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An Exercise on Discrete-Time Intertemporal Optimization

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Abstract

This paper, using the different alternative methods of dynamic optimization - the *Lagrange*/*Kuhn-Tucker* (*LKT*) method, the *substitution* method, the *Hamiltonian* method, and the *dynamic programming* approach - derives the conditions that must be satisfied by the solution to the so-called Ramsey problem, hopefully in a way that can be understood by advanced undergraduate economics students. This is done by assuming that time is discrete and that, for simplicity but without loss of generality, there are only three periods.

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Keywords: Ramsey problem, dynamic optimization, Lagrange method, Substitution method, Hamiltonian method, dynamic programming.

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1. Introduction

The deterministic infinite horizon *Ramsey* model is one of the two workhorses in graduate macroeconomics.[1] In this model, the problem of the benevolent social planner who is endowed with perfect foresight, "how much should a nation save?" (Ramsey, 1928), is an intertemporal/dynamic optimization problem. Specifically, the problem is to choose the path of capital accumulation and therefore the consumption path/plan, or vice-versa, in order to maximize the lifetime utility of an infinitely-lived representative individual/family/dynasty subject to some constraints and boundary conditions.

A crucial part of characterizing/deriving the solution to the intertemporal problem turns out to be either the so-called *capital-Euler equation* or the so-called *consumption-Euler equation*. The capital-*Euler* equation, along with two boundary conditions, yields the optimal path of capital accumulation which, given the resource constraint, yields the optimal consumption path. Equivalently, the so-called *consumption-Euler equation* and the resource constraint, along with two boundary conditions, yield the optimal path of capital accumulation and consumption, the so-called *saddle path*.

Different alternative methods can used to derive/characterize the solution to the Ramsey problem. These alternative methods shall be the focus of this paper.

Although the original Ramsey (1928) problem is in continuous-time, here we use the discrete-time formulation of the problem[2]. Also, for easier tractability but without loss of generality, we assume that there only three periods. The results, derived using the different alternative methods, should give an idea on how to generalize the model to the case where the

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[1] The other work-horse is the overlapping generations (OLG) model.

[2] However, it should be noted if the problem is in discrete-time rather than in continuous-time, we have to have to decide which 'price' to use – whether this period's 'price' or the next period's 'price' – to value the capital stock carried over to the next period.

number of periods of time is large but finite and then to the case where the number of periods of time is infinite. In addition, the discrete-time results should provide intuition for their respective continuous-time analogues. This approach, which does not require mathematical sophistication, should enable us to characterize the solution to the problem.

Thus, the objective of this paper is to derive/characterize, using the different alternative methods, the solution to the Ramsey problem hopefully in a way that can be understood by undergraduate economics students.

Section 2 simply presents a deterministic discrete-time infinite-horizon *Ramsey* problem and specifies the assumptions behind the model/problem. Section 3 assumes that there are only three periods and derives the conditions that must be satisfied by the solution to the optimization problem using four alternative methods/approaches: the *Lagrange*/*Kuhn-Tucker* (*LKT*) method, the *substitution* method, the *Hamiltonian* method, and the *dynamic programming* approach. Section 4, using a concrete example, presents the solution to the problem not only for the three-period case but also for the finitely large horizon case as well as for the infinite horizon case. Finally, Section 5 gives some concluding remarks.

2. The *Ramsey* Problem

The deterministic discrete-time infinite-horizon *Ramsey* problem is

$$\left.\begin{array}{lll} \max\limits_{\{c_t\}_0^\infty} \left\{ U = \sum\limits_{t=0}^{\infty} \beta^t u(c_t) \right\}, & 0 < \beta \equiv 1/(1+\rho) < 1 & \\ s.t.\ k_{t+1} - k_t = f(k_t) - c_t - \delta k_t, & 0 < \delta < 1 & t = 0,1,\ldots \\ \quad k_0 > 0 \ given & & \\ \quad TVC & & \end{array}\right\}, \qquad (1)$$

where U is lifetime utility; ρ is the positive subjective time discount rate or pure rate of time preference which measures impatience to consume and thus $\beta (\equiv 1/(1+\rho))$ is the subjective time-preference or discount factor; c_t is per capita consumption during period *t*; $u(c_t)$ is the period-*t* utility (felicity) function which is assumed to exhibit $u'(c_t) > 0$ and $u''(c_t) < 0$ and satisfy the Uzawa-Inada conditions ($\lim_{C_t \to 0} u'(c_t) = \infty$ which ensures that $c_t > 0$, and $\lim_{c_t \to \infty} u'(c_t) = 0$); k_t (k_{t+1}) is the capital-labor ratio or per capita capital as of the beginning of period *t* (*t* + 1); per capita output during period *t* is given by the period-*t* per capita production function $f(kK_t)$ which is assumed to exhibit $f'(k_t) > 0$ and $f''(k_t) < 0$ and satisfy the Uzawa-Inada conditions ($f(0) = 0$ which since output cannot be produced without capital, $\lim_{k_t \to 0} f'(k_t) = \infty$ which ensures that $f(k_t) - \delta k_t \geq 0$ and $\lim_{k_t \to \infty} f'(k_t) = 0$), δ is the rate of physical capital depreciation, *t* and *t* + 1 are the successive discrete periods of time, and the time horizon begins at *t* = 0.[3]

The term { } or the first equation in (1) is the objective functional which is additively separable because it is a sum of functions. Each $u(c_t)$ in the sum is weighted by β^t which declines as *t* increases, indicating declining weights for future utilities. The second equation in (1) contains the sequence of resource constraints, one constraint for each period *t*. The last two are the 'boundary' conditions. $k_0 > 0$ is the initial condition on per capita capital; k_0 is given by history and cannot be chosen. *TVC* is the transversality condition.

Note that the assumptions about $u(c_t)$ and $f(k_t)$ have led to a simpler problem (1) because they imply that: $c_t > 0$ $(t = 0,1,2)$, $k_t > 0$ $(t = 1,2,)$, and the resource constraints always

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[3] In the literature, a problem such as this one is called the Ramsey model. For the original problem, see Ramsey (1928); for a Ramsey-like objective function, see Barro and Sala-i-Martin (2006, pp.214-215) for the Ramsey-like objective function. The decentralized version of the Ramsey model is also known as the Ramsey-Cass(1965)-Koopmans (1965) model.

For a discrete-time formulation of the one-sector neoclassical growth model, see Takayama (1973), Obstfeld and Rogoff (1996); for a continuous-time formulation, see Takayama (1973), Blanchard and Fischer (1989), Barro and Sala-i-Martin (2004). Also see Cass (1965) and Koopmans (1965).

bind; therefore, the non-negativity constraints on c_t and k_t can be ignored. In addition, they ensure that the *FOC*s for optimization which are necessary are also sufficient.

Although the *Ramsey* model is an infinite horizon model, the usual strategy is to consider a finite horizon (t = 0,1,2,…, *T* where *T* is the final period). The terminal condition in this finite-horizon case) may either be imposed as in the case of the *TVC* in (1) but here it is replaced by the non-negativity constraint $K_T \geq 0$ which will make the derivation of the terminal condition as part of the *K-T FOC*s and thereby provide intuition for the imposed/asserted *TVC* in the infinite horizon case.

3. Alternative Methods of Intertemporal Optimization

In this section, we derive the necessary *FOC*s, the conditions that must be satisfied by the solution to the optimization problem, using four alternative methods/approaches: the *Lagrangian* method, the *substitution* method, the *Hamiltonian* method, and the *dynamic programming* approach.

For easier tractability but without loss of generality, we assume here that there only three periods and that there is no population growth.[4] In this case, the problem, (1), becomes

$$\left.\begin{array}{ll} \max\limits_{c_0, c_1, c_2} \left\{ U = \sum\limits_{t=0}^{2} \beta^t u(c_t) = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) \right\}, & \\ s.t. \quad k_{t+1} - k_t = f(k_t) - \delta k_t - c_t, & (t = 0,1,2) \\ \qquad k_3 \geq 0, & \\ \qquad k_0 > 0 \ \textit{given}. & \end{array}\right\}. \qquad (2)$$

As shown in the next section, each of the four alternative methods of dynamic optimization yields a set of conditions consisting of a system of two non-linear first-order difference equations

[4] The results in the case where there is positive population growth are presented in the Appendix.

$$\frac{u'(c_{t-1})}{u'(c_t)} = \beta\left(f'(k_t) + 1 - \delta \right), \quad t = 1,2 \tag{3.1}$$

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t, \quad t = 0,1,2 \tag{3.2}$$

where $\left(f'(k_t) + 1 - \delta \right)$ is the marginal product of capital adjusted for physical capital depreciation, and the other notations are as defined before. (3.1) and (3.2). along with the boundary conditions,

$$k_3 = 0\,, \tag{3.3.1}$$

$$k_0 > 0\, given\,, \tag{3.3.2}$$

are the conditions that must be satisfied by the solution to the problem, (2). Whether the horizon is finite or infinite, these conditions (with the time span and the *TVC* modified accordingly) apply.

These conditions are presented at the outset not only to minimize repetitions but also to provide some intuition on the results.

(3.2), which we now refer to as 'the' resource constraint, is also called the per capita *capital accumulation* equation. It says that per capita net investment $(k_{t+1} - k_t)$ equals per capita net output $(f(k_t) - \delta k_t)$ minus per capita consumption c_t.

(3.1) is a difference equation showing the relationship between $u'(c_{t-1})$ and $u'(c_t)$ and, thus, between c_{t-1} and c_t, for t = 1,2. In macroeconomics, it is known as the *consumption-Euler* equation.

(3.1) may be rewritten as

$$-u'(c_{t-1}) + \beta u'(c_t)\left(f'(k_t) + 1 - \delta \right) = 0, \quad t = 1,2\,,$$

which states that, at the optimum, the net change in utility arising any consumption reallocation is zero. Note that one unit less of per capita consumption in period $t-1$ means having one unit more of per capita capital in period t and thus $\left(f'(k_t)+1-\delta\right)$ units more of per capita output to consume in period t. The effect on maximized utility of reducing per capita consumption in period $t-1$ by one unit is $-u'(c_{t-1})$ and that of increasing per capita consumption in period t by $\left(f'(k_t)+1-\delta\right)$ units is $u'(c_t)\left(f'(k_t)+1-\delta\right)$; but since this gain is occurs in period t, it must be discounted, yielding $\beta u'(c_t)\left(f'(k_t)+1-\delta\right)$. At the optimum, the sum of the two effects must be zero, i.e., the net discounted gain from any consumption reallocation is zero. Equivalently,

$$u'(c_{t-1}) = \beta u'(c_t)\left(f'(k_t)+1-\delta\right), \qquad t=1,2 \tag{3.1'}$$

i.e., the cost in utility for foregoing one more unit of per capita consumption in period $t-1$ and thus saving one more unit of per capita capital for period t, $u'(c_{t-1})$,

is equal to $\beta u'(c_t)\left(f'(k_t)+1-\delta\right)$, the discounted gain in utility from the increase in units of per capita consumption in period t which is due to the increase in output in period t made possible by one more unit of per capita capital in period t.

(3.1) may also be rewritten as

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = f'(k_t)+1-\delta, \qquad t=1,2 \tag{3.1''}$$

i.e., the marginal rate of substitution (*MRS*) between per capita consumption in periods $t-1$ and t ($u'(c_{t-1})/\beta u'(c_t)$) is equal to the marginal rate of transformation (*MRT*), from production, between per capita consumption in periods $t-1$ and t ($f'(k_t)+1-\delta$).[5]

In macroeconomics, (3.1) [or (3.1') or (3.1'')] is known as the *consumption-Euler*

[5] An advantage of a continuou-tme formulation over discrete-time formulation is that results come out neatly and have straightforward interpretation. Fo instance, the term $f'(k_t)-\delta+1$ in a this discrete-time formulation would simply be $f'(k_t)-\delta$ in a continuous time formulation.

equation, also called the *Keynes-Ramsey* rule (Blanchard and Fischer (1989)).

3.1 *Lagrange* (or *Lagrange-Kuhn-Tucker*) Method

In the *Lagrangian* method, the objective function and the constraints are combined into a single function called the *Lagrangian*.[6] Letting $\mu_{t+1} > 0$ as the Lagrange multiplier for the period-t resource constraint and υ as the *Lagrange* multiplier for the non-negativity constraint on the terminal stock of per capita capital, the *Lagrangian* of the full problem over all three periods (2) is

$$L = \sum_{t=0}^{2} \left\{ \beta^t u(c_t) + \mu_{t+1}\left[f(k_t) - \delta k_t - c_t - \left(k_{t+1} - k_t\right)\right]\right\} + \upsilon k_3, \tag{4}$$

where $k_0 > 0$ is given. The yet undetermined variable μ_{t+1} is interpreted as the marginal value as of time 0, or the shadow price in present value terms, of k_{t+1} in period *t*.[7]

The *FOC*s for optimality are:

$$\left.\begin{array}{llll} \partial L/\partial c_t = 0 & \Rightarrow & \beta^t u'(c_t) = \mu_{t+1}, & t = 0,1,2 \\ \partial L/\partial k_t = 0 & \Rightarrow & \mu_t = \mu_{t+1}\left(f'(k_t) + 1 - \delta\right), & t = 1,2 \\ \partial L/\partial \mu_{t+1} = 0 & \Rightarrow & k_{t+1} - k_t = f(k_t) - c_t - \delta k_t, & t = 0,1,2 \\ \left.\begin{array}{l} \partial L/\partial k_3 = 0 \quad \Rightarrow \quad \mu_3 = \upsilon \\ \upsilon \geq 0,\ \ \partial L/\partial \upsilon = k_3 \geq 0,\ \ \upsilon(\partial L/\partial \upsilon) = \upsilon k_3 = 0 \end{array}\right\rangle & \Rightarrow \mu_3 k_3 = k_3 = 0. & & \end{array}\right) \tag{5}$$

The *FOC*s with respect to c_t say that the discounted marginal utility of per capita consumption in period *t* ($\beta^t u'(c_t)$) is equal to the shadow price in present value terms of k_{t+1} in period *t* (μ_{t+1}).

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[6] Obstfeld and Rogoff (1996) also use the Lagrangian method.

[7] It should be noted if the problem is in discrete-time rather than in continuous-time, we have to have to decide which 'price' to use – whether this period's 'price' or the next period's 'price' – to value the capital stock carried over to the next period. Takayama (1973) uses this period's price. Here, following Dixit (1980), we use the next period's price.

The *FOC*s with respect to μ_{t+1} simply recover the sequence of resource constraint (the second equation in (2)). Note that the resource constraint always binds and its multiplier μ_{t+1} is positive.

In the *Lagrangian* method, the terminal condition can be derived as part of the *FOC*s. The first-order *K*uhn-Tucker condition associated with the constraint $k_3 \geq 0$ is $\upsilon(\partial L / \partial \upsilon) = \upsilon k_3 = 0$, with $\upsilon \geq 0$ and $\partial L / \partial \upsilon = k_3 \geq 0$. Substituting out for υ, which is equal to μ_3, this condition can be written as

$$\mu_3 k_3 = 0 \ .$$

This boundary condition on the terminal per capita capital, k_3, says that if the stock of per capita capital left is positive ($k_3 > 0$), then its shadow price must be zero ($\mu_3 = 0$) or, if the stock of per capita capital at the terminal time has a positive unit value ($\mu_3 > 0$), then no per capita capital must be left ($k_3 = 0$). But $\mu_3 > 0$ since $\mu_3 = u'(c_2)$ and $u'(c_2)$ >0, and $u'(c_2) > 0$ since $c_t > 0$. Thus, the condition $\mu_3 k_3 = 0$ is reduced to the terminal condition

$$k_3 = 0 \ .$$

The remaining *FOC* for the problem relates to k. But there is a problem of $k_t (t = 1,2)$ [or $k_{t+1}(t = 0,1)$] appearing in two terms of the sum (RHS of (4)). Unlike the $c's$ which have the same time subscript *t* and the $\mu's$ which have the same time subscript *t* + 1, k appears in the Lagrangian (4) as k_t and k_{t+1}. This is so because k is a dynamic variable. To avoid confusion, we therefore write the problem in expanded form:

$$\begin{aligned} L = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \mu_1\left[f(k_0) - c_0 - \delta k_0 - (k_1 - k_0)\right] \\ + \mu_2\left[f(k_1) - c_1 - \delta k_1 - (k_2 - k_1)\right] + \mu_3\left[f(k_2) - c_2 - \delta k_2 - (k_3 - k_2)\right] + \upsilon k_3. \end{aligned} \tag{4'}$$

The partial derivatives of the Lagrangian with respect to k_1 and k_2 are

$$\partial L/\partial k_1 = 0 \qquad \Rightarrow \quad \mu_1 = \mu_2\left(f'(k_1)+1-\delta\right),$$
$$\partial L/\partial k_2 = 0 \qquad \Rightarrow \quad \mu_2 = \mu_3\left(f'(k_2)+1-\delta\right),$$

which can be written in a compact form as

$$\partial L/\partial k_t = 0 \qquad \Rightarrow \quad \mu_t = \mu_{t+1}\left(f'(k_t)+1-\delta\right), \quad t = 1,2$$

which is the second equation in (5). The *FOC*s with respect to $k_t (t = 1,2)$ captures how shadow price of per capita capital, μ, changes from one period to another.

Notice that in the *Lagrangian* set-up, c_0 $[c_1]$ is chosen directly and k_1 $[k_2]$ indirectly, since k_1 $[k_2]$ adjusts indirectly as a result of the choice of c_0 $[c_1]$. Specifically, at the beginning of period 0 [1], k_0 $[k_1]$ is already given and choosing c_0 $[c_1]$ to maximize the *Lagrangian* and therefore U indirectly leads to the choice of k_1 $[k_2]$; in period 2, the final period, k_2 is already a given and the choice of c_2 becomes trivial because of the terminal condition $k_3 = 0$. But the *Lagrangian* is also maximized with respect to $k_t (t = 1,2; k_0 > 0 \, given, k_3 = 0)$ in order to yield an additional *FOC*, a condition that is needed since $\mu_{t+1} (t = 0,1,2)$ needs to be determined as well.

The *FOC*s with respect to c, k, and μ determine the optimal sequence for all three variables - (c_0, c_1, c_2), $(k_1, k_2; k_0 \text{ given and } k_3 = 0)$, and (μ_1, μ_2, μ_3). But the usual practice is to combine the *FOC*s with respect to c and the *FOC*s with respect to k as follows,

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = \frac{\mu_t}{\mu_{t+1}} = \left(f'(k_t)+1-\delta\right), \qquad t = 1,2$$

so as to get the consumption-*Euler* equation ((3.1) or (3.1’) or (3.1’’)).

Thus, the solution to (2) - $k_t (t = 1,2)$ and $c_t (t = 0,1,2)$ - must satisfy the consumption-*Euler* equation (3.1 or 3.1’ or 3.1’’), the period resource constraints (3.2), the initial condition

(3.3.1), and the terminal condition (3.3.2)).

3.2 *Substitution* Method

In the *substitution* method, the problem, (2), is converted into an unconstrained maximization problem in the choice variables $k_t\ (t=1,2)$.[8] Each of the period- t resource constraint which is binding (and therefore hold as an equality) is used to solve for the choice variable c_t as a function of k_t and k_{t+1},

$$c_t = f(k_t) - \delta k_t - (k_{t+1} - k_t), \quad t = 0,1,2$$

and this function is used to substitute for c_t in the objective function in (2). Next, in this case where the horizon is finite, the terminal condition ($k_3 = 0$) is imposed. The resulting function is then maximized with respect to $k_t\ (t=1,2)$,

$$\max_{k_1,k_2} \left\{ U = \sum_{t=0}^{2} \beta^t u\underbrace{\left(f(k_t) - \delta k_t - (k_{t+1} - k_t)\right)}_{c_t} \right\}, \tag{6}$$

where $k_0 > 0$ is taken as given and $k_3 = 0$ is imposed.

However, as in the *Lagrangian* method, there is a problem of $k_t\ (t=1,2)$ appearing in two terms of the sum (see (6') below). Specifically, k_1 appears in the c_0 and c_1 terms and k_2 appears in the c_1 and c_2 terms. This is so because the RHS of (6) is a dynamic problem. Thus, (6) should be rewritten in its expanded form:

$$\max_{k_1,k_2} \left\{ \begin{aligned} U = u\underbrace{\left(f(k_0) - \delta k_0 - (k_1 - k_0)\right)}_{c_0} + \beta u\underbrace{\left(f(k_1) - \delta k_1 - (k_2 - k_1)\right)}_{c_1} \\ + \beta^2 u\underbrace{\left(f(k_2) - \delta k_2 - (k_3 - k_2)\right)}_{c_2} \end{aligned} \right\}, \tag{6'}$$

[8] Obstfeld and Rogoff (1996) use this so-called substitution method.

where $k_0 > 0$ *given* and $k_3 = 0$.

The *FOC*s wrt $k_t\,(t = 1,2)$ are

$$\partial U/\partial k_1 = u'\underbrace{\left(f(k_0) - \delta k_0 - (k_1 - k_0)\right)}_{C_0}(-1) + \beta u'\underbrace{\left(f(k_1) - \delta k_1 - (k_2 - k_1)\right)}_{C_1}(f'(k_1) + 1 - \delta) = 0,$$

$$\partial U/\partial k_2 = \beta u'\underbrace{\left(f(k_1) - \delta k_1 - (k_2 - k_1)\right)}_{C_1}(-1) + \beta^2 u'\underbrace{\left(f(k_2) - \delta k_2 - (k_3 - k_2)\right)}_{C_2}(f'(k_2) + 1 - \delta) = 0,$$

which can now be written in a compact form as

$$\begin{aligned}\partial U/\partial k_t = \beta^{t-1}&\underbrace{u'\left(f(k_{t-1}) - \delta k_{t-1} - (k_t - k_{t-1})\right)}_{c_{t-1}}(-1)\\ &+ \beta^t u'\underbrace{\left(f(k_t) - \delta k_t - (k_{t+1} - k_t)\right)}_{c_t}(f'(k_t) + 1 - \delta) = 0, \qquad t = 1,2\end{aligned} \tag{7.1}$$

yielding the Euler equation

$$\frac{u'\overbrace{\left(f(k_{t-1}) - \delta k_{t-1} - (k_t - k_{t-1})\right)}^{c_{t-1}}}{\beta u'\underbrace{\left(f(k_t) - \delta k_t - (k_{t+1} - k_t)\right)}_{c_t}} = f'(k_t) + 1 - \delta, \qquad t = 1,2 \tag{7.2}$$

where $k_0 > 0$ is given and $k_3 = 0$.[9] Clearly, the *FOC*s on $k_t\,(t = 1,2)$ can be rewritten to yield the consumption-*Euler* equation ((3.1) or (3.1’) or (3.1’’)).

As in the *Lagrangian* method, the ‘solution’ to (2) - $k_t\,(t = 1,2)$ and $c_t\,(t = 0,1,2)$ - must satisfy the consumption-*Euler* equation, the period resource constraints, the initial condition, and the terminal condition.

In the *substitution* method, the unconstrained problem of choosing $k_t\,(t = 1,2)$, with $k_0 > 0$ given and $k_3 = 0$ imposed, to maximize U can be thought of as one where $c_t\,(t = 0,1,2)$ is optimally chosen. This is so because at the beginning of period 0 [1], k_0 [k_1] is

9 (7.2) is a second-order difference equation in *k*.

already given and therefore choosing k_1 [k_2] to maximize U implicitly pins down the optimal c_0 [c_1]; at the beginning of period 2, the final period, k_2 is also already given and it is the terminal condition $k_3 = 0$ which implicitly pins down c_2.

3.3 *Hamiltonian* Method

Here, the *Lagrangian* function (4) for the problem (2), rewritten below:[10]

$$L = \sum_{t=0}^{2} \left\{ \beta^t u(c_t) + \mu_{t+1}\left[f(k_t) - \delta k_t - c_t \right] - \mu_{t+1}\left(k_{t+1} - k_t\right) \right\} + \upsilon k_3 . \qquad (8)$$

is used to derive the so-called '*Hamiltonian* recipe' for dynamic optimization.[11]

In the discussion of the *Lagrangian* method in the previous subsection, it is shown that the *FOC*s with respect to $c_t \ (t = 0,1,2)$ and the *FOC*s with respect to $\mu_{t+1} \ (t = 0,1,2)$, which recover the period resource constraints, can easily be derived even without expanding the *Lagrangian* because the $c's$ have the same time subscript *t* and the $\mu's$ have the same time subscript *t* + 1.[12] But this is not so in the case of k which appears as k_t and k_{t+1}. Specifically, $k_t \ (t = 1,2)$ appears in two terms of the sum in (8). For instance, k_1 appears as $\mu_2 k_1$ in the term *t* = 1 and as $-\mu_1 k_1$ in the term *t* = 0 and, thus, getting $\partial L / \partial k_1$ is not that straightforward unless the *Lagrangian* is written in expanded form.

After setting up the Lagrangian, the next step then in making the so-called Hamiltonian recipe is to make $k_t \ (t = 1,2)$ appear only in one term of the sum in (8). Specifically, the term

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[10] Having derived the terminal condition using the *LKT* method, the term υk_3 in (8) can be dropped and the terminal condition can be simply asserted or imposed in order to simplify the problem. However, it is retained in (8).

[11] For details, see Arrow and Kurz (1970), Dixit (1980), Takayama (1973), Dorfman (1969), Intrilligator (1971), Kamien and Schwartz (1981), Chiang (1992), and Pontryagin et al (1962). Blanchard and Fischer (1989) and Barro and Sala-i-Martin (2004) use the Hamiltonian most of the time.

[12] Again, Takayama (1973) uses this period's price; here, following Dixit (1980), we use the next period's price.

$-\sum_{t=0}^{2}\mu_{t+1}\left(k_{t+1}-k_t\right)$ should be rewritten as follows:

$$\begin{aligned}-\sum_{t=0}^{2}\mu_{t+1}(k_{t+1}-k_t) &= \sum_{t=0}^{2}\mu_{t+1}(k_t-k_{t+1}) \\ &= \mu_1(k_0-k_1)+\mu_2(k_1-k_2)+\mu_3(k_2-k_3) \\ &= (\mu_2-\mu_1)k_1+(\mu_3-\mu_2)k_2-(\mu_3k_3-\mu_1k_0),\end{aligned}$$

or, in a compact form,

$$-\sum_{t=0}^{2}\mu_{t+1}(k_{t+1}-k_t) = -(\mu_3k_3-\mu_1k_0)+\sum_{t=1}^{2}(\mu_{t+1}-\mu_t)k_t\ , \tag{8.1}$$

where k_0 and k_3 are *not* choice variables.

The next step is to define a function, called the *Hamiltonian*, as follows:

$$H(k_t,c_t,\mu_{t+1})\equiv\beta^t u(c_t)+\mu_{t+1}\left[f(k_t)-\delta k_t-c_t\right]. \tag{9}$$

The choice of c_t affects k_{t+1} via the period-*t* resource constraint, and this effect of c_t on k_{t+1} equals its effect on $(f(k_t)-\delta k_t-c_t)$. Multiplying this effect by μ_{t+1} gives the resulting change in the objective function. The product $\mu_{t+1}(f(k_t)-c_t-\delta k_t)$ which captures such resulting change is then added to the term $\beta^t u(c_t)$, thereby yielding the *Hamiltonian* (9).[13]

Using (8.1) and the *Hamiltonian* (9), the *Lagrangian* (8) can now be rewritten as

[13] (9), the present (discounted) value Hamiltonian can be rewritten as $H(k_t,c_t,\mu_{t+1})\equiv\beta^t\left\{H^{\mathrm{cv}}(k_t,c_t,\lambda_{t+1})\right\}$, where $H^{\mathrm{cv}}(k_t,c_t,\lambda_{t+1})\equiv u(c_t)+\lambda_{t+1}[f(k_t)-\delta k_t-c_t]$ is the current-value Hamiltonian and $\lambda_{t+1}\equiv\mu_{t+1}/\beta^t$ is the marginal value, in terms current 'utils', of $(k_{t+1}-k_t)$ in period *t*. $H^{\mathrm{cv}}(k_t,c_t,\lambda_{t+1})$ can be interpreted as per capita net national product, $u(c_t)$ as per capita consumption, and $\lambda_{t+1}(f(k_t)-\delta k_t-c_t)$, equal to $\lambda_{t+1}(k_{t+1}-k_t)$ when the period-*t* is binding, as per capita net investment (all measured in current 'utils').

$$L = \sum_{t=0}^{2} \left\{ \underbrace{\beta^t u(c_t) + \mu_{t+1} \left[f(k_t) - \delta k_t - c_t \right]}_{H(...)} - \mu_{t+1} \left(k_{t+1} - k_t \right) \right\} + \upsilon k_3$$
$$= \sum_{t=0}^{2} H(k_t, c_t, \mu_{t+1}) + \sum_{t=1}^{2} (\mu_{t+1} - \mu_t) k_t - (\mu_3 k_3 - \mu_1 k_0) + \upsilon k_3 \tag{10}$$
$$= \sum_{t=1}^{2} \left\{ H(k_t, c_t, \mu_{t+1}) + (\mu_{t+1} - \mu_t) k_t \right\} + u(c_0) + \mu_1 \left[f(k_0) - \delta k_0 - c_0 \right] - (\mu_3 k_3 - \mu_1 k_0) + \upsilon k_3,$$

where $k_0 > 0$, which is historically given, is *not* a choice variable. Now, one can now easily get the *FOC*s with respect to $k_t \, (t = 1,2)$. Although it is now μ which has different time subscripts t and $t + 1$, this does not pose a problem provided because the period-t resource constraint is satisfied. Note that the *FOC*s with respect to $\mu_{t+1} \, (t = 0,1,2)$ merely recover the period-t resource constraint. With a formulation such as (10), the problem of choosing $c_t \, (t = 0,1,2)$ and $k_t \, (t = 1,2)$ is now a single-period optimization problem.[14]

The *FOC*s are

$$\left.\begin{array}{ll} \partial L / \partial c_t = \partial H / \partial c_t = 0, & t = 0,1,2 \\ \partial L / \partial k_t = \partial H / \partial k_t + (\mu_{t+1} - \mu_t) = 0, & t = 1,2 \\ \partial L / \partial \mu_{t+1} = \partial H / \partial \mu_{t+1} - (k_{t+1} - k_t) = 0, \, t = 0,1,2 & \\ \partial L / \partial k_3 = -\mu_3 + \upsilon = 0, & \\ \upsilon \geq 0, \partial L / \partial \upsilon = k_3 \geq 0, \upsilon (\partial L / \partial \upsilon) = 0, & \\ k_0 > 0 \; given, & \end{array}\right\} \tag{11}$$

which can be rewritten as

---

[14] $c_t \, (t = 0,...,T)$ is the 'control' variable in period t, $k_t \, (t = 0,...,T+1)$ is the 'state' variable in period t,, $\mu_{t+1} \, (t = 0,...,T)$ is the 'co-state' or 'auxiliary' variable or Lagrange multiplier in period t, $U(c_0,...,c_T)$ is the objective function, $(f(k_t) - \delta k_t - c_t) - (k_{t+1} - k_t) \geq 0 \; (t = 0,...,T; k_0 \; given)$ is the period-t constraint linking the control variable c_t and the state variable k_t in period t or the 'transition' equation in period t; in this paper, $t = 0,1,2$ .and thus $T = 2$.

$$\left.\begin{array}{ll}\partial H/\partial c_t = 0, & t=0,1,2\\ \mu_{t+1}-\mu_t = -\partial H/\partial k_t, & t=1,2\\ k_{t+1}-k_t = \partial H/\partial \mu_{t+1}, & t=0,1\\ \mu_3 k_3 = 0,\ k_0>0\ given. & \end{array}\right\} \quad (11')$$

The *FOC*s for the maximization of the objective function s.t. to the period resource constraints and the boundary conditions (see (2)), are: for each period *t* (= 0,1,2), c_t maximizes the *Hamiltonian* $H(c_t, k_t, \mu_{t+1})$ and the changes in k_t and μ_{t+1} obey the difference equations in (11') and must satisfy the initial condition $k_0 > 0$ given and the terminal condition $k_3 = 0$. This is Pontryagin et al's *maximum principle* (see Pontryagin et al (1962), Arrow and Kurz (1970), and Dixit (1980)).

Using the *Hamiltonian* recipe and the *Hamiltonian*, the *FOC*s, (11'), can be rewritten as

$$\left.\begin{array}{lll}\partial H/\partial c_t = 0: & \beta^t u(c_t) = \mu_{t+1}, & t=0,1,2\\ \mu_{t+1}-\mu_t = -\partial H/\partial k_t: & \mu_{t+1}-\mu_t = -\mu_{t+1}\left(f'(k_t)-\delta\right), & t=1,2\\ k_{t+1}-k_t = \partial H/\partial \mu_{t+1}: & k_{t+1}-k_t = f(k_t)-\delta k_t - c_t, & t=0,1,2\\ \mu_3 k_3 = 0,\ k_0>0\ given, & & \end{array}\right\}, \quad (12)$$

which are of course the same as those using the *Lagrangian* method (see (5)).[15]

The *FOC*s with respect to k_t (*t* = 1,2), the second equation in (12), indicate that

$$(\mu_{t+1}-\mu_t)+\mu_{t+1}\left(f'(k_t)-\delta\right)=0,$$

i.e., the change in the value of a unit of per capita capital or "capital gains" $(\mu_{t+1}-\mu_t)$ plus the value of the marginal net return on a unit of per capita capital or "dividends" $(\mu_{t+1}(f'(k_t)-\delta))$ must be zero.[16] When k_t $(t=1,2)$ is optimal, the overall return $(((\mu_{t+1}-\mu_t)+\mu_{t+1}(f'(k_t)-\delta))$

[15] Note that the second equation in (12) can be rewritten as the second equation in (5) since $\mu_{t+1}-\mu_t = -\mu_{t+1}\left(f'(k_t)-\delta\right) \Rightarrow \mu_t = \mu_{t+1}\left(f'(k_t)+1-\delta\right) \Rightarrow \mu_t/\mu_{t+1} = \left(f'(k_t)+1-\delta\right).$

[16] These are in present (discounted) value terms. Since $\lambda_{t+1} \equiv \mu_{t+1}/\beta^t$, the $\mu_{t+1}-\mu_t$ equation can be converted

should be zero. In other words, the shadow prices take values that do not allow for an excess return from holding the stock; this is an intertemporal no-arbitrage condition.[17]

Again, the *FOC*s with respect to c_t and the *FOC*s with respect to k_t, the first and second equations in (12), can be combined

$$\mu_{t+1} - \mu_t = -\mu_{t+1}\left(f'(k_t) - \delta\right)$$

$$\underbrace{\beta^t u'(c_t)}_{\mu_{t+1}} - \underbrace{\beta^{t-1} u'(c_{t-1})}_{\mu_t} = -\underbrace{\beta^t u'(c_t)}_{\mu_{t+1}}\left(f'(k_t) - \delta\right)$$

$$\beta^{t-1} u'(c_{t-1}) = \beta^t u'(c_t)\left(f'(k_t) + 1 - \delta\right)$$

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = f'(k_t) + 1 - \delta, \qquad t = 1,2$$

to yield the consumption-*Euler* equation ((3.1) or (3.1') or (3.1'')), as shown above.

Thus, as in the *Lagrangian* method and the *substitution* method, the solution to (2) – $k_t\ (t = 1,2)$ and $c_t\ (t = 0,1,2)$ – must satisfy the consumption-*Euler* equation, the period resource constraints, the initial condition, and the terminal condition.

3.4 *Dynamic Programming*

In the *Hamiltonian* method, the full problem over all periods is reduced to a single-period (static) optimization problem. In contrast, in the *dynamic programming* approach, the full *T*-period intertemporal problem is broken into *T* separate static (two-period, but effectively single-period) optimization problems. This method of optimization over time as a sequence/succession of static optimization problems is known as *Bellman's Dynamic Programming*.[18]

To see how this approach works, choose any *t* and consider the decision about c_t in period *t*. Any particular choice of c_t will lead to next period's per capita capital stock k_{t+1} (see the

---

into a $\lambda_{t+1} - \lambda_t$ equation, and the interpretation is exactly that one given in Finance.

[17] See Dixit (1980), Kamien and Schwartz (1981), and Dorfman (1969).

[18] For details, see Arrow and Kurz (1970), Dixit (1980), Sargent (1987), and Bellman (1957).

period *t* budget constraint and note that k_t is the per capita capital stock as of the beginning of period *t* and therefore is taken as given during period *t*). Thereafter, it remains to solve the sub-problem starting at *t* + 1, and achieve the maximum value $V_{t+1}(k_{t+1})$. Then the total value starting with k_t at *t* can be broken down into two terms: $u(c_t)$ that accrues at once, and $\beta V_{t+1}(k_{t+1})$ that accrues thereafter. The choice of c_t should maximize the sum of these two terms, i.e., $u(c_t) + \beta V_{t+1}(k_{t+1})$ for this one *t*. This is *Bellman's principle of optimality*: "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." (Bellman (1957, p. 83)).[19]

With finite (infinite) horizon, this involves choosing a finite (an infinite) sequence of per capita consumption or per capita capital accumulation, one for each period *t*. But the problem of solving for a finite (an infinite) sequence can be replaced by the problem of solving for a single unknown function *W*, a value function.

3.4.1 *Dynamic Programming* by Backward Recursion

Here, the full 3-period intertemporal problem can be broken into 3 separate static (single-period) problems. The sequence of problems can be solved either forward or backward. With finite horizon, it is easier to solve the problem backward, as illustrated below:

[19] In other words, "an individual who plans to optimize starting tomorrow can do no better today than to optimize taking the future optimal plans as given" (Obstfeld and Rogoff (1996)).

$$V(k_0) \equiv \max_{\substack{c_0, c_1, c_2 \\ k_0\, given, k_3=0}} \left\{ u(c_0) + \beta u(c_1) + \beta^2 u(c_2) \right\}$$

$$\equiv \max_{\substack{c_0 \\ k_0\, given}} \left\{ u(c_0) + \beta \underbrace{\left[\max_{\substack{c_1 \\ k_1\, given}} \left(u(c_1) + \beta \underbrace{\left(\max_{\substack{c_2 \\ k_2\, given, k_3=0}} \left(u(c_2) + \beta \overbrace{V(k_3)}^{=0} \right) \right)}_{V(k_2)} \right) \right]}_{V(k_1)} \right\}, \quad (13)$$

where each maximization problem is subject to the relevant period constraint.[20] (13), which shows a method of optimization over time as a succession of static optimization problems is the essence of Bellman's dynamic programming.

Period- T (Last Period) Problem: *Period-* 2 *Problem*. The period-2 problem is

$$W(k_2) \equiv \max_{\substack{c_2 \\ s.t. \\ k_3 - k_2 = f(k_2) - c_2 - \delta k_2 \\ k_2\ given, k_3 = 0}} \left\{ u(c_2) + \beta W(k_3) \right\} \equiv \max_{\substack{c_2 \\ s.t. \\ k_3 - k_2 = f(k_2) - c_2 - \delta k_2 \\ k_2\ given, k_3 = 0}} u(c_2)\ , \quad (14.1)$$

since $k_3 = 0$. The RHS of (14.1) is a straightforward static optimization problem, yielding the optimal choice[21]

$$c_2{*} = f(k_2) + (1-\delta)k_2\ , \quad (14.2)$$

and the maximum value function

$$V(k_2) \equiv u(c_2{*}) = u[f(k_2) + (1-\delta)k_2] \quad (14.3)$$

which can be used in the *T*-1 problem. Note from (14.3) that

$$\partial V(k_2)/\partial k_2 = u'(c_2)(f'(k_2) + 1 - \delta). \quad (14.4)$$

Period- T-1 *Problem*: *Period*-1 *problem*. The period-1 problem is given by the RHS of the

---

[20] c_t is the 'control' variable in period *t*, k_t is the 'state' variable in period *t*, $u(c_t)$ is the one-period return function in period *t*, and $(f(k_t) - c_t - \delta k_t) = (k_{t+1} - k_t)$ is the one-period transition' equation in period *t*, as in note 10. See Sargent (1987).

[21] The period- 2 resource constraint, noting that $k_3 = 0$ and k_2 is given as of the beginning of period 2, yields the value of $c_2{*}$ (14.2).

equation below,

$$V(k_1) \equiv \max_{\substack{c_1 \\ s.t. k_2 - k_1 = f(k_1) - c_1 - \delta k_1 \\ k_1 \; given}} \left\{ u(c_1) + \beta V(k_2) \right\}. \tag{15.1}$$

Differentiating (15.1) with respect to c_1 and setting the result equal to zero gives

$$\frac{\partial V(k_1)}{\partial c_1} = u'(c_1) + \beta \frac{\partial V(k_2)}{\partial k_2} \frac{\partial k_2}{\partial c_1} = 0$$

and noting from the period-1 resource constraint that $\partial k_2 / \partial c_1 = -1$ yields the *FOC* wrt c_1,

$$u'(c_1) = \beta \frac{\partial V(k_2)}{\partial k_2} \; . \tag{15.2}$$

Combining (14.4), $\partial V(k_2) / \partial k_2 = u'(c_2)(f'(k_2) + 1 - \delta)$, and (15.2) gives

$$\frac{u'(c_1)}{\beta u'(c_2)} = f'(k_2) + 1 - \delta , \tag{15.3}$$

and gives the value function

$$V(k_1) \equiv u(c_1^*) + \beta V(k_2) , \tag{15.4}$$

where

$$\begin{aligned} \frac{\partial V(k_1)}{\partial k_1} &= u'(c_1) \underbrace{\left[(f'(k_1) + 1 - \delta) - \frac{\partial k_2}{\partial k_1} \right]}_{= \partial c_1 / \partial k_1 , \text{using the period-1 resource constraint}} + \beta \frac{\partial V(k_2)}{\partial k_2} \frac{\partial k_2}{\partial k_1} \\ &= u'(c_1)(f'(k_1) + 1 - \delta) + \underbrace{\left[\beta \frac{\partial V(k_2)}{\partial k_2} - u'(c_1) \right]}_{=0, \text{ using the } FOC \text{ on } c_1 (15.2)} \frac{\partial k_2}{\partial k_1} \\ &= u'(c_1)(f'(k_1) + 1 - \delta), \end{aligned} \tag{15.5}$$

$$\frac{\partial V(k_1)}{\partial c_1} = u'(c_1)\frac{\partial c_1}{\partial k_1} + \beta\frac{\partial V(k_2)}{\partial k_2}\underbrace{\left[\left(f'(k_1)+1-\delta\right)-\frac{\partial c_1}{\partial k_1}\right]}_{=\frac{\partial k_2}{\partial k_1},\text{, using the period-1 resource constraint}}$$

$$= \beta\frac{\partial V(k_2)}{\partial k_2}\left(f'(k_1)+1-\delta\right) + \underbrace{\left[u'(c_1)-\beta\frac{\partial V(k_2)}{\partial k_2}\right]}_{=0\text{, using the FOC on } c_1 \text{ (15.2)}}\frac{\partial c_1}{\partial k_1} \qquad (15.6)$$

$$= \beta\frac{\partial V(k_2)}{\partial k_2}\left(f'(k_1)+1-\delta\right).$$

Period-0 *Problem.* The period-0 problem is given by the RHS of the equation below,

$$V(k_0) \equiv \max_{\substack{c_0 \\ s.t.\, k_1-k_0=f(k_0)-c_0-\delta k_0 \\ k_0\ given}} \left\{u(c_0)+\beta V(k_1)\right\}. \qquad (16.1)$$

Using (16.1),

$$\frac{\partial V(k_0)}{\partial c_0} = u'(c_0) + \beta\frac{\partial V(k_1)}{\partial k_1}\frac{\partial k_1}{\partial c_0} = 0 \ .$$

Noting from the period-1 resource constraint that $\partial k_1 / \partial c_0 = -1$, (16.2) yields the *FOC* wrt c_0,

$$u'(c_0) = \beta\frac{\partial V(k_1)}{\partial k_1} \ . \qquad (16.2)$$

Combining (15.5), $\partial V(k_1)/\partial k_1 = u'(c_1)(1+f'(k_1)-\delta)$, and (16.2) yields

$$\frac{u'(c_0)}{\beta u'(c_1)} = f'(k_1)+1-\delta, \qquad (16.3)$$

and gives the value function

$$V(k_0) \equiv u(c_0{*}) + \beta V(k_1) \ , \qquad (16.4)$$

where $V(k_0)$ is the maximized value of lifetime utility $U*$ when the starting level of per capita capital is k_0, since

$$
\begin{aligned}
V(k_0) &\equiv u(c_0^*) + \beta[\underbrace{u(c_1^*) + \underbrace{\beta u(c_2^*)}_{=V(k_2),\ from(14.3)}}_{=V(k_1),\ from(15.4)}] \\
&= u(c_0^*) + \beta u(c_1^*) + \beta^2 u(c_2^*) \\
&= U^*,
\end{aligned}
$$

where $u(c_0^*) = u(c_0(k_0))$, $u(c_1^*) = u(c_1(k_1(k_0)))$, and $u(c_2^*) = u(c_2(k_2(k_1(k_0))))$.[22]

Notice that (15.2) and (16.2) can be written in a compact form as

$$
u'(c_t) = \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}}, \quad t = 0,1 \quad \text{or} \quad u'(c_{t-1}) = \beta \frac{\partial V(k_t)}{\partial k_t}, \quad t = 1,2
$$

which is the *FOC* wrt c. Similarly, notice that (15.3) and (16.3) can be written in compact form as

$$
\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta, \quad t = 0,1 \quad \text{or} \quad \frac{u'(c_{t-1})}{\beta u'(c_t)} = f'(k_t) + 1 - \delta, \quad t = 1,2
$$

the consumption-*Euler* equation.

3.4.2 *Dynamic Programming*: Recipe

Step 1. Write the problem in terms of the Bellman equation. The optimization problem, (2), can be written as

$$
V(k_t) \equiv \max_{c_t} \left(u(c_t) + \beta V(k_{t+1}) \right), \tag{17}
$$

subject to $k_{t+1} - k_t = f(k_t) - \delta k_t - c_t$ k_t *given*, $k_0 > 0$ *given*, $k_{T+1} = 0$. $V(k_t)$ is the value of today's per capita capital stock k_t and $\beta V(k_{t+1})$ is the value of tomorrow's per capita capital stock k_{t+1}, where $V(k_{t+1})$ is an unknown function. (17) is known as the Bellman equation while $V(.)$ known as the Bellman value function.

Step 2. Derive the FOCs on c_t. Using the Bellman equation (17) in period *t* and the

[22] This can be shown using iterative procedure.

period-t resource constraint to substitute for k_{t+1}, and differentiating with respect to c_t and setting the result equal to zero,

$$\frac{\partial V(k_t)}{\partial c_t} = u'(c_t) + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial c_t} = 0,$$

yield, noting from the period-t resource constraint that $\partial k_{t+1} / \partial c_t = -1$, the *FOC* wrt c_t:

$$u'(c_t) = \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}}, \; t = 0,1. \tag{18.1}$$

Foregoing a unit of per capita consumption in period t or carrying over a unit of per capita capital to the next period has cost and benefit. At an optimum, the marginal cost $u'(c_t)$ must equal the marginal benefit $\beta(\partial V(k_{t+1}) / \partial k_{t+1})$. Note however that $\partial V(k_{t+1}) / \partial k_{t+1}$ is unknown since $V(k_{t+1})$ is an unknown function.

Step 3. Derive the envelope relation between $\partial V(k_t) / \partial k_t$ and $\partial V(k_{t+1}) / \partial k_{t+1}$. Using again the Bellman equation (17) in period t and the period-t resource constraint to substitute for k_{t+1}, and differentiating with respect to k_t and applying the envelope theorem on the result,

$$\begin{aligned}\frac{\partial V(k_t)}{\partial k_t} &= u'(c_t) \frac{\partial c_t}{\partial k_t} + \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} \underbrace{\left[\left(f'(k_t) + 1 - \delta \right) - \frac{\partial c_t}{\partial k_t} \right]}_{= \frac{\partial k_{t+1}}{\partial k_t}, \text{using the period}-t \text{ resource constraint}} \\ &= \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} \left(f'(k_t) + 1 - \delta \right) + \underbrace{\left[u'(c_t) - \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} \right]}_{=0, \text{ using the FOC on } c_t \text{ (18.1)}} \frac{\partial c_t}{\partial k_t},\end{aligned}$$

yield

$$\frac{\partial V(k_t)}{\partial k_t} = \beta \left(f'(k_t) + 1 - \delta \right) \frac{\partial V(k_{t+1})}{\partial k_{t+1}} \; . \tag{18.2}$$

(18.2) is the envelope relation between $\partial V(k_t) / \partial k_t$ and the yet unknown $\partial V(k_{t+1}) / \partial k_{t+1}$.

Step 3. Derive the consumption-Euler equation using the FOCs and the envelope relation. Using the *FOC* wrt c_t (18.1) lagged one period, the envelope result (18.2), and the *FOC* wrt c_t again,

$$u'(c_{t-1}) = \beta \underbrace{\left(\beta \left(\frac{\partial V(k_{t+1})}{\partial k_{t+1}} \right) \left(f'(k_t) + 1 - \delta \right) \right)}_{=\partial V(k_t)/\partial k_t} \frac{\partial k_t}{\partial c_{t-1}}$$

$$= \beta \underbrace{\left(\beta \left(\frac{\partial V(k_{t+1})}{\partial k_{t+1}} \right) \underbrace{\frac{\partial k_t}{\partial c_{t-1}}}_{=-1} \right)}_{u'(c_t)} \left(f'(k_t) + 1 - \delta \right), \quad t = 1, 2,$$

yields for t = 1, 2,

$$u'(c_{t-1}) = \beta u'(c_t)\left(f'(k_t) + 1 - \delta \right) \text{ or } \frac{u'(c_{t-1})}{\beta u'(c_t)} = f'(k_t) + 1 - \delta \text{ or } \frac{u'(c_{t-1})}{u'(c_t)} = \beta\left(f'(k_t) + 1 - \delta \right), \quad (18.3)$$

which is exactly the consumption-*Euler* equation derived earlier. This equation, as in the other three methods presented earlier, is part of the set of conditions that must be satisfied by the solution to (2).

Recall that the *FOC* wrt c_t is $\mu_{t+1} = \beta^t u'(c_t)$ using either the *Lagrangian* method or the *Hamiltonian* method and $u'(c_t) = \beta(\partial V(k_{t+1}) / \partial k_{t+1})$ using the *dynamic programming* approach. Along with the defintion $\lambda_{t+1} \equiv \frac{\mu_{t+1}}{\beta^t}$, these imply that

$$\underbrace{\beta^t \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}}}_{u(c_t)} = \beta^t \underbrace{u(c_t)}_{\lambda_{t+1}} = \beta^t \lambda_{t+1} \equiv \mu_{t+1} \Leftrightarrow \beta \frac{\partial V(k_{t+1})}{\partial k_{t+1}} = u(c_t) = \lambda_{t+1} \equiv \frac{\mu_{t+1}}{\beta^t}, \quad (19.1)$$

where μ_{t+1} is the shadow price of k_{t+1} in present value terms while λ_{t+1} is the shadow price of k_{t+1} in current value terms or in terms of current utility in period *t*. That $u(c_t) = \lambda_{t+1}$ is not surprising since (2) is an aggregative neoclassical model where there is only one type of good in the economy, which can be consumed or invested.[23]

Finally, note also that

$$
\begin{aligned}
&W(k_t) = \frac{V(k_t)}{\beta} \Leftrightarrow V(k_t) = \beta W(k_t) \\
&\Rightarrow W(k_t) = \frac{V(k_t)}{\beta} \Leftrightarrow V(k_{t+1}) = \beta W(k_{t+1}),
\end{aligned}
\tag{19.2}
$$

where $W(.)$ is a current-value value function while $V(.)$ is a present-value value function (Arrow and Kurz (1970)).

4. An Example with A Closed-Form Solution

This section presents the solution to a simplified intertemporal problem. Here we want the optimization problem to have a closed-form solution, so we consider a special case, following Brock and Mirman (1972), where the following are assumed: a *Cobb-Douglas* production ($f(k_t) = Ak_t^{\alpha}, A > 0, 0 < \alpha < 1$), a logarithmic utility function ($u(c_t) = \ln c_t$), no population growth ($n = 0$), and full physical capital depreciation ($\delta = 1$). In this case, the resource constraint is simply $Ak_t^{\alpha} = c_t + k_{t+1}$.

Based on the *FOC*s derived above, the solution in this special case must satisfy a system of first-order non-linear difference equations,

$$
\frac{c_{t+1}}{c_t} = \beta\alpha A k_{t+1}^{\alpha-1} \;\Leftrightarrow\; c_{t+1} - c_t = [\beta\alpha A_{t+1}^{\alpha-1} - 1]c_t, \qquad t = 0,1,... \tag{20.1}
$$

$$
k_{t+1} = Ak_t^{\alpha} - c_t, \qquad t = 0,1,2,... \tag{20.2}
$$

or, equivalently, by combining (20.1) and (20.2), a single second-order non-linear difference equation

[23] An example is coconut.

$$\frac{\overbrace{k_{t+1}^{\alpha}-k_{t+2}}^{c_{t+1}}}{\underbrace{k_{t}^{\alpha}-k_{t+1}}_{c_t}}=\beta\alpha AK_{t+1}^{\alpha-1}, \qquad t=0,1,... \tag{21}$$

with two boundary conditions [$k_0 > 0\, given$ as the initial condition and, with infinite horizon (finite horizon), $\lim_{t\to\infty}\mu_{t+1}k_{t+1}=0$ $(k_{T+1}=0)$ as the *TVC* (terminal condition)].

In this special case, it can be shown that the optimal per capita consumption sequence/path and per capita capital sequence/path are: [24]

(i) when there are only two periods,

$$c_0=\frac{1}{1+\alpha\beta}Ak_0^{\alpha}, \quad c_1=Ak_1^{\alpha},\ k_1=\frac{\alpha\beta}{1+\alpha\beta}Ak_0^{\alpha},\ k_0>0\, given,\ k_2=0 \ ; \tag{22.1}$$

(ii) when there are three periods,

$$\left.\begin{array}{l} c_0=\dfrac{1}{1+\alpha\beta+(\alpha\beta)^2}Ak_0^{\alpha}, \quad c_1=\dfrac{1}{1+\alpha\beta}Ak_1^{\alpha}, \quad c_2=Ak_2^{\alpha}, \\ k_1=\dfrac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}Ak_0^{\alpha}, \quad k_2=\dfrac{\alpha\beta}{1+\alpha}Ak_1^{\alpha}, \quad k_0>0\, given,\ k_3=0 \end{array}\right\} ; \tag{22.2}$$

(iii) when the horizon is finite, in general,[25]

$$c_t=(1-\alpha\beta)\frac{1}{1-(\alpha\beta)^{T-t+1}}Ak_t^{\alpha},\ k_{t+1}=\alpha\beta\frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}Ak_t^{\alpha},\ k_0>0\, given,\ k_{T+1}=0\ ; \tag{22.3}$$

(iv) and when the horizon is infinite,[26]

$$c_t=(1-\alpha\beta)Ak_t^{\alpha},\ k_{t+1}=\alpha\beta Ak_t^{\alpha}\ \ k_0>0\, given,\ \lim_{T\to\infty}\mu_{T+1}k_{T+1}=0\ (TVC)\ . \tag{22.4}$$

---

[24] The results presented below can be derived using iterative procedure. Another procedure is the guess-and-verify method or the method of undetermined coefficients which will work only in two classes of specifications of preferences and constraints: (i) linear constraints and quadratic preferences or (ii) Cobb-Douglas constraints and logarithmic preferences (Sargent, (1987, p. 22)).

[25] The 3-period results can be extended to *T*-period results by noting that $1+(\alpha\beta)+(\alpha\beta)^2+...+(\alpha\beta)^{T-t-1}+(\alpha\beta)^{T-t}=(1-(\alpha\beta)^{T-t+1})/(1-(\alpha\beta))$ and that $\alpha\beta+...+(\alpha\beta)^{T-t}=\alpha\beta(1+(\alpha\beta)+(\alpha\beta)^{T-t-1}=(1-(\alpha\beta)^{T-t})/(1-\alpha\beta)$.

[26] The results under finite horizon can be extended to infinite horizon by taking the limit as $T\to\infty$.

When there are only two periods (t = 0,1) – the present (today) and the future (tomorrow), there is only a single consumption-*Euler* equation applicable between period 0 and period 1 and two resource constraints, for periods 0 and 1. As $k_0 > 0$ is given and $k_2 = 0$ must be satisfied, these three equations determine the three unknowns: c_0, c_1, k_1 (see 22.1). The optimal consumption choice (c_0, c_1) may be illustrated graphically. Specifically, it is given by the point of tangency between the intertemporal production possibility frontier (derived from the period resource constraints) and the highest possible intertemporal indifference curve.

When the horizon is infinite, the dynamic equations ((20.1) and (20.2)) can be illustrated in a phase diagram, and the solution we are looking for is actually given by the saddle path and the steady-state is given by the saddlepoint. Alternatively, (21) can be used to get the optimal path of capital accumulation which, given the resource constraint, yields the optimal consumption path. Specifically: $c_t = (1-\alpha\beta)Ak_t^{\alpha}$ (first equation in (22.4)) defines the *saddle path* (the relation between c_t and k_t along the optimal trajectory), the optimal value of c_0 (equal to $(1-\alpha\beta)Ak_0^{\alpha}$) that places the system on the saddle path, and the consumption function (c_t as a $(1-\alpha\beta)$ proportion of Ak_t^{α}; $k_{t+1} = \alpha\beta Ak_t^{\alpha}$ (second equation in (22.4)) where $k_0 > 0$ *given* and $\lim_{t\to\infty} \mu_{t+1}k_{t+1} = 0$) defines the optimal path of capital accumulation or the optimal capital sequence.

Thus, in this version of the Ramsey model, the answer to the question "how much should a nation save?" (Ramsey, 1928) is $k_{t+1} = \alpha\beta Ak_t^{\alpha}$ or, how much should a nation consume per capita is given by the optimally derived per capita consumption function, $c_t = (1-\alpha\beta)Ak_t^{\alpha}$.

5. *C*oncluding Remarks

This paper, using four alternative methods/approaches - the *Lagrangian* method, the *substitution* method, the *Hamiltonian* method, and the *dynamic programming* approach - has derived the conditions that must be satisfied by the solution to an intertemporal problem, specifically the deterministic discrete-time *Ramsey* problem.[27] A crucial part of characterizing/deriving the solution is either the so-called capital-*Euler* equation or the so-called consumption-*Euler* equation. For easier tractability but without loss of generality, results were derived assuming that there only three periods. However, as shown, the results generalize to the case where the number of periods of time is large but finite and to the case where the number of periods of time is infinite.

Note that in the *Ramsey* model the social planner is endowed with rational expectations (perfect foresight in this case because the model is deterministic (not stochastic), so that $E[c_{t+1}] = c_{t+1}$). With rational expectations (*RE*), the subjective expectation is the same as the mathematical expectation and the implication is that the expected value of future variable (here, $E[c_{t+1}]$, also c_{t+1} because of perfect foresight) depends on all the parameters of the model (here, A and α in the production function and the subjective time-discount factor β). This is why, in implementing/testing the model, "the hallmark of rational expectations is cross-restriction across equations".

[27] Barro and Sala-i-Martin (2004, pp. 604-617) provides a Hamiltonian recipe for dynamic optimization in continuous time, both finite and infinite horizons. Blanchard and Fischer (1989) discusses the assumptions and workings of the Ramsey model (pp. 38-47), ruling out of explosive paths in the Ramsey model (p. 75), and the local behavior of capital around the steady state in the Ramsey model (pp. 75-76). Obstfeld and Rogoff (1996) also discuss the method of Lagrange multipliers (pp. 715-718) and dynamic programming (pp. 718-721).
Arrow and Kurz (1970), Dorfman (1969), Dixit (1980), and Kamien and Scwartz (1981) provide intuition for the results.
For the mathematically inclined, Intrilligator (1971) and Takayama (1973), and Chiang (1992) are excellent readings.

Appendix

With population growth, $L_{t+1}=(1+n)L_t$, the period-*t* resource constraint becomes

$$k_{t+1}-k_t=(f(k_t)-c_t-(n+\delta)k_t)/(1+n), \quad t=0,1,2$$

and the changes in the results are summarized below (with the equation numbers in the text preserved but denoted A).

Substitution method: $$\max_{k_1,k_2} U=\sum_{t=0}^{2}u\underbrace{\left(f(k_t)-(n+\delta)k_t-(1+n)(k_{t+1}-k_t)\right)}_{c_t\text{, from the period}-t\text{ resource constraint}}. \quad \text{(A6)}$$

Lagrangian method: $$L=\sum_{t=0}^{2}\left\{\beta^t u(c_t)+\mu_{t+1}\left[\frac{f(k_t)-c_t-(n+\delta)k_t}{1+n}\right]-\mu_{t+1}(k_{t+1}-k_t)\right\}+\upsilon k_3. \quad \text{(A8)}$$

Hamiltonian method: $$H(k_t,c_t,\mu_{t+1})\equiv\beta^t u(c_t)+\mu_{t+1}\left[\frac{f(k_t)-c_t-(n+\delta)k_t}{1+n}\right]. \quad \text{(A9)}$$

$$\begin{aligned}
L&=\sum_{t=0}^{2}\left\{\underbrace{\beta^t u(c_t)+\mu_{t+1}\left[\frac{f(k_t)-c_t-(n+\delta)k_t}{1+n}\right]}_{H(\ldots)}-\mu_{t+1}(k_{t+1}-k_t)\right\}+\upsilon k_3\\
&=\sum_{t=0}^{2}H(k_t,c_t,\mu_{t+1})+\sum_{t=1}^{2}(\mu_{t+1}-\mu_t)k_t-(\mu_3k_3-\mu_1k_0)+\upsilon k_3 \qquad \text{(A10)}\\
&=\sum_{t=1}^{2}\{H(k_t,c_t,\mu_{t+1})+(\mu_{t+1}-\mu_t)k_t\}\\
&\quad+u(c_0)+\mu_1\left[\frac{f(k_0)-(n+\delta)k_0-c_0}{1+n}\right]-(\mu_3k_3-\mu_1k_0)+\upsilon.
\end{aligned}$$

Dynamic Programming method: The results are the same as those in the text except that now $\partial k_{t+1}/\partial c_t=-1/(1+n)$ instead of $\partial k_{t+1}/\partial c_t=-1$.

The set of conditions now become

$$\frac{u'(c_{t-1})}{u'(c_t)}=\beta\left(\frac{1+f'(k_t)-\delta}{1+n}\right), \quad t=1,2 \qquad \text{(A3.1)}$$

$$k_{t+1}-k_t=\frac{f(k_t)-c_t-(n+\delta)k_t}{1+n}, \quad t=0,1,2 \qquad \text{(A3.2)}$$

$$k_3=0, \qquad \text{(A3.3.1)}$$

$$k_0>0 \textit{ given}. \qquad \text{(A3.3.2)}$$

where, now, the marginal product of per capital is adjusted not only for physical capital depreciation but also for population growth, $(1+f'(k_t)-\delta)/(1+n)$, and $(1+n)\beta^t u'(c_t)=\mu_{t+1}$.

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