $p_i^j = x_j$. Then we have the following result:

PROPOSITION 1. *The greater the spatial clustering of players, the faster is the evolution of strategies.*

Proof. Solving equation (3) using review rates given by (5a) and (5b) and probabilities of switching $p_j^i = x_i$ and $p_i^j = x_j$, we obtain the following re-scaled Replicator Dynamics:

$$\dot{x}_i = \delta \beta [u(e^i, x) - u(x, x)] x_i \quad (6a)$$

$$\dot{x}_j = \delta \beta [u(e^j, x) - u(x, x)] x_j \quad (6b)$$

which are faster the larger δ is. Note that δ increases with intensity λ_1 when $N > [\lambda_1 V(S)]$ and decreases when $N < [\lambda_1 V(S)]$. However, on the whole, there are more chances for δ to be closer to one with greater (than with less) spatial clustering of players. More precisely, denoting $\tilde{N} = [\lambda_1 V(S)]$, the range of possible N values $[\tilde{N}, \infty)$ for which $\frac{\partial \delta}{\partial \lambda_1} > 0$, is greater than the range $[0, \tilde{N}]$ for which $\frac{\partial \delta}{\partial \lambda_1} < 0$. Also, given two spatial Poisson distributions with intensities λ_{1a} and λ_{1b} , where $\lambda_{1a} < \lambda_{1b}$, and denoting a value $\underline{N} = [\max(\lambda_{1a}, \lambda_{1b}) V(S)]$ for which both $\frac{\partial \delta}{\partial \lambda_{1a}} > 0$ and $\frac{\partial \delta}{\partial \lambda_{1b}} > 0$ whenever $N > \underline{N}$, it is evident that the sum $\sum_{N \geq \underline{N}} e^{-\lambda_{1a} V(S)} \frac{[\lambda_{1a} V(S)]^N}{N!} = 1 - F(\underline{N})$ is less than $\sum_{N \geq \underline{N}} e^{-\lambda_{1b} V(S)} \frac{[\lambda_{1b} V(S)]^N}{N!} = 1 - G(\underline{N})$, where $F(\underline{N})$ is the c.d.f. of r.v. $X(S)_1^{\lambda_{1a}}$, and $G(\underline{N})$ the c.d.f. of r.v. $X(S)_1^{\lambda_{1b}}$. In other words, $X(S)_1^{\lambda_{1b}}$ first-order stochastically dominates $X(S)_1^{\lambda_{1a}}$, i.e. $X(S)_1^{\lambda_{1b}} \succ_{FSD} X(S)_1^{\lambda_{1a}}$.

■

Suppose, instead, that the decision to keep or switch strategies depends on some inherent bias or preference for a strategy. Specifically, let $X(S)_2$ be a spatial Poisson-distributed random variable that counts the number of players in space S

that inherently prefer strategy i to j , and $NA \leq N$ its particular realisation for a population of size N .² Let λ_2 denote the intensity parameter of the process, and $\lambda_2 V(S)$ the mean occurrence in volume $V(S)$. Thus, the probability that a player has a preference for i can be captured by:

$$\eta = Pr[X(S)_2 = NA] = \frac{e^{-\lambda_2 V(S)} [\lambda_2 V(S)]^{NA}}{NA!}. \quad (14)$$

Now suppose that $p_j^i = \eta$ and $p_i^j = 1 - \eta$, that is, η is the probability of a player switching from j to i inasmuch as it is the probability of having a preference for i , while $(1 - \eta)$ is the probability of switching from i to j inasmuch as it is the probability of not having a preference for i .³ Now unlike imitating the first man on the street, the switching probabilities here are not dependent on expected payoffs. Switching in this case can then be thought of as mutant behavior, and the probabilities of switching as mutation rates. Solving equation (3) using review rates given by (5a) and (5b) and mutation rates $p_j^i = \eta$ and $p_i^j = 1 - \eta$ obtains the following dynamics:

$$\dot{x}_i = \delta\{\alpha(\eta - x_i) + \beta[u(e^i, x)x_i - \eta u(x, x)]\} \quad (6a)$$

$$\dot{x}_j = \delta\{\alpha(1 - \eta - x_j) + \beta[u(e^j, x)x_j - (1 - \eta)u(x, x)]\} \quad (6b)$$

which are regular, since the weighted sum of the growth rates is zero, or $\dot{x}_i + \dot{x}_j = 0$. (Note that (6b) can be re-written as $\dot{x}_j = \delta\{\alpha(x_i - \eta) + \beta[\eta u(x, x) - u(e^i, x)x_i]\}$

²The succeeding analysis analogously holds if the bias is for strategy j , that is, if $X(S)_2$ instead counts the number of players that inherently prefer strategy j to i .

³One can imagine that players have propensity to switch to the other strategy equal to one. Thus, it is as if potential switching is 100%, but actual switching depends on whether or not the player has a bias.

using the fact that $x_i + x_j = 1$ and that $u(x, x) = u(e^i, x)x_i + u(e^j, x)x_j$.

The dynamics (6a) and (6b), however, are not payoff-monotonic (i.e. $u(e^i, x) > u(e^j, x) \not\Rightarrow \frac{\dot{x}_i}{x_i} > \frac{\dot{x}_j}{x_j}$). Instead, strategy i spreads in the population, i.e. $\dot{x}_i > 0$, whenever:

$$\eta > x_i \frac{[\alpha - \beta u(e^i, x)]}{[\alpha - \beta u(x, x)]}. \quad (7a)$$

On the other hand, j spreads, i.e. $\dot{x}_j > 0$, while i decreases, i.e. $\dot{x}_i < 0$, whenever⁴:

$$\eta < x_i \frac{[\alpha - \beta u(e^i, x)]}{[\alpha - \beta u(x, x)]}. \quad (7b)$$

We thus have the following results:

PROPOSITION 2. *Mutation behavior can determine the equilibrium of the evolutionary game.*

Proof. Note that the stationary points of the system (7a) and (7b) are at $x_i = 0$, $x_i = 1$ and $\eta = x_i \frac{[\alpha - \beta u(e^i, x)]}{[\alpha - \beta u(x, x)]}$. Which particular equilibrium is reached depends on mutation rate η . To see this, note that if:

a. $0 < \eta < 1$:

When conditions 7a is met, then it is met at each succeeding time period until $\eta = x_i \frac{[\alpha - \beta u(e^i, x)]}{[\alpha - \beta u(x, x)]}$, or when x_i exactly reaches the value $x_i^* = \eta \frac{[\alpha - \beta u(x, x)]}{[\alpha - \beta u(e^i, x)]}$ (at which point $\dot{x}_i = 0$). Once x_i increases, it always keeps increasing until x_i^* , that is, η is binding, since $x_i \frac{[\alpha - \beta u(e^i, x)]}{[\alpha - \beta u(x, x)]}$ increases with x_i . Note that x_i^* is possibly a

⁴To obtain 7a and 7b, re-write 6a as $\delta\{\eta[\alpha - \beta u(x, x)] - \alpha x_i + \beta x_i u(e^i, x)\}$, and note that the sum of the last two terms inside the brackets $\{\}$ is always non-positive, while the first term inside $\{\}$ is always non-negative, since $\alpha \geq \beta \max[u(e^i, x), u(e^j, x)]$. Thus, $\dot{x}_i$ is non-negative whenever condition 7a is met, and non-positive whenever 7b is met.

mixed-state equilibrium, unless $\eta[\alpha - \beta u(x^*, x^*)] = [\alpha - \beta u(e^i, x^*)]$, in which case $x_i^* = 1$.

b. $\eta = 1$:

In this case, 7a always holds as $t \rightarrow \infty$, and the system *asymptotically* approaches the pure equilibrium $x_i = 1$ since $x_i \frac{[\alpha - \beta u(e^i, x)]}{[\alpha - \beta u(x, x)]} \rightarrow 1$ as $x_i \rightarrow 1$.

c. $\eta = 0$:

In this case, 7b always holds as $t \rightarrow \infty$, and the system *asymptotically* approaches the pure equilibrium $x_i = 0$ since $x_i \frac{[\alpha - \beta u(e^i, x)]}{[\alpha - \beta u(x, x)]} \rightarrow 0$ as $x_i \rightarrow 0$. ■

PROPOSITION 3. *The greater the spatial clustering of biased players, the greater the chance that their preferred strategy is selected as the equilibrium of the evolutionary game.*

Proof. It is evident that larger mutation rate η makes it easier for (7a) to be satisfied at initial t_0 , and thus, for the preferred strategy to keep growing before (a possibly mixed) equilibrium is reached. In turn, η is closer to one as biased players are more clustered, whenever $NA > [\lambda_2 V(S)]$. Analogous to the proof of Proposition 1, there are more instances when $NA > [\lambda_2 V(S)]$ than when $NA < [\lambda_2 V(S)]$, since given $\underline{NA} = [\lambda_2 V(S)]$, the range $[\underline{NA}, \infty)$ is larger than $[0, \underline{NA}]$. Also, $X(S)_2^{\lambda_{2b}} \succsim_{FSD} X(S)_2^{\lambda_{2a}}$, where the superscripts of r.v. $X(S)_2$ refer to the intensity of the particular spatial Poisson distribution from which they are drawn, and assuming $\lambda_{2b} > \lambda_{2a}$. ■

3. CONCLUSIONS

In this paper, we show how the Imitation Dynamic and, by extension, the Replicator Dynamic, can approximate the result from local interaction games that the selection of, and the rate of convergence to, equilibrium can depend on the partic-

ular structure of the network of players. In addition, not only do we introduce mutation behavior in the dynamic, but motivate it with a plausible scenario in which some players may have inherent preferences for a particular strategy. This is in contrast with most evolutionary games in which mutation is modeled as some fixed rate that describes random experimentation or mistake of players in the calculation of expected payoffs.⁵ Blume (2003) and Bergin and Lipman (1996) show that if mutation rates are fixed and 'state-independent', the long-run equilibrium corresponds to selection of the risk-dominant strategy (in coordination games) since this strategy has a deeper basin of attraction. That is, mutation does not affect the equilibrium in that the risk-dominant strategy always survives mutation behavior.⁶ In our model, however, mutation rates are not the same for all players since they can have different preferences. As a consequence, equilibrium selection depends on the mutation or the nature of the preferences. Using dynamics (6a) and (6b), it is straightforward to illustrate that for the coordination game, the risk-dominant strategy may die out if enough players are biased against it.

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⁵See, e.g. Foster and Young (1990), Friedman (1991), Fudenberg and Harris (1992), Kandori, Mailath and Rob (1993), Young (1993). For an interpretation of mutation behavior, see, for instance, Amir and Berninghaus (1996).

⁶One exception is Robson and Vega-Redondo (1996), who show that the equilibrium corresponds to the pareto-dominant strategy.

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