



UP School of Economics Discussion Papers

Discussion Paper No. 2013-07

July 2013

'Time Inconsistency': The Phillips Curve Example (An Analysis for Intermediate Macroeconomics)

by

Fidelina B. Natividad-Carlos

School of Economics, University of the Philippines

UPSE Discussion Papers are preliminary versions circulated privately to elicit critical comments. They are protected by Republic Act No. 8293 and are not for quotation or reprinting without prior approval.

‘Time Inconsistency’: The Phillips Curve Example
(An Analysis for Intermediate Macroeconomics)

Fidelina B. Natividad-Carlos*

Abstract

This paper provides the algebra and a panel diagram to attempt to examine the so-called inflation- unemployment (or Phillips curve, or aggregate supply) example, the most popular example in the literature when introducing the concept of “time inconsistency” or “dynamic inconsistency”. The resulting panel diagram (along with the derivations presented in the appendices) is used to analyze the different possible outcomes, depending on the scenarios – rule or pre-commitment, cheating, and equilibrium – and find out whether there is indeed “time inconsistency” or “dynamic inconsistency” in the said example.

JEL: E31, E52, E61

Keywords: Philips curve, aggregate supply, time inconsistency, dynamic inconsistency, short-run optimal policy, long-run optimal policy, rational expectations, rules vs discretion.

* School of Economics, University of the Philippines, Diliman, Quezon City 1101. The author is grateful to the PCED for financial support.

1. Introduction

Inconsistency between “the optimal plan of future behaviour chosen as of a given time” and “the optimizing future behaviour of the individual” arises because *either* preferences change over time [Strotz’s (1956) problem of “intertemporal tussle” due to non-exponential discounting of future felicities] *or* constraints change over time [Kydland and Prescott’s (1977) “inconsistency of optimal plans”]. With either changing preferences or changing constraints, plans at different dates will not necessarily coincide and therefore are ‘inconsistent’ but all are optimal (although one is optimal while the others are suboptimal) because each plan is derived from maximization of an objective function reflecting preferences subject to some constraint/s.

The issue of ‘time inconsistency’ is a not a mere intellectual curiosity. For instance, ‘time inconsistency’ in the tradition of Strotz (revived by Phelps and Pollak (1968) and made popular by Laibson (1997)) can help explain a wide range of economic phenomena: consumption/ saving/ growth, self-regulation/control, job search, choice of retirement, information acquisition, investment in human capital, procrastination, addiction, etc. On the other hand, Kydland and Prescott’s (1977) idea help explain, in the case of monetary policy, why “the pursuit of the first-best tends to push the economy away from the second best of a rule with low inflation, and toward the third best of discretionary policy with high inflation” (Barro 1986, p 28). It can also explain ‘hold-up’ problems that may lead to underinvestment and other suboptimality results.

The focus of the paper shall be on Kydland and Prescott’s idea of ‘inconsistency’, also using the inflation-unemployment (or Phillips curve or the aggregate supply) as an example. In the macro literature on the topic, one can find inconsistent/confusing terminologies as well as looseness in the use of the word equilibrium. [For the record, Kydland and Prescott (1977) did not use the phrase ‘time inconsistency’. Calvo’s (1978) used “time inconsistency” while

Fischer's (1980) used "dynamic inconsistency"]. Using equations and graphs, the paper will clarify if indeed there is 'time inconsistency' or dynamic inconsistency in the said example.

The paper is organized as follows.

Section 2 presents the Phillips curve example and examines the different possible outcomes.

Specifically, section 2.1 describes the components of the model. In this Phillips curve example, the supply side of the economy is summarized by the Phillips curve or the aggregate supply curve – short run and long run – while demand side is summarized by some monetary policy.

There are two agents in the economy, the private sector (or the wage setters) and the policy maker, whose objective/s and preferences as described by their respective utility functions. The private sector sets nominal wages on the basis of its expected inflation rate, and their expectations are formed rationally. The utility of the private sector is maximized when its expected inflation rate equals the actual inflation rate chosen by the policy maker. The rational expectations equilibrium (REE) line of private sector coincides with the vertical Phillips curve or vertical AS. The policy maker, on the other hand, has control over inflation, and the choice of the inflation rate depends on (is independent of) the private sector's expected inflation rate when his/her preferences are quadratic (quasi-linear, as shown in the next section). The rule for the choice of the inflation rate is summarized by the so-called optimal policy (*OP*) line. Section 2.2 provides two alternative graphical derivations of optimal policy line. [The derivations of the equation for the *OP* line using alternative methods - equating the slopes, substitution method, equating marginal cost and marginal benefit, and Lagrangian method – are presented in the Appendix A.] Section 2.3 clarifies the distinction among short run optimal points, *REE*, and *the* long run optimal point.

Section 2.4 presents a three-panel diagram and use it to illustrate graphically the possible outcomes – rule or commitment, cheating, and equilibrium. [The derivations are presented in Appendix B.] In the literature, the rule solution is labeled as time inconsistent while the equilibrium solution is labeled as time consistent. However, the terms “time inconsistency” or “dynamic inconsistency” are a misnomer because there is actually no inconsistency in the so called “time inconsistency” or “dynamic inconsistency” problem. Choices are different because the policy maker faces different constraints depending on the private sector’s on inflationary expectations, as shown in the panel diagrams. Different optimization problems yield different solutions or, graphically, different points of intersections. Section 3 replaces the quadratic utility function of the policy maker in by a quasi-linear function. Another possible outcome – over-expected case - is added, and the four possible outcomes are interpreted using simple game theory. That the terms “time inconsistency” or “dynamic inconsistency” are a misnomer is even more visible when the possible outcomes are analyzed using simple game theory. Section 4 discusses briefly the distinction between rules and discretion. And, section 4 concludes.

2. The Inflation-Output Example when The Policy Maker’s Utility Function is Quadratic

2.1 The Model

The model is based on Kydland and Prescott (1977), Barro and Gordon (1983a, 1983b), and Blanchard and Fischer (1989). In this set up, the economy is described by the following: short-run and long-run Phillips curves or aggregate supply curves, private sector’s objective and expectations formation, and policy maker’s objectives and preferences.

Phillips curve or Aggregate Supply. Output in the short-run obeys the expectational Phillips curve,

$$y_t = \bar{y} + \alpha(\pi_t - \pi_t^e), \quad \alpha > 0, \quad (1)$$

which can be rewritten as

$$\pi_t = \underbrace{\left(\pi_t^e - \frac{1}{\alpha} \bar{y} \right)}_{\text{v. intercept}} + \underbrace{\left(\frac{1}{\alpha} \right)}_{\text{slope}} y_t, \quad (1')$$

where y is output (in natural logarithm), \bar{y} is the full-employment level or the natural rate of output (in natural logarithm), π is the actual inflation rate, π^e is the expected inflation rate as embodied in predetermined nominal wages, α is a parameter measuring the sensitivity of output to inflation surprise $(\pi - \pi^e)$, and subscript t is time or period.

(1) or (1'), the short-run Phillips curve (SRPC) or the short-run aggregate supply (SRAS) curve is one of the constraints faced by the policy maker. It is a relationship between output and inflation which assumes sticky-price adjustment and/or sticky-wage adjustment. (1) shows that output is above (equal to; below) the full-employment level whenever the actual inflation rate is higher than (equal to; lower than) the expected inflation rate.

In a $y - \pi$ diagram, the slope of the SRAS indicates that the policy maker has to accept $(1/\alpha)$ unit/s of inflation to gain a unit increase in output¹. Since a SRAS is drawn for a given π^e , an increase in π^e from π^T to π_E increases the v. intercept and shifts the SRAS upward and to the left (see (1') and Figure 1).²

Output will equal its full-employment level when there are no inflationary surprises, In the long-run, on the average, wage-setters (the private sector) will neither underpredict nor overpredict the inflation rate, i.e., $\pi^e = \pi$, and output is given by

¹ (1) can also be rewritten as $\pi_t^e = (1/\alpha)\bar{y} + \pi_t - (1/\alpha)y$, and its analogue is the consumer's budget line. Since here the constraint is upward sloping, $d\pi/dy$ is (-) "price" of y /"price" of π = the (-) "price" of y in terms of π = $(1/\alpha)/1$

² As α increases (decreases), the v. intercept increases, the slope decreases, and the SRAS pivots around the point where it intersect the LRAS; this means that the SRAS becomes flatter (steeper).

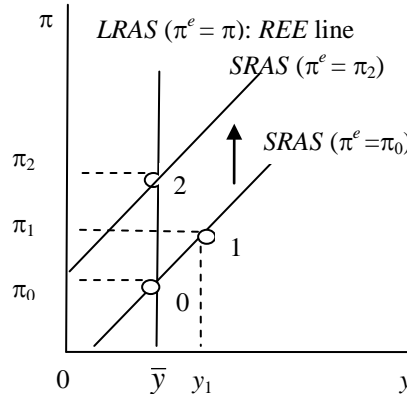
$$y = \bar{y}, \quad (2)$$

i.e., output is equal to its full-employment level. Note, however, that if even in the short run the private sector is able to exactly predict the inflation rate, i.e., $\pi_t^e = \pi_t$, then

$$y_t = \bar{y}, \quad (2)$$

output is equal to its full-employment level even in the short run. The long-run Phillips curve (*LRPC*) or the long-run aggregate supply (*LRAS*) curve (2) is vertical at \bar{y} . The intersection of a given *SRAS* with the *LRAS* shows the π^e associated with it (see Figure 1).

Figure 1. Short-run vs Long-run Aggregate Supply



The Private Sector's Expectations. It is assumed that the private sector forms its expectations rationally and that its utility function is given by

$$U_t^P = -(\pi_t - \pi_t^e)^2, \quad (3)$$

where U_t^P is the private sector's utility. (3) shows that deviations of π from π^e , whether positive or negative, negatively affect the utility of the private sector, $U^P = 0$ only $\pi^e = \pi$, and $U^P < 0$ when $\pi^e \neq \pi$.

The private sector chooses the expected inflation rate π^e so as to maximize (3), and its utility is maximized when

$$\partial U^P / \partial \pi = -2(\pi - \pi^e) = 0 \Leftrightarrow \pi^e = \pi \quad (3.1)$$

where

$$\pi_t^e \equiv E_{t-1} \pi_t. \quad (3.2)$$

(3.1) is the rational expectations equilibrium (*REE*), or perfect foresight equilibrium because there is no uncertainty in the model, condition for the private sector which is derived from the private sector's utility maximization while (3.2), that the expected inflation rate at time t , π_t^e , is the rational expectation of π_t based on all the available information as of the end of $t-1$ or just the beginning of time t , $E_{t-1} \pi_t$, is the *RE* assumption.

Even if the private sector forms expectations rationally, it may underpredict or underpredict the inflation rate but in the long-run, on the average, $\pi^e = \pi$; if in the short run it fully anticipates the inflation rate, then $\pi_t^e = \pi_t$.

In a $\pi^e - \pi$ diagram, the *REE* line is the $\pi^e = \pi$ line or the 45° line; in a $y - \pi$ diagram, the *REE* line coincides with the vertical *LRAS*.

The Policy Maker's Utility Function. The objective function of the policy maker is given by

$$U_t = -a(y_t - y^T)^2 - b(\pi_t - \pi^T)^2, \quad 0 < a, b < \infty, y^T > \bar{y} > 0, \pi^T > 0, \quad (4)$$

where U is the utility of the policy maker, y^T is the target output level, π^T is the target inflation rate, a is the weight on output deviations, b is the weight on inflation deviations which is a measure of the degree of the policy maker's inflation aversion, and the other variables are as defined before.³

³ The policy maker's objective function may either be a utility function or a loss function. Two forms of the policy maker's objective function are used in the literature: quasi-linear and quadratic. See Walsh (2010, p. 271-273).

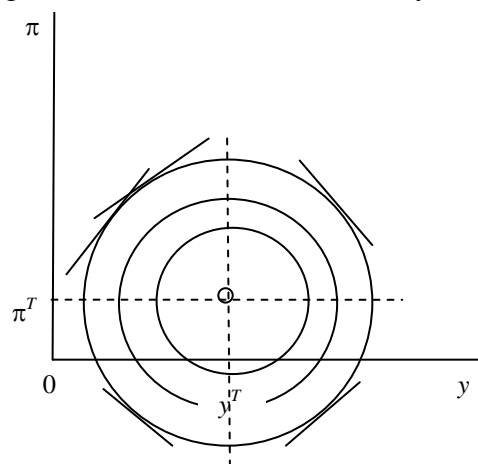
The policy maker has output and inflation targets, and his/her utility is negatively affected by deviations from these targets. The utility function is quadratic in the deviation of output from the target level $(y - y^T)$ and the deviation of inflation rate from the target rate $(\pi - \pi^T)$, implying that the policy maker has a symmetrical attitude to positive and negative deviations.

Distortions/imperfections in labor and product markets (due to taxes, monopoly unions, or monopolistic competition) make the full-employment or natural rate of employment of output too low. Thus, output target is higher than the full-employment level of output, i.e., $y^T > \bar{y}$.

It is assumed that the policymaker manipulates or has control over the inflation rate π and that it is set after the private sector sets nominal wages based on their expectations about the inflation rate π^e .

The preferences of the policy maker, as summarized by his/her utility function, can be illustrated graphically using contours (see Figure 1). $U = 0$ only at the bliss point where $(y, \pi) = (y^T, \pi^T)$, and $U < 0$ when $y \neq y^T$ and $\pi \neq \pi^T$.⁴ The farther is the indifference curve

Figure 2. Preferences of the Policy Maker



⁴ Note also that a change in y^T and/or π^T will shift the entire indifference map.

from the bliss point, the lower the utility level.

The shape of the contours depend on parameters a and b .⁵ In the region where $(\pi - \pi^T) > 0$ and $(y - y^T) < 0$, a relatively steep (flat) indifference curve indicates that the policymaker is willing to accept a larger (smaller) increase in inflation for a given increase in output.⁶

Setting the total differential of U equal to zero,

$$dU = 0 = -2a(y - y^T)dy - 2b(\pi - \pi^T)d\pi, \quad (4.1)$$

gives the slope of the indifference ellipse,

$$\left(\frac{d\pi}{dy} \right)_{dU=0} = - \frac{a(y - y^T)}{b(\pi - \pi^T)}. \quad (4.2)$$

The indifference curves are positively sloped when $(\pi - \pi^T) > 0$ and $(y - y^T) < 0$ and become steeper (flatter) as the weight of output deviation a increases (decreases) and/or the measure policymaker's inflation aversion b decreases (increases).

The absolute value of the slope of the indifference curve $|d\pi/dy|_{IC}$, called the marginal rate of 'substitution' of π for y , $MRS_{\pi \text{ for } y}$, indicates the addition to (reduction in) π that the policy maker is *willing* to accept in order to have one more (less) unit of y and yet the he/she

⁵ Each indifference curve is a concentric circle, a circle with (y^T, π^T) at its center when the policy maker is equally concerned about output and inflation from targets, i.e., when $a/b = 1$. On the hand, each indifference curve is a concentric oval, an ellipsoid with a vertical (horizontal) orientation and (y^T, π^T) at its center when the policymaker attaches more (less) weight on deviations in inflation rate than on deviations in output, i.e., when $a/b > 1$ (< 1); in this case, each indifference is relatively steep (flat). See Carlin and Soskice (2006, p. 143).

⁶ The slope of the indifference curve depends on the region. In the region to the NW of the bliss point, $(\pi - \pi^T) > 0$ and $(y - y^T) < 0$; the other regions are to the SE, SW, and NE of the bliss point.

remains on the same utility level. The $MRS_{\pi \text{ for } y}$ declines (increases) as y increases when

$$(y - y^T) < 0 \text{ } (> 0).$$

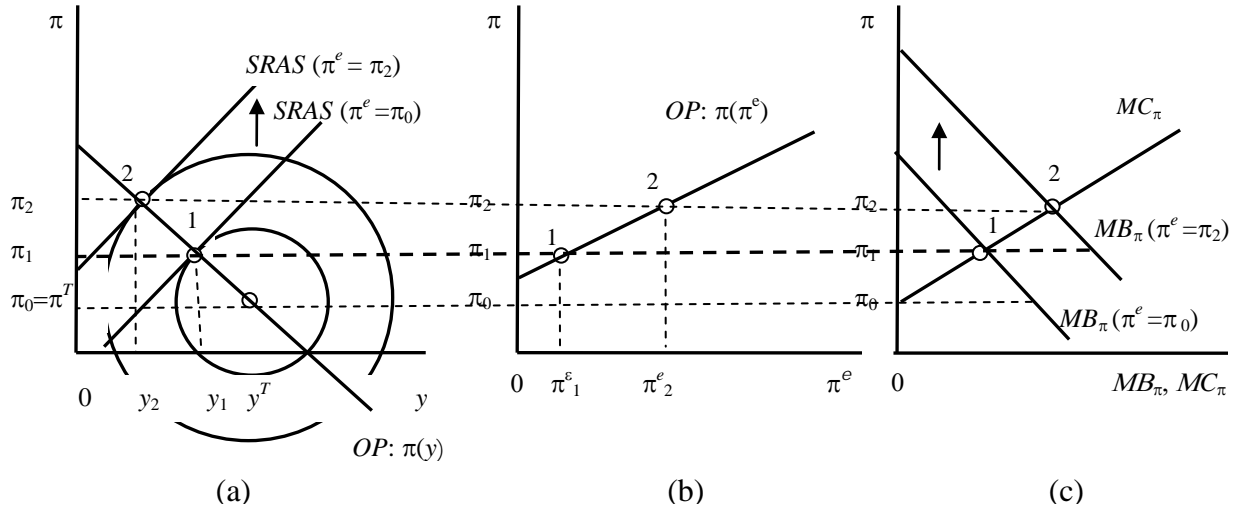
2.2 Deriving the Policy Maker's Short-Run Optimal Policy (OP) Line

The policy maker chooses output and inflation to maximize his/her utility function (4) subject to the constraint given by the SRAS (1):

$$\begin{aligned} \max_{y, \pi} \{ & U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \} \\ \text{s.t. } & y = \bar{y} + \alpha(\pi - \pi^e) \end{aligned}$$

The solution to this problem yields the condition for optimal policy (OP). The OP line can be illustrated in either the $y - \pi$ diagram or the $\pi^e - \pi$ diagram.⁷

Figure 3. Alternative Derivations of the OP Line



The alternative ways to get the equation for the OP line are: the method of equating slopes, the method of equating marginal benefit and marginal cost, the substitution method, and the Lagrangian method (see Appendix).

⁷ The analogue of the SRAS (1) or (1') rewritten as $\pi^e - (1/\alpha)\bar{y} = -(1/\alpha)y + \pi$ is the budget line the analogue of the downward sloping OP line in $y - \pi$ diagram is the income-consumption curve while the analogue of the upward sloping OP line in $\pi^e - \pi$ diagram is the Engel curve.

Method of equating slopes. Equating the slope of the *SRAS* (1') and the slope of the indifference curve (4.2),

$$\underbrace{-\frac{a(y - y^T)}{b(\pi - \pi^T)}}_{\left(\frac{d\pi}{dy}\right)_{IC}} = \underbrace{\frac{1}{\alpha}}_{\left(\frac{d\pi}{dy}\right)_{PC}}, \quad (5)$$

yields

$$y - y^T = \frac{b}{a\alpha}(\pi - \pi^T) \quad \Leftrightarrow \quad \pi = \underbrace{\pi^T + \frac{a\alpha}{b}y^T}_{\text{v. intercept} > 0} - \underbrace{\frac{a\alpha}{b}y}_{\text{slope} < 0}. \quad (6.1)$$

which is the equation for *OP* line in the $y - \pi$ diagram. This *OP* line is a locus of y, π combinations for which the policy maker's objective function (4) is maximized subject to the *SRAS* constraint (1).⁸ As shown in Figure 1, it is a locus of all optimal y, π combinations, i.e., a locus of points of tangency between the indifference curve of the policy maker and the *SRAS*.⁹

To see why a point on the *OP* line such as point 1 is an optimal point, consider points 0 and 1 on the *SRAS* constraint drawn given that $\pi^e = \pi_0$ (Figure 3(a)). Starting at point 0, the policy maker can increase his/her utility by increasing π from π_0 to π_1 , i.e., by moving to point 1 which lies on a higher indifference.

The *OP* line - $\pi(y)$ line - can also be derived graphically. Assume that initially the point of tangency is point 1. An increase in π^e from π_0 to π_2 shifts the *SRAS* to the left (i.e., output decreases at each π), and the new point of tangency is at point 2, where the optimal y is lower

⁸ The $\pi(y)$ line as the *OP* line is also called the “social expansion path” (Heijdra and van der Ploeg (2002), or the *MR* line (Carlin and Soskice, 2006).

⁹ The *OP* line in either diagram (i) becomes flatter (steeper) as: the measure of inflation aversion b increases (decreases), the weight on output deviations a decreases (increases), and the sensitivity of output to inflation surprise α decreases (increases) and (ii) shifts up as y^T increases and/or π^T increases.

and the optimal π is higher, and the policy maker's utility level is lower since y is an economic good while π is an economic bad. Connecting points of tangencies 1 and 2 yields the downward sloping $\pi(y)$ line as the OP line in the $y - \pi$ diagram (see Figure 3(a)).

Since the optimal y decreases and the optimal π increases as π^e increases, there IS a positive relationship between π^e and optimal π , This means that $\pi(\pi^e)$ line as the OP line in the $\pi^e - \pi$ is upward sloping.

Note that both points 1 and 2 are optimal points, but point 2 is sub-optimal as compared to point 1.

The $\pi(y)$ equation (6.1) as the equation for the OP line can be converted into $\pi(\pi^e)$ equation. Combining (6.1) and (1),

$$\pi = \pi^T + \frac{a\alpha}{b} y^T - \frac{a\alpha}{b} \overbrace{(\bar{y} + \alpha(\pi - \pi^e))}^{=y, \text{ using (1)}},$$

and rearranging/simplifying,

$$\pi(1 + \frac{a\alpha^2}{b}) = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) + \frac{a\alpha^2}{b}\pi^e$$

yields the equation for the OP line in $\pi^e - \pi$ diagram,

$$\pi = \underbrace{\frac{b}{b + a\alpha^2} \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y})}_{\text{v.intercept} > 0} + \underbrace{\frac{a\alpha^2}{b + a\alpha^2} \pi^e}_{0 < \text{slope} < 1}. \quad (6.2)$$

In the $\pi^e - \pi$ diagram, the OP line is upward sloping but flatter than the 45° line (see Figure 3(b)).¹⁰

Method of equating marginal benefit and marginal cost. The $\pi(\pi^e)$ equation (6.2) can also be derived by equating the MB_π equation and the MC_π equation (see Appendix)¹¹,

$$MB_\pi = -2a\alpha(\alpha(\pi - \pi^e) - (y^T - \bar{y})) \Leftrightarrow \pi = \underbrace{\pi^e + \frac{1}{\alpha}(y^T - \bar{y})}_{\text{v.intercept}} - \underbrace{\frac{1}{2a\alpha^2}MB_\pi}_{\text{slope}}, \quad (7.1)$$

$$MC_\pi = 2b(\pi - \pi^T) \Leftrightarrow \pi = \underbrace{\pi^T}_{\text{v.intercept}} + \underbrace{\frac{1}{2b}MC_\pi}_{\text{slope}}, \quad (7.2)$$

and $(\partial U / \partial y)(\partial y / \partial \pi) = MB_\pi$ = marginal utility, or marginal benefit of higher inflation through higher output while $\partial U / \partial \pi = (-) MC_\pi$ = marginal utility of higher inflation, or the negative of the marginal cost of higher inflation (Obstfeld and Rogoff (1996, p. 637)).

The $\pi(\pi^e)$ line can also be derived using the $MB_\pi, MC_\pi - \pi$ diagram (Figure 3(c)). The intersection between a specific MB_π curve and the MC_π curve determines a specific optimal inflation rate. Assume that the initial intersection is at point 1. As π^e increases, the upward

¹⁰ The OP line (the term used by Blanchard and Fischer, 1989) in the $\pi^e - \pi$ diagram, like the OP line in the $y - \pi$ diagram, (i) becomes flatter (steeper) as b increases (decreases), a decreases (increases), and α decreases (increases) and (ii) shifts up as y^T increases and/or π^T increases.

The OP line in either diagram becomes flatter (steeper) as: the measure of inflation aversion b increases (decreases), the weight on output deviations a decreases (increases), and the sensitivity of output to inflation surprise α decreases (increases).

Note also that, in either a $y - \pi$ diagram or a $\pi^e - \pi$ diagram, an increase in y^T and/or an increase in π^T will increase the v. intercept of the OP line, i.e., it will shift up the OP line.

¹¹ The MB_π curve will shift as y^T and, more importantly, as π^e changes and will become flatter as a increases and/or α increases. The MC_π curve will shift only when π^T changes and will become flatter as b increases.

sloping MC_π curve stays the same but the downward sloping MB_π curve shifts to the left (i.e., the marginal benefit of higher inflation through output decreases at each π) and, at the initial optimal π , $MB_\pi > MC_\pi$; thus, the optimal π must increase to restore equality between MB_π and MC_π , now at point 2. Thus, there is a positive relationship between π^e and π along the OP line. (see Figure 3(b)).

2.3 Short Run Optimal Points, REE , and the Long-Run Optimal Point

At the outset, before discussing the possible solutions/outcomes, a distinction must be made among short-run optimal points, rational expectations equilibrium, and the long-run optimal point. Short run optimal points are points on the OP line while REE points are points on the REE line or the $LRAS$.

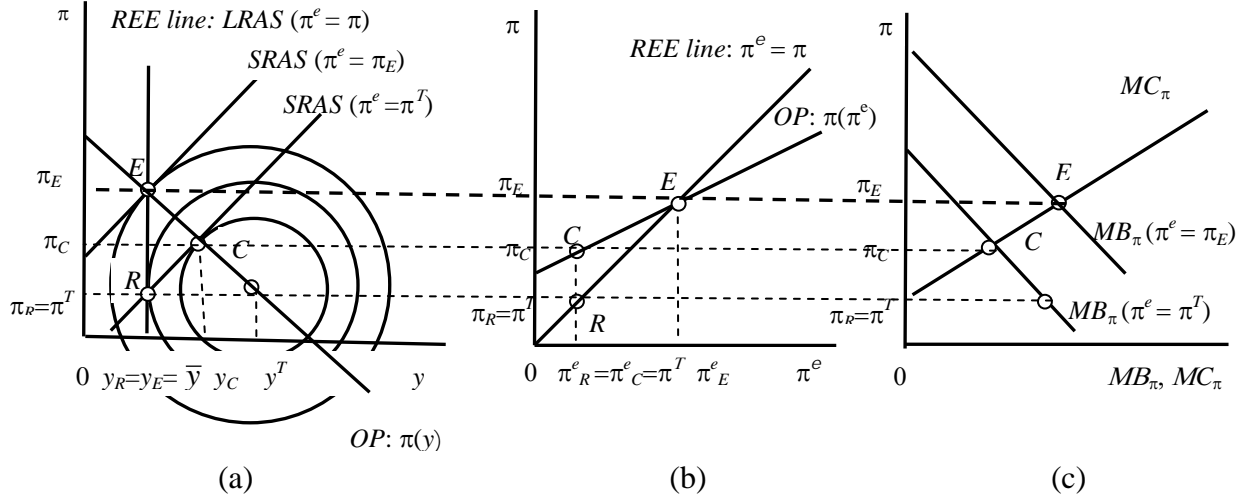
The point of tangency between the $LRAS$ and an indifference curve is *the* long-run optimal point; if, even at time t , $\pi_t^e = \pi_t$, and such tangency occurs, then *the* long-run optimal point is attained even at time t .

In Kydland and Prescott's (1977) terminologies, *the* long-run optimal point is the point of "optimal equilibrium" while point the intersection between the OP line and the REE line is the point of "consistent equilibrium." In the literature, other terminologies are used to refer to these points, as the discussion below will show.

2.4 Possible Outcomes

Figure 4 illustrates the possible outcomes using three alternative diagrams. The possible outcomes (solutions) - represented by points R for rule or commitment, C for cheating, and E for equilibrium - arise from different scenarios (mathematical problems).

Figure 4. Possible Outcomes Illustrated using Alternative Diagrams



*Rule or Commitment*¹². In Figure 4(a), consider only the *LRAS* or the *REE line* and the indifference map. Suppose that the policy maker can commit to an inflation rate, π^T , and sets $\pi = \pi^T$, and that the private sector also expects that the inflation rate would be π^T . In this scenario, using the equation for the *SRAS* (1), output under commitment is $y_R = \bar{y}$.

Point R represents the solution to maximizing (4) s.t. the vertical *LRAS* (2), since $y = \bar{y} + \alpha(\pi - \pi^e)$, $\pi = \pi^e$, $\pi^e = E\pi$, $\pi = \pi^T$. In Figure 4(a), point R is *the* point of long-run optimal equilibrium because it is *the* point of tangency between an indifference curve and the *LRAS* or the *REE line*. This is Kydland and Prescott's (1977) point of "optimal equilibrium".

Note that here there is no short-run optimization involved, as the policy maker chooses $\pi = \pi^T$ which is not on the *OP* line. In particular, the solution is given by a point, R , that lies *on* the *REE line* but *off* the *OP* line, as shown in Figures 4(a) and 49(b). Since the *OP* line is a locus of points of intersection between *MB $_{\pi}$* curves and the *MC $_{\pi}$* curve and point R is not on the *OP*

¹²□□ The rule or commitment solution is also referred to as "equilibrium under the Taylor rule" (Sorensen and Whitta-Jacobsen (2010, p. 661), or commitment or "optimal policy under precommitment" (Walsh 2010, p. 281).

line, it follows that $MB_{\pi} \neq MC_{\pi}$; specifically, with $\pi = \pi^T$, $MB_{\pi} > MC_{\pi}$, as shown in Figure 4(c). Thus, the ‘commitment’ solution is consistent with *REE* but not optimal in the short-run

In a game without commitment, the “commitment” solution cannot be an equilibrium (Fudenberg and Tirole (1991, p. 76).

*Cheating*¹³. Here, the private sector mistakenly believes that the inflation rate would be $\pi^e = \pi^T$, i.e.,

$$\pi^e = \pi_C^e = \pi^T, \quad (9.1)$$

but the policy maker will choose π based on the $\pi(\pi^e)$ equation (A4.2), the equation for the *OP* line. Substituting (B2.1) into (A4.2) yields the solution for the inflation rate under cheating

$$\pi_C = \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}). \quad (9.2)$$

Substituting (B2.1) and (B2.2) into (A2) gives the solution for output under cheating,

$$y_C = \bar{y} + \frac{a\alpha^2}{b + a\alpha^2} (y^T - \bar{y}). \quad (9.3)$$

In this cheating scenario, once the private sector has committed itself to $\pi^e = \pi^T$, the policy maker will choose the $\pi = \pi_C$ to maximize its utility. The cheating solution, point *C*, is optimal since it is an intersection between the MB_{π} curve and the MC_{π} curve (see Figure 4(c)) and a point on the *OP* line (see Figures 4(a) and 4(b)). Point *C* however is a point *on* the *OP* line but *off* the *REE* line, i.e., the cheating solution is optimal but not consistent with *RE*. Thus, point *C* cannot be the equilibrium of the model where the private sector forms expectations rationally.

*Equilibrium*¹⁴. In a model where the problem is given by

¹³ The cheating solution is also referred to as “cheating outcome with surprise inflation” (Sorensen and Whitta-Jacobsen (2010, p. 661), or cheating solution (Heijdra and van der Ploeg 2002, p. 241), or ‘fooling’ solution (Blanchard and Fischer, 1989).

$$\max_{y, \pi} \left\{ U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \right\} \quad s.t. \quad y = \bar{y} + \alpha(\pi - \pi^e)$$

and

$$\max U^P = -(\pi - \pi^e)^2,$$

or equivalently

$$\left. \begin{aligned} \max_{y, \pi} \quad & \left\{ U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \right\} \\ s.t. \quad & y = \bar{y} + \alpha(\pi - \pi^e) \\ & \pi = \pi^e \\ & \pi^e \equiv E\pi \end{aligned} \right\}, \quad (P3)$$

the equilibrium solution is given by point E, the intersection between the *OP* line and the *REE* line in the $\pi^e - \pi$ diagram, or the intersection between the *OP* line $\pi(y)$ and the *REE* line or the *LRAS* in $y - \pi$ diagram, or the equality between MB_π and MC_π (see Figures 5a, 5b, and 5c).

The inflation rate chosen optimally by the policy maker is a function of inflationary expectations. Since expectations are rational, and since the private sector or the wage-setters know that inflation rate will be set based on (6.2), the equilibrium expected inflation rate must satisfy

$$\begin{aligned} \pi^e &= E\pi \\ &= E \left[\frac{b}{b + a\alpha^2} \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2} \pi^e \right] \end{aligned}$$

Using (6.2) for π and noting that $E\pi^T = \pi^T$, $E\bar{y} = \bar{y}$, and $E(\pi^e) = E(E\pi) = E\pi = \pi^e$,

¹⁴ This equilibrium solution, Kydland and Prescott's (1977) 'consistent equilibrium', is also referred to as "time-consistent rational expectations equilibrium" (Sorensen and Whitta-Jacobsen (2010, p. 662), or equilibrium under discretion (Walsh (2010, p. 279-280), solution under discretion (Heijdra and van der Ploeg, p 239), or "consistent" solution (Kydland and Prescott, 1977).

$$\pi^e = E \left[\underbrace{\frac{b}{(b+a\alpha^2)}\pi^T + \frac{a\alpha}{(b+a^2)}(y^T - \bar{y}) + \frac{a\alpha^2}{(b+a\alpha^2)}\pi^e}_{\text{using (6.2) for } \pi} \right]$$

$$\pi^e \left(1 - \frac{a\alpha^2}{b+a\alpha^2} \right) = \frac{b}{(b+a\alpha^2)}\pi^T + \frac{a\alpha}{(b+a\alpha^2)}(y^T - \bar{y})$$

$$\pi^e = \frac{b+a\alpha^2}{b} \left[\frac{b}{(b+a\alpha^2)}\pi^T + \frac{a\alpha}{(b+a\alpha^2)}(y^T - \bar{y}) \right]$$

yields the equilibrium expected inflation rate,

$$\pi_E^e = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}). \quad (10.1)$$

Substitution of (10.1) into (6.2) yields the equilibrium inflation rate,

$$\pi_E = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}), \quad (10.2)$$

which is the same as the equilibrium expected inflation rate (10.1). The equilibrium actual inflation rate relative to the target inflation rate is higher: (i) the greater is the wedge between the target output level and the full employment level, (ii) the greater is a/b , the weight the policy maker put on output stabilization relative to inflation stabilization, and (iii) the greater the value of α .

Using (1), (10.1), and (10.2), the equilibrium level of output is

$$y_E = \bar{y}. \quad (10.3)$$

Comparing the Different Possible Outcomes. Comparing the results,

$$\overbrace{\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})}^{\pi_E} > \overbrace{\pi^T + \frac{a\alpha}{b+a\alpha^2}(y^T - \bar{y})}^{\pi_C} > \overbrace{\pi^T}^{\pi_R} > 0, \quad (11.1)$$

$$\overbrace{\bar{y} + \left(\frac{a\alpha^2}{b+a\alpha^2} \right)(y^T - \bar{y})}^{y_C} > \overbrace{\bar{y}}^{y_E = y_R} > 0, \quad (11.2)$$

$$0 < \overbrace{-a\left(\frac{b+a\alpha^2}{b}\right)(y^T - \bar{y})^2}^{U_E} < \overbrace{-a(y^T - \bar{y})^2}^{U_R} < \overbrace{-a\left(\frac{b}{b+a\alpha^2}\right)(y^T - \bar{y})^2}^{U_C}, \quad (11.3)$$

$$\overset{U_E^P=U_R^P}{\tilde{0}} < \overbrace{-\left(\frac{a\alpha}{b+a\alpha^2}(y^T - \bar{y})\right)^2}^{U_C^P}, \quad (11.4)$$

i.e., in terms of the utility of the policy maker, the third-best outcome is the equilibrium outcome (point E), the second-best outcome is the ‘rules’ or ‘commitment’ outcome (point R), and the first-best outcome is the cheating outcome (point C), as shown in (11.3) and Figure 4.

The inflation bias, given by

$$\begin{aligned} \text{inflation bias} &\equiv \pi_E - \pi_R \\ &= \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) - \pi^T \\ &= \frac{a\alpha}{b}(y^T - \bar{y}), \end{aligned}$$

arises in this model because the policy maker has an incentive to cheat,

$$\begin{aligned} \text{incentive to cheat} &\equiv U_C - U_R = -a\left(\frac{b}{b+a\alpha^2}\right)(y^T - \bar{y})^2 - (-)a\left(\frac{b+a\alpha^2}{b}\right)(\bar{y} - y^T)^2 \\ &= \left(\underbrace{\frac{b+a\alpha^2}{b}}_{>1} - \underbrace{\frac{b}{b+a\alpha^2}}_{<1}\right)a(y^T - \bar{y})^2 > 0 \\ &= \left(\frac{a\alpha^2(2b+a\alpha^2)}{b(b+a\alpha^2)}\right)a(y^T - \bar{y})^2 > 0 \end{aligned}$$

due to because to his/her inability to precommit to its target inflation rate once the private sector have formed its expectations. The incentive to cheat (or temptation to cheat) is larger and thus the amount of inflation bias will be larger, the larger is α , the larger is a , the smaller is b , and the larger is $(y^T - \bar{y})$.

There will be inflation that is higher than π^T as a result of the optimizing behavior of the policy maker and the maximizing behavior of the private sector. Had the policy maker chosen to stick to the rule ($\pi = \pi_R = \pi^T$) and had the private sector expected this ($\pi^e = \pi_R^e = \pi^T$), the private sector would have the same pay-off ($U_R^P = U_E^P = 0$) but the policy maker would have a higher pay-off ($U_R > U_E$). Why would the policy maker choose to have a higher inflation rate? It can not credibly commit itself to what is the optimal plan of inflation because this plan is not time consistent.

3. The Inflation-Output Example when the Policy Maker's Utility Function is Quasi-Linear

The model is the same as in the previous section but now (4) is replaced by

$$U_t = a(y_t - y^T) - b(\pi_t - \pi^T)^2, \quad a, b > 0, y^T, \pi^T > 0, \quad (12)$$

i.e., it assumed now that the utility function of the policy maker is quasi-linear.¹⁵ The policy likes to have output that is higher than the full employment level and, as in the quadratic case, dislikes inflation deviations, whether positive or negative, from the target/desired level.

Using (12) and (1'), either the method of equating slopes which give

$$\frac{a}{\underbrace{2b(\pi - \pi^T)}_{\left(\frac{d\pi}{dy}\right)_{IC}}} = \frac{1}{\underbrace{\alpha}_{\left(\frac{d\pi}{dy}\right)_{PC}}}, \quad (13)$$

or the substitution method which give

$$\frac{\partial U}{\partial \pi} = 0: \quad \underbrace{\frac{a\alpha}{MB_\pi = \frac{\partial U}{\partial y} \cdot \frac{\partial y}{\partial \pi}}}_{\left(\frac{d\pi}{dy}\right)_{IC}} + \underbrace{-2b(\pi - \pi^T)}_{(-)MC_\pi = \frac{\partial U}{\partial \pi}} = 0, \quad (14)$$

¹⁵ It does not matter for decisions whether U is linear in $y - y^T$, or $y - \bar{y}$, or y .

yields the equation for OP line:

$$\pi = \pi^T + (1/2)(a/b)\alpha. \quad (15)$$

In this case, the optimal π is independent of y and therefore π^e (via (1)). This means that the OP line is a horizontal line either in the $y - \pi$ diagram (see Figure 5(a)) or the $\pi^e - \pi$ diagram (Figure 5(b)).

Thus, (12) has a special property. No matter what π^e (or y via (1)) is, or where the position of the $SRAS$ is, the policy maker always maximizes his/her utility by generating the same π .

Notice also that in the $MB_\pi, MC_\pi - \pi$ diagram (see Figure 5(c)) and using (14),

$$MB_\pi = a\alpha, \quad (14.1)$$

$$\pi = \underbrace{\pi^T}_{\text{v.intercept}} + \frac{1}{\underbrace{2b}_{\text{slope}}} MC. \quad (14.2)$$

Equating (14.1) and (14.2), also yields the optimal π (13).

To derive graphically the OP line in the $y - \pi$ diagram (Figure 5(a)), consider an increase in π^e . As π^e increases, the $SRAS$ shifts to the left and, the optimal y is higher but the optimal π remains the same; the points of tangencies yield the horizontal OP line. What this implies in the $\pi^e - \pi$ diagram (Figure 5(b)) is that the OP line is also a horizontal line since π^e increases but the optimal π remains the same. In $MB_\pi, MC_\pi - \pi$ diagram (Figure 5(c)), as π^e increases, the MC_π curve does not shift as in the quadratic case but the MB_π curve also does not shift unlike in the quadratic case and therefore the optimal π , given by their point of intersection also remains the same. This is why both points C and E lie on the intersection point of the MB_π

and MC_π curves.

In summary, in a quadratic (quasi-linear) case: (i) the OP line is downward sloping (horizontal) in the $y - \pi$ diagram and upward sloping (also horizontal) in the $\pi^e - \pi$ diagram, (ii) the MB_π curve is downward sloping (vertical) in the $MB_\pi, MC_\pi - \pi$ diagram and shifts (does not shift) as π^e changes.

The procedure for deriving the solution values for π , y , U , and U^P under different possible outcomes is the same as that presented in the previous section. We can therefore simply summarize the results as follows:

$$\underbrace{\pi_C = \pi_E = \pi_O}_{\pi^T + (1/2)(a/b)\alpha} > \underbrace{\pi_R}_{\pi^T}, \quad (15.1)$$

$$\underbrace{y_C}_{\bar{y} + (1/2)(a/b)\alpha^2} > \underbrace{y_E = y_R}_{\bar{y}} > \underbrace{y_O}_{\bar{y} - (1/2)(a/b)\alpha^2}, \quad (15.2)$$

$$\underbrace{U_O}_{-a(y^T - \bar{y}) - b(-(1/2)(a/b)\alpha)^2} = \underbrace{U_E}_{-a(y^T - \bar{y}) - b((1/2)(a/b)\alpha)^2} < \underbrace{U_R}_{-a(y^T - \bar{y})} < \underbrace{U_C}_{-a(y^T - \bar{y}) + (1/2)(a/b)\alpha^2(1/2)}, \quad (15.3)$$

$$\underbrace{U_O^P}_{-(-(1/2)(a/b)\alpha)^2} = \underbrace{U_C^P}_{-((1/2)(a/b)\alpha)^2} < \underbrace{U_E^P}_0 = \underbrace{U_R^P}_0, \quad (15.4)$$

These results are qualitatively similar to those in the quadratic case. Note that another outcome scenario, $\pi^e > \pi$ where $\pi_O^e = \pi^T + (1/2)(a/b)\alpha$, $\pi_O = \pi^T$, and subscript O stands for over-expected. Point O does not lie in either the OP line or the REE line (see Figure 5). Finally, using (15.3),

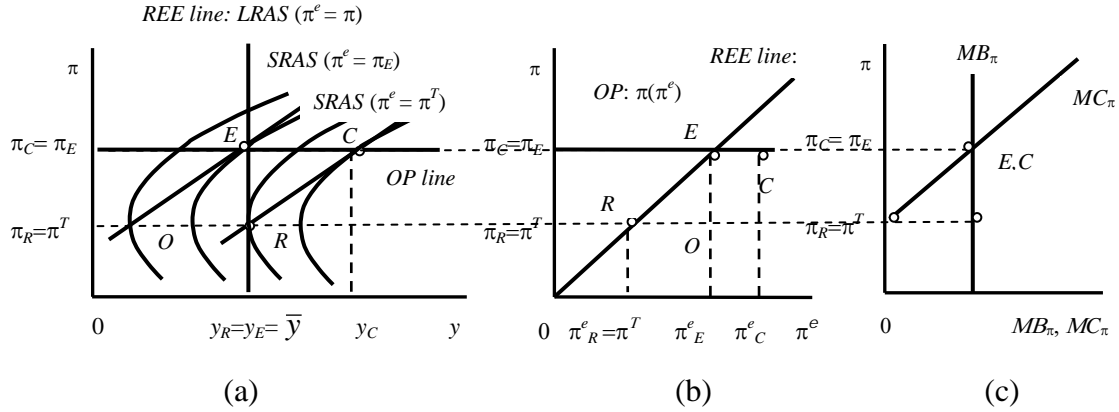
$$\text{incentive to cheat} \equiv U_C - U_R = (1/2)(a/b)\alpha\alpha^2(1/2) > 0,$$

and, using (15.1),

$$\text{inflation bias} \equiv \pi_E - \pi_R = (1/2)(a/b)\alpha > 0.$$

The incentive to cheat (or temptation to cheat) is larger and thus the amount of inflation bias will be larger, the larger is α , the larger is a , and the smaller is b as in the quadratic case but are no longer affected by $(y^T - \bar{y})$ unlike in the quadratic case.

Figure 5. Possible Outcomes: Quasi-Linear Case



The four possible outcomes – points R , C , O and E in Figure 5 – are also analyzed using Table 1 below along with (15.3) and (15.4). Essentially, it is a game between policy maker who chooses the inflation rate and the utility the private sector which chooses the expected inflation rate, where the payoff of each depends on the action of the other. It is assumed that both the policymaker and the private sector have common knowledge and common rationality.

Table 1

		policy maker	
		$\pi = \pi^T$	$\pi = \pi^T + (1/2)(a/b)\alpha$
private sector	$\pi^e = \pi^T$	U_R^P, U_R	U_C^P, U_C
	$\pi^e = \pi^T + (1/2)(a/b)\alpha$	U_O^P, U_O	U_E^P, U_E

This is a game with sequential move, with the private sector moving first but, since the equilibrium solution is derived by backward induction, it is as if the policy maker is the one moving first. The policy maker would reason that if the private sector chooses $\pi^e = \pi^T$, then

he/she should choose $\pi = \pi^T + (1/2)(a/b)\alpha$ since $U_C > U_R$ but if the private sector chooses $\pi^e = \pi^T + (1/2)(a/b)\alpha$, then he/she should choose $\pi = \pi^T + (1/2)(a/b)\alpha$ since $U_E > U_O$. So no matter what the private sector expects, the policy maker would choose $\pi = \pi^T + (1/2)(a/b)\alpha$. Knowing this, the private sector would choose $\pi = \pi^T + (1/2)(a/b)\alpha$ since $U_E^P > U_C^P$.

4. Rules vs Discretion in the Inflation-Output Example

In this inflation-output example, the policy maker, whether under rules or discretion, attempts to optimize; specifically, he/she maximizes his/her objective function subject to the *LRAS* in the case of rules and subject to the *SRAS* in the case of discretion. Whether the policy maker's preferences are quadratic or quasi-linear, policy conducted according to a rule (chosen to pertain to a large number of periods) would lead to a $\pi = \pi_R = \pi^T$ for all t . In contrast, policy choices made in a discretionary period-by-period manner would lead, for all t , to $\pi = \pi_E = \pi^T + (1/2)(a/b)\alpha$ in the quasi-linear case and $\pi = \pi_E = \pi^T + (a/b)\alpha(y^T - \bar{y})$.

According to McCallum (1989), “the rules versus discretion distinction centers on the process by which the π value is determined, not what those values turn out to be”. Specifically, under discretion, the policy maker chooses each period t 's value of π , on the basis of period t 's optimization; this means that the policy maker optimizes in period t so as to choose period t 's (single period's) π . Thus, a discretionary policy is conducted on a period-by-period basis, and therefore the π choices in the different periods are independent from each other. By contrast, In the case of a rule, the policy maker optimizes so as to choose a π rule to be applicable for a large number of periods, not just the current period, a π which he/she merely implements in each period. In short, “rule-type policymaking involves implementation in each period of a formula designed to apply to periods in general, while discretionary policymaking involves freshly made

decisions in each period” (McCallum (1989)).

5. Conclusion

This paper has attempted to provide the algebra and a panel diagram for the ‘time inconsistency: the Phillips curve example’, the most popular example in the literature when introducing the concept. Specifically, the three kinds of diagrams used in the literature – the IC- $LRAS$ - SAS in $y - \pi$ diagram, the $\pi^e = \pi$ line and the OP line in the $\pi^e - \pi$ diagram, and the $MB_\pi, MC_\pi - \pi$ diagram – are placed as panels in a single diagram.

The resulting panel diagram is used to analyze the different possible outcomes, depending on the scenarios – rule or pre-commitment, cheating, and equilibrium. It is shown that: (i) the rule or pre-commitment solution is the long-run optimal point (a point of tangency between the $LRAS$ and an indifference curve) and a REE point, but not a short-run optimal point since it is off the OP line (not a point of tangency between $SRAS$ and indifference curve and an outcome where $MB_\pi > MC_\pi$; (ii) the cheating solution is a point on the OP line (a tangency between another $SRAS$ and another indifference curve and thus a $MB_\pi = MC_\pi$ point) but *off* the REE line, i.e., it is short-run optimal but rational expectations equilibrium does not hold; and, (3) the equilibrium solution is a point on the OP line (a tangency between still another $SRAS$ and still another indifference curve and thus another $MB_\pi = MC_\pi$ point) and on the REE line.

In the literature, the rule solution is labeled as time inconsistent while the equilibrium solution is labeled as time consistent. The terms “time inconsistency” or “dynamic inconsistency” are a misnomer because there is actually no inconsistency in the so called “time inconsistency” or “dynamic inconsistency” problem. Choices are different because the policy maker faces different constraints depending on the private sector’s on inflationary expectations, as shown in the panel diagrams. Different optimization problems yield different solutions or,

graphically, different points of intersections. That the terms “time inconsistency” or “dynamic inconsistency” are a misnomer is even more visible when the possible outcomes are analyzed using simple game theory concepts.

Finally, the terms “time inconsistency” or “dynamic inconsistency” in this Phillips curve example are misleading because the model is a one-period model and therefore a static model.

Appendix A

Derivation of the *OP* Line

The policy maker chooses output and inflation to maximize

$$U_t = -a(y_t - y^T)^2 - b(\pi_t - \pi^T)^2, \quad 0 < a, b < \infty, y^T > \bar{y} > 0, \pi^T > 0 \quad (A1)$$

subject to

$$y_t = \bar{y} + \alpha(\pi_t - \pi_t^e), \quad \alpha > 0. \quad (A2)$$

The solution to this problem yields the condition for optimal policy (*OP*). The *OP* line can be illustrated in either the $y - \pi$ diagram or the $\pi^e - \pi$ diagram.

The alternative ways to get the equation for the *OP* line are: the method of equating slopes, the method of equating marginal benefit and marginal cost, the substitution method, and the Lagrangian method, as shown below.

Method of equating slopes. The slope of the SAS can easily be seen by rewriting (A2) as

$$\pi_t = \underbrace{\left(\pi_t^e - \frac{1}{\alpha} \bar{y} \right)}_{\text{v. intercept}} + \underbrace{\left(\frac{1}{\alpha} \right)}_{\text{slope}} y_t. \quad (A7)$$

The slope of the indifference ellipse is found by setting the total differential of U (A1) equal to zero,

$$\begin{aligned} dU = 0 &= -2a(y - y^T)dy - 2b(\pi - \pi^T)d\pi, \\ \Rightarrow \left(\frac{d\pi}{dy} \right)_{dU=0} &= -\frac{a(y - y^T)}{b(\pi - \pi^T)}. \end{aligned} \quad (A3)$$

Equating the slope of the *SRAS* (from (A7)) and the slope of the indifference curve (A3),

$$\underbrace{-\frac{a(y - y^T)}{b(\pi - \pi^T)}}_{\left(\frac{d\pi}{dy}\right)_{IC}} = \underbrace{\frac{1}{\alpha}}_{\left(\frac{d\pi}{dy}\right)_{PC}},$$

yields

$$y - y^T = \frac{b}{a\alpha}(\pi - \pi^T) \quad \Leftrightarrow \quad \pi = \underbrace{\pi^T + \frac{a\alpha}{b}y^T}_{\text{v.intercept} > 0} - \underbrace{\frac{a\alpha}{b}y}_{\text{slope} < 0}. \quad (\text{A4.1})$$

which is the equation for *OP* line in the $y - \pi$ diagram.

The $\pi(y)$ equation (A4.1) as the equation for the *OP* line can be converted into $\pi(\pi^e)$ equation. Combining (A4.1) and (A2),

$$\pi = \pi^T + \frac{a\alpha}{b}y^T - \frac{a\alpha}{b}\overbrace{(\bar{y} + \alpha(\pi - \pi^e))}^{=y, \text{ using (1)}},$$

and rearranging/simplifying,

$$\pi\left(1 + \frac{a\alpha^2}{b}\right) = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) + \frac{a\alpha^2}{b}\pi^e$$

yields the equation for the *OP* line in $\pi^e - \pi$ diagram,

$$\pi = \underbrace{\frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})}_{\text{v.intercept} > 0} + \underbrace{\frac{a\alpha^2}{b + a\alpha^2}\pi^e}_{0 < \text{slope} < 1}. \quad (\text{A4.2})$$

Lagrangian method. The Lagrangian of the problem is

$$L = -a(y - y^T)^2 - b(\pi - \pi^T)^2 + \lambda[\bar{y} + \alpha(\pi - \pi^e) - y], \quad (\text{A5})$$

and the first-order necessary conditions are

$$\frac{\partial L}{\partial \pi} = -2b(\pi - \pi^T) + \lambda\alpha = 0, \quad (\text{A5.1})$$

$$\frac{\partial L}{\partial y} = -2a(y - y^T) - \lambda = 0, \quad (\text{A5.2})$$

$$\frac{\partial L}{\partial \lambda} = \bar{y} + \alpha(\pi - \pi^e) - y = 0, \quad (\text{A5.3})$$

where λ is the Lagrange multiplier. Plugging in the λ implied by (A5.2) into (A5.1),

$$y - y^T = \frac{b}{a\alpha}(\pi - \pi^T) \Leftrightarrow \pi = \underbrace{\pi^T + \frac{a\alpha}{b}y^T}_{\text{v. intercept}} - \underbrace{\frac{a\alpha}{b}y}_{\text{slope}},$$

also yields (A4.1), the equation for the OP line in the $y - \pi$ diagram. (A5.3) recovers the constraint (A2).

Again, the $\pi(y)$ equation (A4.1) as the equation for the OP line can be converted into $\pi(\pi^e)$ equation (A4.2). Substituting (A1.3) or (A2) for y in the $\pi(y)$ equation (A4.1) yields the $\pi(\pi^e)$ equation (A4.2).

Substitution Method. In this method, the constrained maximization problem is converted into an unconstrained maximization problem. Using (A2) to substitute out for y in (A1), the policy maker's problem becomes

$$\max_{\pi} U = -a(\overbrace{(\bar{y} + \alpha(\pi - \pi^e) - y^T)}^{= y, \text{ using (1)}})^2 - b(\pi - \pi^T)^2$$

and the condition for maximization is

$$\frac{\partial U}{\partial \pi} = 0: \quad \underbrace{-2a\alpha(\alpha(\pi - \pi^e) - (y^T - \bar{y}))}_{=\frac{\partial U}{\partial y} \cdot \frac{\partial y}{\partial \pi}} + \underbrace{-2b(\pi - \pi^T)}_{=\frac{\partial U}{\partial \pi}} = 0, \quad (\text{A6})$$

where

$$\frac{\partial U}{\partial y} \cdot \frac{\partial y}{\partial \pi} \equiv MB_{\pi} = -2a\alpha(\alpha(\pi - \pi^e) - (y^T - \bar{y})) \Leftrightarrow \pi = \underbrace{\pi^e + \frac{1}{\alpha}(y^T - \bar{y})}_{\text{v. intercept}} - \underbrace{\frac{1}{2a\alpha^2}MB_{\pi}}_{\text{slope}}, \quad (\text{A6.1})$$

$$\frac{\partial U}{\partial \pi} \equiv (-)MC_{\pi} = -2b(\pi - \pi^T) \quad \Leftrightarrow \quad \pi = \underbrace{\pi^T}_{v.\text{intercept}} + \underbrace{\frac{1}{2b}}_{slope} MC_{\pi}, \quad (\text{A6.2})$$

and $(\partial U / \partial y)(\partial y / \partial \pi) = MB_{\pi}$ = marginal utility, or marginal benefit of higher inflation through higher output while $\partial U / \partial \pi = (-)MC_{\pi}$ = marginal utility of higher inflation, or the negative of the marginal cost of higher inflation.

Simplifying (A6),

$$\begin{aligned} 2a\alpha[-\alpha(\pi - \pi^e) + (y^T - \bar{y})] &= 2b(\pi - \pi^T) \\ -(a/b)\alpha^2(\pi - \pi^e) + (a/b)\alpha(y^T - \bar{y}) &= (\pi - \pi^T) \\ \pi(1 + (a/b)\alpha^2) &= \pi^T + (a/b)\alpha(y^T - \bar{y}) + (a/b)\alpha^2\pi^e \\ \pi\left(\frac{b + a\alpha^2}{b}\right) &= \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) + \frac{a\alpha^2}{b}\pi^e \\ \pi &= \frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2}\pi^e, \end{aligned}$$

also yields the $\pi(\pi^e)$ equation (A4.2).

The $\pi(\pi^e)$ equation (A4.2) as the equation for the *OP* line can also be converted into

$\pi(y)$ equation (A4.1). Using the *SRAS* (A2) to substitute for π^e in (A4.2),

$$\pi = \frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2} \overbrace{[\pi - (1/\alpha)(y - \bar{y})]}^{\pi^e \text{ from (A2)}}$$

and simplifying,

$$\begin{aligned} \pi\left(1 - \frac{a\alpha^2}{b + a\alpha^2}\right) &= \frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}y^T - \frac{a\alpha}{b + a\alpha^2}y \\ \pi\left(\frac{b}{b + a\alpha^2}\right) &= \frac{b}{b + a\alpha^2}\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - y) \\ \pi &= \pi^T + \frac{a\alpha}{b}(y^T - y) \end{aligned}$$

yields the $\pi(y)$ equation (A4.1).

Equating marginal benefit and marginal cost. Notice from (A3) that the $\pi(\pi^e)$ equation (A4.2) can be derived by equating the MB_π equation (A6.1) and the MC_π equation (A6.2):

$$\underbrace{-2a\alpha(\alpha(\pi - \pi^e) - (y^T - \bar{y}))}_{MB_\pi} = \underbrace{2b(\pi - \pi^T)}_{MB_\pi}$$

$$\pi = \frac{b}{b + a\alpha^2} \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}) \frac{a\alpha^2}{b + a\alpha^2} \pi^e.$$

The Private Sector's *REE* Line and the Vertical *LRAS*

The private sector chooses the expected inflation rate π^e so as to maximize

$$U_t^P = -(\pi_t - \pi_t^e)^2, \quad (A7)$$

and its utility is maximized when

$$\partial U^P / \partial \pi = -2(\pi - \pi^e) = 0 \Leftrightarrow \pi^e = \pi \quad (A7.1)$$

where

$$\pi_t^e \equiv E_{t-1} \pi_t. \quad (A7.1)$$

(3.1) is the rational expectations equilibrium (*REE*), or perfect foresight equilibrium while (A7.1)

is the *RE* assumption. In a $\pi^e - \pi$ diagram, the *REE* line is the $\pi^e = \pi$ line or the 45° line; in a

$y - \pi$ diagram, the *REE* line is the same as the vertical *LRAS*.

Appendix B

Solutions for π, y, U , and U^P Under the Different Scenarios

The solution for the optimal values of π and y are derived using, respectively, (A4.2) and (A2). The value of U is derived using (A1) while the value U^P is derived from (A7).

Rule or Commitment (R). In this scenario,

$$\pi^e = \pi_R^e = \pi^T, \quad (\text{B1.1})$$

$$\pi = \pi_C = \pi^T. \quad (\text{B1.2})$$

which, when substituted into the *SRAS* (A2), yields the output under commitment:

$$y_R = \bar{y} + \alpha(\overbrace{\pi^T}^{\pi=\pi_R} - \overbrace{\pi^T}^{\pi^e=\pi_R^e}) = \bar{y}. \quad (\text{B1.3})$$

The utility of the policy maker, using (A1), (8,2), and (B1.3), is

$$U_R = -a(\overbrace{\bar{y}}^{y=y_R} - y^T)^2 - b(\overbrace{\pi^T}^{\pi=\pi_R} - \pi^T)^2 = -a(\bar{y} - y^T)^2, \quad (\text{B1.4})$$

while the utility of the private sector, using (A7), (B1.1), and (B1.2), is

$$U_R^P = -(\overbrace{\pi^T}^{\pi=\pi_R} - \overbrace{\pi^T}^{\pi^e=\pi_R^e})^2 = 0. \quad (\text{B1.5})$$

Note that in this scenario, the problem is reduced to maximizing (A1) subject to the vertical *LRAS* (A7.1),

$$\begin{aligned} & \max_{\pi} \left\{ U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \right\} \\ & \text{s.t.} \quad y = \bar{y} \end{aligned} \quad (\text{P1})$$

which yields the optimal choice $\pi = \pi^T$.

Cheating (C). In this scenario, the problem is given by

$$\left. \begin{aligned} & \max_{y, \pi} \left\{ U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \right\} \\ & s.t. \quad y = \bar{y} + \alpha(\pi - \pi^e) \\ & \quad \pi^e = \pi^T \end{aligned} \right\}. \quad (P2)$$

This is so because the private sector's expectation is given by

$$\pi^e = \pi_C^e = \pi^T, \quad (B2.1)$$

but the policy maker will choose π based on the $\pi(\pi^e)$ equation (A4.2), the equation for the OP line. Substituting (B2.1) into (A4.2) yields the solution for the inflation rate under cheating

$$\begin{aligned} \pi_C &= \frac{b}{b + a\alpha^2} \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2} \overbrace{(\pi^T)}^{\pi^e = \pi_C^e} \\ &= \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}). \end{aligned} \quad (B2.2)$$

Substituting (B2.1) and (B2.2) into (A2) gives the solution for output under cheating,

$$\begin{aligned} y_C &= \bar{y} + \alpha \left(\overbrace{\left(\pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}) \right)}^{\pi = \pi_C} - \overbrace{\pi^T}^{\pi^e = \pi_C^e} \right), \\ &= \bar{y} + \frac{a\alpha^2}{b + a\alpha^2} (y^T - \bar{y}). \end{aligned} \quad (B2.3)$$

Using (A1), (B2.2) for π , and (B2.3) for y , the utility of the policy maker under cheating is

$$\begin{aligned}
U_c &= -a \left(\overbrace{\left(\bar{y} + \frac{a\alpha^2}{b+a\alpha^2} (y^T - \bar{y}) \right)}^{y=y_c} - y^T \right)^2 - b \left(\overbrace{\left(\pi^T + \frac{a\alpha}{b+a\alpha^2} (y^T - \bar{y}) \right)}^{\pi=\pi_c} - \overbrace{\pi^T}^{\pi^e=\pi_c^e} \right)^2 \\
&= -a \left(\left(\frac{a\alpha^2}{b+a\alpha^2} - 1 \right) (y^T - \bar{y}) \right)^2 - b \left(\left(\frac{a\alpha}{b+a\alpha^2} \right) (y^T - \bar{y}) \right)^2 \\
&= - \left[a \left(\frac{-b}{b+a\alpha^2} \right)^2 + b \left(\frac{a\alpha}{b+a\alpha^2} \right)^2 \right] (y^T - \bar{y})^2 \\
&= - \left(\frac{ab^2 + b(a\alpha)^2}{(b+a\alpha^2)^2} \right) (y^T - \bar{y})^2 \\
&= -a \left(\frac{b(b+a\alpha^2)}{(b+a\alpha^2)^2} \right) (y^T - \bar{y})^2
\end{aligned}$$

and thus

$$U_c = -a \left(\frac{b}{b+a\alpha^2} \right) (y^T - \bar{y})^2. \quad (\text{B2.4})$$

And, using (A7), (B2.1), and (B2.2), the utility of the private sector is given by

$$\begin{aligned}
U_c^p &= - \left(\overbrace{\pi^T + \frac{a\alpha}{b+a\alpha^2} (y^T - \bar{y})}^{\pi=\pi_c} - \overbrace{\pi^T}^{\pi^e=\pi_c^e} \right)^2 \\
&= - \left(\frac{a\alpha}{b+a\alpha^2} (y^T - \bar{y}) \right)^2 < 0.
\end{aligned} \quad (\text{B2.5})$$

Equilibrium (E). In this scenario, the problem is given by

$$\begin{aligned}
&\max_{y, \pi} \left\{ U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \right\} \\
&\text{s.t.} \quad y = \bar{y} + \alpha(\pi - \pi^e) \\
&\quad \pi = \pi^e \\
&\quad \pi^e \equiv E\pi
\end{aligned} \quad (\text{P3})$$

or, equivalently, the policy maker's problem is

$$\max_{y, \pi} \left\{ U = -a(y - y^T)^2 - b(\pi - \pi^T)^2 \right\} \quad \text{s.t.} \quad y = \bar{y} + \alpha(\pi - \pi^e) \quad (\text{P3.1})$$

and the private sector's problem is

$$\max U^P = -(\pi - \pi^e)^2 \quad \text{where} \quad \pi^e = E\pi . \quad (\text{P3.2})$$

The inflation rate chosen optimally by the policy maker is a function of inflationary expectations . Since expectations are rational, and since the private sector knows that inflation rate will be set based on (A4.2), the equilibrium expected inflation rate must satisfy

$$\begin{aligned} \pi^e &= E\pi \\ &= E \left[\frac{b}{b + a\alpha^2} \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2} \pi^e \right] \end{aligned}$$

Using (A4.2) for π and noting that $E\pi^T = \pi^T$, $E\bar{y} = \bar{y}$, and $E(\pi^e) = E(E\pi) = E\pi = \pi^e$,

$$\begin{aligned} \pi^e &= E \left[\underbrace{\frac{b}{(b + a\alpha^2)} \pi^T + \frac{a\alpha}{(b + a\alpha^2)} (y^T - \bar{y}) + \frac{a\alpha^2}{(b + a\alpha^2)} \pi^e}_{\text{using (6.2) for } \pi} \right] \\ \pi^e \left(1 - \frac{a\alpha^2}{b + a\alpha^2} \right) &= \frac{b}{(b + a\alpha^2)} \pi^T + \frac{a\alpha}{(b + a\alpha^2)} (y^T - \bar{y}) \\ \pi^e &= \frac{b + a\alpha^2}{b} \cdot \left[\frac{b}{(b + a\alpha^2)} \pi^T + \frac{a\alpha}{(b + a\alpha^2)} (y^T - \bar{y}) \right] \end{aligned}$$

yields the equilibrium expected inflation rate,

$$\pi_E^e = \pi^T + \frac{a\alpha}{b} (y^T - \bar{y}) . \quad (\text{B3.1})$$

Substitution of (B3.1) into (A4.2),

$$\begin{aligned} \pi_E &= \frac{b}{b + a\alpha^2} \pi^T + \frac{a\alpha}{b + a\alpha^2} (y^T - \bar{y}) + \frac{a\alpha^2}{b + a\alpha^2} \underbrace{\left(\pi^T + \frac{a\alpha}{b} (y^T - \bar{y}) \right)}_{=\pi^e = \pi_E^e} \\ &= \left(\frac{b + a\alpha^2}{b + a\alpha^2} \right) \pi^T + \left(\frac{ba\alpha + a\alpha^2(a\alpha)}{b(b + a\alpha^2)} \right) (y^T - \bar{y}) \\ &= \pi^T + \left(\frac{a\alpha(b + a\alpha^2)}{b(b + a\alpha^2)} \right) (y^T - \bar{y}) \end{aligned}$$

yields the equilibrium inflation rate,

$$\pi_E = \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}), \quad (\text{B3.2})$$

which is the same as the equilibrium expected inflation rate (B3.1).

Using (A2), (B3.1), and (B3.2),

$$y_E = \bar{y} + \alpha \left(\overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) \right)}^{\pi = \pi_E} - \overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) \right)}^{\pi^e = \pi_E^e} \right)$$

the equilibrium level of output is

$$y_E = \bar{y}. \quad (\text{B3.3})$$

Using (A1), (B3.3), (B3.2), and (B3.1),

$$\begin{aligned} U &= -a \left(\overbrace{\bar{y}}^{y=y_E} - y^T \right)^2 - b \left(\overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) \right)}^{\pi = \pi_E} - \overbrace{\pi^T}^{\pi^e = \pi_E^e} \right)^2 \\ &= -\left(a + \frac{b(a\alpha)^2}{b^2}\right)(\bar{y} - y^T)^2 \\ &= -\frac{ab^2 + b(a\alpha)^2}{b^2}(\bar{y} - y^T)^2 \\ &= -ab \left(\frac{b + a\alpha^2}{b^2} \right) (\bar{y} - y^T)^2 \\ &= -ab \left(\frac{b + a\alpha^2}{b^2} \right) (y^T - \bar{y})^2 \end{aligned}$$

and thus the utility of the policy maker is

$$U_E = -a \left(\frac{b + a\alpha^2}{b} \right) (y^T - \bar{y})^2. \quad (\text{B3.4})$$

Using (A1), (B3.1), and (B3.2), the utility of the private sector is

$$U_E^P = - \left(\overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) \right)}^{\pi = \pi_E} - \overbrace{\left(\pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) \right)}^{\pi^c = \pi_E^c} \right)^2 = 0. \quad (\text{B3.5})$$

Comparing the Different Possible Inflation-Output Outcomes. Comparing the results,

$$\overbrace{\pi^T + \frac{a\alpha}{b}(y^T - \bar{y})}^{\pi_E} > \overbrace{\pi^T + \frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y})}^{\pi_C} > \overbrace{\pi^T}^{\pi_R} > 0, \quad (\text{B4.1})$$

$$\overbrace{\bar{y} + \left(\frac{a\alpha^2}{b + a\alpha^2} \right)(y^T - \bar{y})}^{y_C} > \overbrace{\bar{y}}^{y_E = y_R} > 0, \quad (\text{B4.2})$$

$$0 < \overbrace{-a \left(\frac{b + a\alpha^2}{b} \right)(y^T - \bar{y})^2}^{U_E} < \overbrace{-a(y^T - \bar{y})^2}^{U_R} < \overbrace{-a \left(\frac{b}{b + a\alpha^2} \right)(y^T - \bar{y})^2}^{U_C}, \quad (\text{B4.3})$$

$$\overbrace{0}^{U_E^P = U_R^P} < \overbrace{- \left(\frac{a\alpha}{b + a\alpha^2}(y^T - \bar{y}) \right)^2}^{U_C^P}. \quad (\text{B4.4})$$

Inflation bias and incentive to cheat. The inflation bias,

$$\begin{aligned} \text{inflation bias} &\equiv \pi_E - \pi_R \\ &= \pi^T + \frac{a\alpha}{b}(y^T - \bar{y}) - \pi^T \\ &= \frac{a\alpha}{b}(y^T - \bar{y}), \end{aligned} \quad (\text{B5.1})$$

arises in this model because the policy maker has an incentive to cheat,

$$\begin{aligned}
\text{incentive to cheat} &\equiv U_C - U_R = -a \left(\frac{b}{b + a\alpha^2} \right) (y^T - \bar{y})^2 - (-)a \left(\frac{b + a\alpha^2}{b} \right) (\bar{y} - y^T)^2 \\
&= \left(\underbrace{\frac{b + a\alpha^2}{b}}_{>1} - \underbrace{\frac{b}{b + a\alpha^2}}_{<1} \right) a (y^T - \bar{y})^2 > 0 \\
&= \left(\frac{a\alpha^2(2b + a\alpha^2)}{b(b + a\alpha^2)} \right) a (y^T - \bar{y})^2 > 0.
\end{aligned} \tag{B5.2}$$

References

- Barro, Robert J. and Gordon, David B. (1983a). A positive theory of monetary policy in a natural rate model. *Journal of Political Economy* 91(4):589-610.
- Barro, Robert J. and Gordon, David B. (1983b). Rules, discretion, and reputation in a model of monetary policy. *Journal of Monetary Economics* 12:101-120.
- Blanchard, Olivier J. and Fischer, Stanley (1989). *Lectures on Macroeconomics*. MIT Press, Cambridge, Mass.
- Carlin, Wendy and Soskice, David. (2005). *Macroeconomics: Imperfections, Institutions, and Policies*. Oxford University Press.
- Calvo, Guillermo A. (1978). On the time consistency of optimal policy in a monetary economy. *Econometrica* 46 (6); 1411-1428.
- Cukierman, Alex (1992). *Central Bank strategy, Credibility, and Independence*. MIT Press: Cambridge, Mass.
- Fischer, Stanley (1980). Dynamic inconsistency, cooperation, and the benevolent dissembling government. *Journal of Economic Dynamics and Control* 2:93-107.
- Fudenberg, Drew and Tirole, Jean (1991). *Game Theory*. MIT Press: Cambridge, Mass.
- Gartner, Manfred. (2009). *Macroeconomics*. 3e. PrenticeHall.
- Heijdra, Ben J. and van der Ploeg, Frederick. (2002). *Foundations of Modern Macroeconomics*. Oxford University Press.
- Kydland, Finn E. and Prescott, Edward C. (1977). Rules rather than discretion: the inconsistency of optimal plans. *Journal of Political Economy* 85(4): 473-491.
- McCallum, Bennett T. (1989). *Monetary Economics: Theory and Policy*. MacMillan: NY.
- Obstfeld, Maurice and Rogoff, Kenneth (1996). *Foundations of International Macroeconomics*. MIT Press: Cambridge, Mass.
- Pollak, R. A. (1968). Consistent planning. *Review of Economic Studies* 35 (4): 201–208.
- Romer, David. (2006). *Advanced Macroeconomics*. McGraw-Hill.
- Strotz, Robert H. (1956). Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies* 23 (4): 165–180.
- Simons, Henry C. (1936). Rule versus authorities in monetary policy. *Journal of Political Economy* 44 (1): 1–30.
- Walsh, Carl E. (2010). *Monetary Theory and Policy*. MIT Press: Cambridge, Mass.