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Gatekeeper Versus Auctioneer:
A Non-Tatonnement Result

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GATEKEEPER VERSUS AUCTIONEER: A NON-TATONNEMENT RESULT

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Abstract

A non-tatonnement process is described using the simplest demand-and-supply model, involving the following: uniformly distributed agents; random matching of buyers and sellers; and a universal permission to engage in mutually acceptable trade at non-equilibrium prices. A sufficient condition is then stated where expected welfare gains are paradoxically greater when the number of market agents is restricted, compared to when all traders are allowed to participate.

1. Define market demand and supply in the familiar partial-equlibrium model as follows:

$$P(x) = a - bx \tag{1}$$
$$R(y) = c + ey \tag{2}$$

where buyers with unit-demands, indexed by x, are uniformly distributed in the interval $X = [0, 1]$, with $P(x)$ representing their bid-prices. Unit-sellers indexed by y, on the other hand, are similarly distributed in $Y = [0, 1]$ with $R(y)$ representing their respective offer-prices. Then it is well-known that the equilibrium price and quantity are the following:

$$x^* = y^* = \left(\frac{a-c}{b+e}\right) \leq 1 \text{ and } P^* = R^* = \left(\frac{ae-bc}{b+e}\right). \tag{3}$$

That is, all sellers and buyers with indexes in the interval $[0, x^*]$ are able to transact at the common equilibrium price, P^*, while those in $(x^*, 1]$ are unable to exchange. This process results in a maximum of social surplus (i.e., the sum of consumers' and producers' surpluses) equal to $½(a-c)^2/(b+e)$.

Equally well-known, however, is the fact that the above results are straightforward only in a process of *tatonnement* [originally Walras 1954(1926): 84], involving: (i) a common price being centrally quoted and used to evaluate the feasibility of all notional and actual transactions and (ii) a prohibition on the execution of any actual private exchanges until and unless the equilibrium price in (3) has been discovered and announced. A process such as that described is observed only in the most *formally organised* exchanges, e.g., stock exchanges.[1] Leijonhuvud [1968] rediscovered this fact and was apparently the first

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[1] Indeed, to explain *tatonnement*, Walras [1954(1926)] explicitly used the example of the buying and selling of French Rentes in the Paris Bourse.

to coin the metaphor of the "Walrasian auctioneer" to give a realistic representation of what, for Walras, was always an ideal process.[2]

2. The question has therefore perennially been raised (albeit in different ways[3]), whether the above picture of organised exchange with centrally provided price information and a credibly enforced prohibition of otherwise mutually beneficial trades is the best representation of Smith's "invisible hand". After all, a full specification of the *institutional requirements* of a *tatonnement* process significantly circumscribes – if it does not undermine – the argument for markets as self-regulating mechanisms that rely primarily on private information and permit actions (within the bounds of law) based solely on considerations of self-interest.

3. Here instead we ask what welfare results arise when one drops the *tatonnement* assumption and allows instead for the simplest consideration of an unorganized market. The model we consider has the following characteristics: (i) uniformly distributed unit-buyers and -sellers with reservation bid- and offer-prices as described in (1)-(2); (ii) random bilateral matching of buyers with sellers[4]; and (iii) a universal permission to exchange when a buyer's bid-price is greater than or equal to a seller's offer-price. We then ask: in such conditions, what is the expected value of the difference between the bid and the ask when *all* agents are allowed to participate?

4. Under the conditions stated, if two agents x and y meet, voluntary trade takes place if the buyer's bid price (which is assumed equal to his reservation price) is greater than or equal to the (randomly matched) seller's offer or reservation price, i.e., if $P(x) \geq R(y)$. The nonnegative gain from such a trade is equal to

$$V(x, y) = (a - bx) - (c + ey) = (a - c) - (bx + ey).$$

Random matches that yield $V(x, y) < 0$ obviously do not take place and are not counted towards the surplus. In general, then, the *expected* value of social gains V across accomplished trades takes the form

$$\int_{x\in X} f_x(x) \int_{y\in Y} f_y(y)V(x, y)dydx\,, \qquad (4)$$

[2] The formal description of the rules to be followed by a "super-auctioneer", however, and its use "to give some flesh to an abstraction" in a general competitive model is probably first found in the definitive work of Arrow and Hahn [1971:264; 266-270].
[3] It suffices here to note that one of the very earliest to raise doubts regarding the *tatonnement* process was Arrow himself, who noted a "logical gap" in competitive price theory: "It isn't explained whose decision it is to change prices in accordance with [the 'Law of Supply and Demand']. Each individual participant in the economy is supposed to take prices as given and determine his choices as to purchases and sales accordingly; there is no one left over whose job it is to make a decision on price" [Arrow 1959:43]. An even earlier critique was Goodwin's [1951] gloss of Walras, which Jaffe mentions [Walras 1954(1926):169, endnotes 11 and 12].
[4] The random character of the trades assumed here makes this model a member of the family of statistical equolibirum models described by Foley [1999].

where the limits of integration are defined by the number of traders x and y, and f_x and f_y are the probability density functions of buyers and sellers, respectively, which are identically equal to the standard uniform distribution and everywhere equal to 1.

5. An arbitrary agent x is willing to trade with any supplier y as long as the latter's offer-price is no greater than x's demand price, i.e.,

$c + ey \leq a - bx$, or $y \leq (a - bx - c)/e$.

This means all suppliers in the interval $[0, (a - bx - c)/e]$ are equally potential exchange partners of x. Remembering that $f_x = f_y = 1$ everywhere, the inner integral of (4) can be written as

$$\int_0^{\frac{(a-bx-c)}{e}} (a-bx)-(c+ey)\,dy = \left[ay - bxy - cy - \tfrac{1}{2}ey^2\right]_0^{(a-bx-c)/e} \quad (5)$$

$$= \frac{1}{2e}(a-bx-c)^2 \qquad \text{(see Note 1 for details)} \quad (6)$$

The expression (6) represents the expected value of trades for an arbitrary buyer x. Integrating over all potential buyers gives the expected value of total social gains described in (4). Now the *last buyer* for whom any exchange is feasible is that whose bid-price is at least as great as the lowest-offer price, which is c (See Figure). So the upper limit of integration of x is given as

$a - bx \geq c$, or $\qquad x \leq (a - c)/b$

Performing the outer integral operation in (4) in this case, (again remembering $f_x = 1$ everywhere) one obtains

$$\int_0^{\frac{a-c}{b}} \frac{1}{2e}(a-bx-c)^2 dx \quad (7)$$

$$= \frac{1}{2e}\left[-\frac{1}{3b}(a-bx-c)^3\right]_0^{\frac{a-c}{b}}$$

$$= -\frac{1}{6be}\left[(a-b\left(\frac{a-c}{b}\right)-c)^3 - (a-b(0)-c)^3\right]$$

$$= \frac{(a-c)^3}{6be} \quad (8)$$

Expression (8) therefore is the computed expected total social gain from random feasible exchange *along the entire lengths of the demand and supply curves*.

6. Suppose, however, it was possible to restrict the market participants only to those in the intervals $X' = Y' = [0, (a - c)/(b + e)]$. As a consequence, the common value of the uniform distributions of buyers and sellers would no no longer be unity, but rather $f_x = f_y = (b + e)/(a - c) > 1$ (i.e., remembering that $(a - c)/(b + e) < 1$).

Using this information, we again proceed to evaluate the inner integral in (4):

$$\int_0^{(a-c)/(b+e)} \left(\frac{b+e}{a-c}\right)[(a-bx)-(c+ey)]\,dy$$
$$=\left(\frac{b+e}{a-c}\right)\left[ay-bxy-cy-\tfrac{1}{2}ey^2\right)\Big[_0^{(a-c)/(b+e)}$$
$$=\left(\frac{b+e}{a-c}\right)\left[a\frac{(a-c)}{(b+e)}-bx\frac{(a-c)}{(b+e)}-c\frac{(a-c)}{(b+e)}-\tfrac{1}{2}e\frac{(a-c)}{(b+e)^2}^2\right]$$
$$=\left[a-bx-c-\tfrac{1}{2}e\frac{(a-c)}{(b+e)}\right]$$
This is then integrated over all *x* in the relevant domain:

$$\frac{(b+e)}{(a-c)}\int_0^{(a-c)/(b+e)} a-bx-c-\tfrac{1}{2}\frac{e(a-c)}{(b+e)}\,dx$$

$=\tfrac{1}{2}(a-c)>0$. (see Note 2 for details) (9)

7. The expression (9) should be compared to (8), and we can ask whether conditions exist where

$\tfrac{1}{2}(a-c) > (a-c)^3/6be.$ (10)

A response is given by the following:

> **Theorem**: (*Sufficient condition*). Suppose (i) $a - b \leq c$ and (ii) $c + e \geq a$ with at least one inequality holding strictly. (See Figure.) Then
>
> ½ $(a - c) > (a - c)^3/6be$.
>
> That is, the expected social gains from trade are greater when exchange is *restricted* to the interval [0, $(a - c)/(b + e)$] for both buyers and sellers, compared to those when *all* agents represented over the entire length of the demand and supply curves are allowed to trade.

Proof:

Statements (i) and (ii) imply, respectively, that $(a - c) \leq b$ and $(a - c) \leq e$, and if at least one inequality holds strictly,

$be > (a - c)^2$.

Multiplying both sides by $(a - c)/6$, and the left side again by 3 leaves the sense of the inequality unchanged, yielding the desired result upon rearrangement:
$$\frac{be(a-c)}{6} > \frac{(a-c)^3}{6}$$
$$\frac{3be(a-c)}{6} > \frac{(a-c)^3}{6}$$
$$\tfrac{1}{2}(a-c) > \frac{(a-c)^3}{6be} \quad \blacksquare$$

For illustration we consider the following parameter values:

Parameters	Case 1	Case 2	Case 3
a	10	14	14
b	7	7	4
c	4	5	5
e	7	7	6
Expected gains:			
Restricted trade	3	4.5	4.05
Unrestricted trade	0.73	2.48	5.06

Both sufficient conditions hold in Case 1 but neither holds in Cases 2 and 3. Nonetheless the expected gains to restricted exchange are greater in both Cases 1 and 2, but not in Case 3. This verifies that the stated conditions are indeed sufficient but not necessary. Graphically, one may imagine that the inequality (10) is more likely to hold if – relative to the equlibrium quantity – the nontrading sections of the demand and supply curves (which are therefore "irrelevant" to the outcome) are fairly lengthy.

5. We therefore conclude that where buyers and sellers are randomly matched and are constrained in their actions only by the feasibility of mutually beneficial trade, *restricting the number of market agents can under certain conditons lead to greater expected welfare gains* than pure "laissez-faire", i.e., allowing all comers to participate in trade.

It is as if entrance tickets were to be issued by a gatekeeper to a physical market, with the assignment of tickets being made to depend on whether one has a high (resp. low) bid price (resp. offer price) – versus another scenario where any and all buyers and sellers are allowed to enter. The gatekeeper in this case replaces the Walrasian auctioneer and trading at "false prices" is allowed – indeed, there is no requirement here to arrive at a unified price.

The superior expected welfare gains from restricting the number of agents are the result of reducing the likelihood of trades in which the difference between bid and offer prices is not very large (and therefore where the expected social surplus is small). The same principle is allows a perfectly discriminating monopolist to achieve maximum social surplus (albeit unevenly distributed) by rationing buyers according to their reservation prices.

Optimally rationing the participation of agents may seem to present its own information problems, since it ideally requires the gatekeeper to know, and agents to reveal, their reservation prices. On the other hand, there may in practice be social institutions and signals that sift out, however imperfectly, low bids and high offers, e.g., significant entrance-fees at auctions and market reputations.❑

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Note 1

$$\left[ay - bxy - cy - \tfrac{1}{2}ey^2\right]_0^{(a-bx-c)/e}$$

$$= \frac{1}{e}\left[a(a-bx-c) - bx(a-bx-c) - c(a-bx-c)\right] - \tfrac{1}{2}e\frac{(a-bx-c)^2}{e^2}$$

$$= \frac{1}{e}\left[a^2 - abx - ac - bx(a-bx-c) - ac + bcx + c^2 - \tfrac{1}{2}(a-bx-c)^2\right]$$

$$= \frac{1}{e}\left[a^2 - abx - ac - abx + b^2x^2 + bxc - ac + bcx + c^2) - \tfrac{1}{2}(a-bx-c)^2\right]$$

$$= \frac{1}{e}\left[a^2 - 2abx - 2ac + b^2x^2 + 2bxc + c^2) - \tfrac{1}{2}(a-bx-c)^2\right]$$

$$= \frac{1}{e}\left[(a-bx-c)^2 - \tfrac{1}{2}(a-bx-c)^2\right]$$

$$= \frac{1}{2e}(a-bx-c)^2.$$

Note 2

$$\frac{(b+e)}{(a-c)}\int_0^{(a-c)/(b+e)} a - bx - c - \tfrac{1}{2}\frac{e(a-c)}{(b+e)}\,dx$$

$$= \frac{(b+e)}{(a-c)}\left[ax - \tfrac{1}{2}bx^2 - cx - \tfrac{1}{2}\frac{e(a-c)}{(b+e)}x\right]_0^{(a-c)/(b+e)}$$

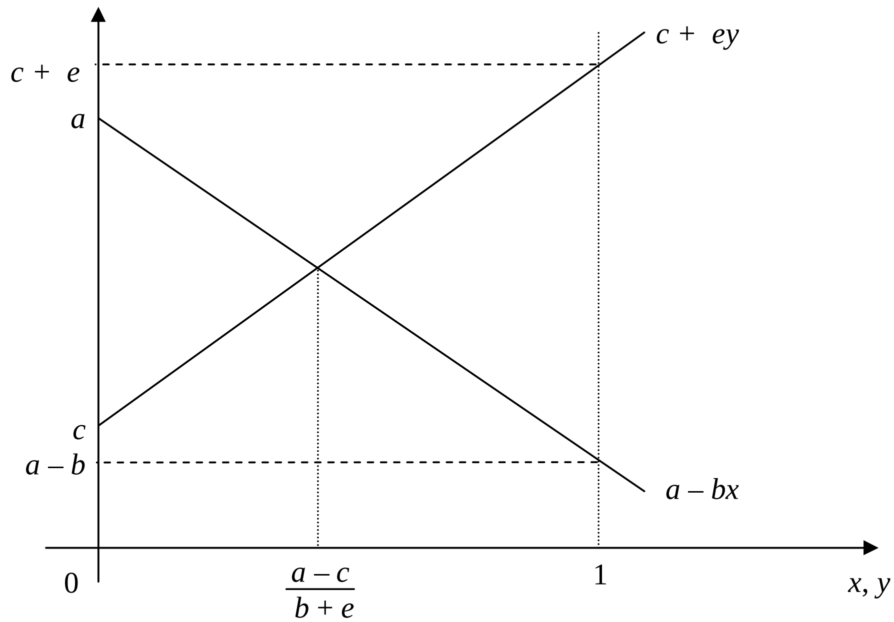
$$= \frac{(b+e)}{(a-c)}\left[a\frac{(a-c)}{(b+e)} - \tfrac{1}{2}b\frac{(a-c)^2}{(b+e)^2} - c\frac{(a-c)}{(b+e)} - \tfrac{1}{2}e\frac{(a-c)^2}{(b+e)^2}\right]$$

$$= \left(a - \tfrac{1}{2}b\frac{(a-c)}{(b+e)} - c - \tfrac{1}{2}e\frac{(a-c)}{(b+e)}\right)$$

$$= \left((a-c) - \tfrac{1}{2}(b+e)\frac{(a-c)}{(b+e)}\right)$$

$$= \tfrac{1}{2}(a-c) > 0.$$

Figure



Condition (i) in the theorem implies that the highest bid-price *a* is lower than the highest offer-price ($c + e$), while condition (ii) requires the lowest offer-price *c* to be greater than the lowest bid price ($a - b$).